PRACTICAL GEOMETRY AND GRAPHICS

D. A. LOW

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WITH OVER 800 ILLUSTRATIONS AND OVER 700 EXERCISES

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PREFACE

An endeavour has been made to provide in this work a fairly complete course of instruction in practical geometry for technical students. The field covered is a very wide one, but by adopting a concise style and by endeavouring to make the illustrations "talk" the work has been kept of reasonable size.

A special feature has been made of the illustrations. These are very numerous, and they have been carefully planned so that even the most complicated of them should be distinct and clearly show the constructions used, notwithstanding that most of them are comparatively small. Pictorial projections have been freely used in dealing with solid geometry, and these in many cases are just as useful as actual models in illustrating the relative positions of points, lines, and planes in space. Nevertheless teachers are recommended to provide models for class use in teaching solid geometry, and the student should also make small cardboard models for himself in studying many of the problems, especially in the earlier stages of his work.

The student cannot hope to master the subject of practical geometry unless he works on the drawing board a large number of examples. Hence, another special feature of this work is the large collection of exercises. These exercises have been prepared and selected with great care, and it will be found that they not only provide ample practice for the student but in many cases they amplify the text. About ninety per cent. of the exercises are original and the remainder have been selected from the examination papers of the Board of Education. To economize space many of the diagrams for the exercises have been placed on a squared ground. It should scarcely be necessary to state that in reproducing these diagrams the squared ground need not be drawn, but by counting the squares the

various points and lines may be plotted from two axes at right angles to one another.

Great credit is due to Mr. J. W. Barrett for the care, intelligence, and skill which he has displayed in making the finished drawings for the illustrations from the author's pencil drawings and sketches.

D. A. L.

East London College (University of London),
September, 1912.

CONTENTS

CHAPTER I

INTRODUCTORY

1. Lines.—Euclid defines a *line* as "that which has length without breadth." A line, therefore, in the strict mathematical sense has no material existence, and the finest "line" that can be drawn on paper is only a rough approximation to a mathematical line. A *straight line* is defined by Euclid as "that which lies evenly between its extreme points." An important property of straight lines is that two straight lines cannot coincide at two points without coinciding altogether. This property is made use of in the applications of a "straight-edge."

A line is also defined as the *locus* or path traced by a moving point. If the direction of motion of the point is constant the locus is a straight line.

In geometry when the term "line" is used it generally stands for "straight line."

2. The Circle—Definitions.—A *circle* is a plane figure contained by one line, which is called the *circumference*, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another: and this point is called the *centre* of the circle.

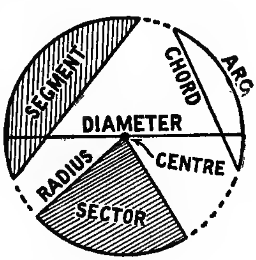


FIG. 1.

The meanings of the terms *radius*, *diameter*, *chord*, *arc*, *sector*, and *segment* of a circle are given in Fig. 1.

The term "circle" is very frequently used when the "circumference" of the circle is meant. Euclid, for example, states that "one circle cannot cut another at more than two points."

"A *tangent* to a circle is a straight line which meets the circumference, but, being produced, does not cut it" (see also Art. 13, p. 12).

3. Angles.—"A *plane angle* is the inclination of two straight lines to one another, which meet together, but are not in the same straight line."

"When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a *right angle*." If it is required to test whether the angle BAC of a set-square, which is nominally a right angle, is really a right angle,

apply the set-square to a straight-edge DE as shown in Fig. 2, and draw the line AB. Next turn the set-square over into the position AB′C′ and draw the line AB′. If AB′ coincides with AB, then the angle BAC is equal to the angle BAC′, and therefore by the definition each is a right angle.

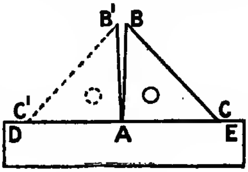


FIG. 2.

An angle which is less than a right angle is called an *acute angle*, and an angle which is greater than a right angle and less than two right angles is called an *obtuse angle*.

If two angles together make up a right angle they are said to be *complementary*, and one is called the *complement* of the other.

If two angles together make up two right angles they are said to be *supplementary*, and one is called the *supplement* of the other.

In measuring angles in practical geometry the unit angle is generally the 1-90th part of a right angle and is called a *degree*. The 1-60th part of a degree is called a *minute*, and the 1-60th part of a minute is called a *second*. 47° 35′ 23″ is to be read 47 degrees, 35 minutes, 23 seconds. The single and double accents which denote minutes and seconds respectively are also used to denote feet and inches respectively, but this double use of these symbols seldom causes any confusion.

Angles are also measured in *circular measure*. AOB (Fig. 3) is an angle, and AB is an arc of a circle whose centre is O. The circular measure of the angle AOB is the ratio $\frac{\text{arc}}{\text{radius}} = \theta$. For two right angles the length of the arc is πr, where $\pi = 3{\cdot}1416$, and the circular measure of two right angles is therefore equal to π. When the arc is equal to the radius the angle is the unit angle in circular measure and is called a *radian*.

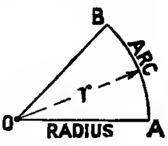


FIG. 3.

If n is the number of degrees in an angle whose circular measure is θ radians, then $\frac{n}{180} = \frac{\theta}{\pi}$, since each of these is the fraction which the angle is of two right angles.

Hence $n = \frac{180\theta}{\pi} = 57{\cdot}2958\theta$, and $\theta = \frac{\pi n}{180} = 0{\cdot}01745n$.

The instrument most commonly used for measuring angles in degrees is the protractor, portions of two forms of which are shown in Fig. 4. The angle AOB on the paper beneath the protractor is seen to be 55°. The protractor may be made of boxwood, ivory, celluloid, or cardboard.

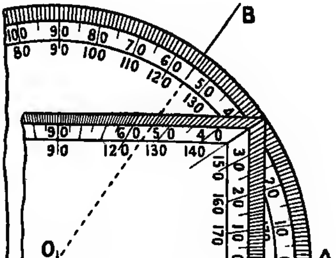


FIG. 4.

A *scale of chords* is a convenient instrument for measuring angles. The way in which the scale is constructed is shown in Fig. 5. ADF is a right angle, and the arc ACF is described from the centre D. The arc ACF is shown divided into 18 equal parts, each part therefore subtends at D an angle of 5°. On the scale of chords AB, lengths are marked off from A equal to the chords of the arcs of the different angles. For example, AE is the chord of the arc which subtends an angle ADE of 40° at the centre D, and this length is marked off on AB by describing the arc EH from the centre A. It will be found that the length of the chord AC for 60° is equal to the radius of the arc ACF Hence in using the scale of chords the radius of the arc is made equal to the distance between the points marked 0 and 60.

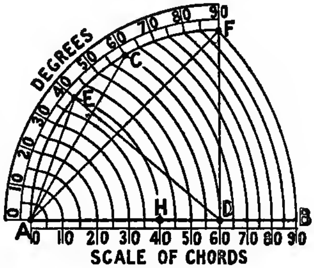


FIG. 5.

The scale shown in Fig. 5 only shows the chords for angles which are multiples of 5° up to 90°, but by making the scale large enough, single degrees, and even fractions of a degree, may be shown. It may also be extended beyond 90°. But when an angle is greater than 90° its supplement, which will be less than 90°, should first be determined. It should be noticed that the divisions on the scale of chords are unequal.

By using the ordinary 45° and 60° set-squares separately and together angles which are multiples of 15° may be constructed with great accuracy. Angles may also be measured and constructed by making linear measurements taken from tables of the trigonometrical ratios of angles.

4. Trigonometrical Ratios of Angles.—The angle A in the diagrams, Fig. 6, is contained by the lines OP and OX. The line OP

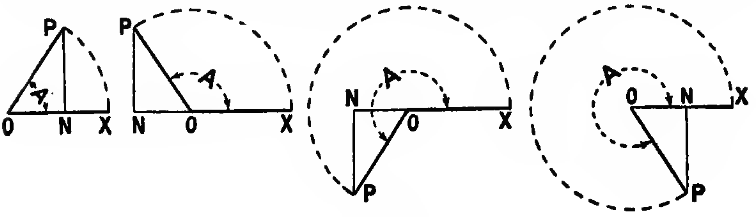


FIG. 6.

is supposed to start from the position OX and sweep out the angle A by moving round O as centre in the direction opposite to that of the hands of a watch. An angle so described is called a *positive angle.* If the motion takes place in the opposite direction the angle described is called a *negative angle.*

PN is perpendicular to OX. PN is positive (+) when it is above

OX, and negative ($-$) when it is below OX. ON is positive when it is to the right of O, and negative when it is to the left of O. OP is always positive.

The most important trigonometrical ratios of an angle are, the *sine*, the *cosine*, the *tangent*, and the *cotangent*. The sine of the angle A is the ratio of PN to OP. Thus, sine A $= \frac{PN}{OP}$. If PN is half of OP, then sine A $= \frac{1}{2}$. If PN $= 2$, and OP $= 3$, then, sine A $= \frac{2}{3}$.

The other important ratios are, cosine A $= \frac{ON}{OP}$, tangent A $= \frac{PN}{ON}$, and cotangent A $= \frac{ON}{PN}$.

In addition to the above the following ratios are also used: cosecant A $= \frac{OP}{PN}$, secant A $= \frac{OP}{ON}$, coversed sine A $= 1 -$ sine A, and versed sine A $= 1 -$ cosine A.

The names of these ratios are abbreviated as follows: sin, cos, tan, cot, cosec, sec, covers, and vers.

The following results are obvious from the definitions:—

$$\cot A = \frac{1}{\tan A},\ \tan A = \frac{1}{\cot A},\ \operatorname{cosec} A = \frac{1}{\sin A},\ \sec A = \frac{1}{\cos A}.$$

The complement of A is $90° - A$, and the supplement of A is $180° - A$.

The following relations between the trigonometrical ratios of an angle and those of its complement and supplement are easily proved:—

$$\sin(90° - A) = \cos A. \qquad \cos(90° - A) = \sin A.$$
$$\tan(90° - A) = \cot A. \qquad \cot(90° - A) = \tan A.$$
$$\sin(180° - A) = \sin A. \qquad \cos(180° - A) = -\cos A.$$
$$\tan(180° - A) = -\tan A. \qquad \cot(180° - A) = -\cot A.$$

It may be noticed that the sine and cosine of an angle can never be greater than 1, but the tangent and cotangent can have any magnitudes.

The problem of constructing an angle which has a given trigonometrical ratio evidently resolves itself into the simple one of constructing the right-angled triangle OPN (Fig. 6) having given the ratio of one side to another. But in general there are two angles less than 360° which have a given trigonometrical ratio.

5. Accuracy in Drawing.—A good serviceable line by an ordinary draughtsman is about 0·005 inch wide, but an expert draughtsman can work with a line of half that width. A good exercise for the student is to try how many separate parallel lines he can draw between two parallel lines which are 0·2 inch apart, as shown in Fig. 7, which is twice full size for the sake of clearness. When the lines are drawn as close as possible together, but distinctly separate, it will be found that the distance between them is about equal to the width of the lines. Hence, if the lines and spaces are about 0·005 inch wide, there will be about 20 lines in a width of 0·2 inch.

The probable error in measuring the distance between two points

on a straight line may be as small as the thickness of the division lines on the measuring scale, or say 0·005 inch.

The error e in the position of the point of intersection O of two straight lines OA and OB (Fig. 8), when the error in the lateral position of one of them is x, is $\frac{x}{\sin \theta}$, where θ is the angle between the lines. If $x = 0{\cdot}005$ inch, $e = 0{\cdot}286$ inch when $\theta = 1°$, $e = 0{\cdot}057$ inch when $\theta = 5°$, and $e = 0{\cdot}029$ inch when $\theta = 10°$.

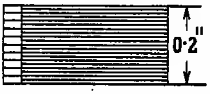


FIG. 7.

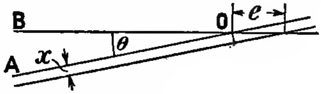


FIG. 8.

Using a radius of 10 inches, an angle may be constructed by marking off the chord from a table of chords with a minimum probable error of from 1-10th to 1-15th of a degree.

Great accuracy in drawing can only be obtained by exercising great care, and by having the best instruments kept in the best possible condition.

Exercises Ia

To test accuracy in drawing and measuring

1. On a straight line about 7 inches long mark a point O near one end. Mark off the following lengths on this line: OA = 1·20 inches, AB = 0·85 inch, BC = 1·35 inches, CD = 1·55 inches, and DE = 0·95 inch, all measured in the same direction along the line. Measure the length OE, which should be 5·90 inches.

2. Take twenty strips of paper about 2¼ inches long, and number them 1 to 20. On an edge of No. 1 mark off a length of 2 inches. On an edge of No. 2 mark off the length from No. 1. On an edge from No. 3 mark off the length from No. 2, and so on to No. 20. Then compare the length on No. 20 with the length on No. 1.

3. Draw a straight line AB (Fig. 9) and mark two points A and B on it 2 inches apart. Draw BC, making the angle ABC = 60°. Make BC 4 inches long. Draw CD, making the angle BCD = 30°. Make CD = CA. Draw DE parallel to AB. Make DE 4 inches long. Draw EF, making the angle DEF = 30°. Make EF = CA. Let DE cut BC at O. A circle with centre O and radius OB should pass through the points D, F, C, E, and A. Also the chords of the arcs BD, DF, FC, CE, and EA should each be 2 inches long.

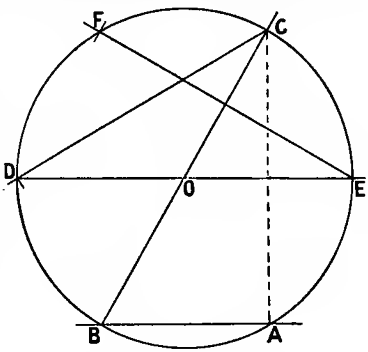


FIG. 9.

4. On a straight line AB (Fig. 10) 2 inches long construct an equilateral triangle ABC, using the 60° set-square. Produce the sides of this triangle as shown. Through the angular points of the triangle ABC draw parallels to the opposite sides forming the triangle DEF. Using the 30° and 90° angles of the set-square, draw lines through D, E, and F, to form the hexagon DHEKFL as

shown. Join FA, EB, and DC. If these lines are produced, they should pass through the points H, L, and K, respectively. Complete the figure by drawing the circles as shown.

5. Draw two lines OA, OA, each 10 inches long, and containing an angle AOA of 88°. Measure accurately and state the distance apart of A, A. What should be the exact distance AA? State any special precautions you may have taken to ensure accuracy in your construction. [B.E.]

6. Take from the tables the chord and tangent of 22°. Construct an angle of 22° by using the value of the chord, and a second angle of 22° by using the tangent. From each of these angles determine the sine of 22°, and compare the mean of the two values with the true value of sine 22°. [B.E.]

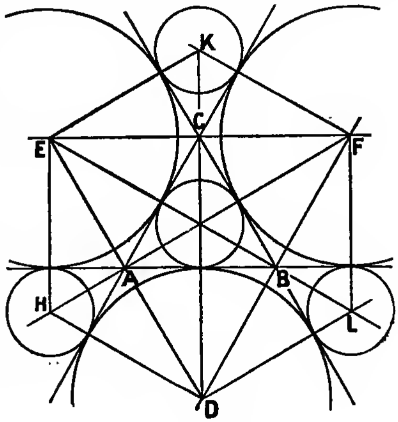


FIG. 10.

7. By aid of your protractor, and without using the tables, find the value of

$$\sin 23° + \tan 23° + \cos 23°.$$

Now take out the true values of the sine, tangent, and cosine from the tables, add them, and calculate the percentage error in your first answer. [B.E.]

8. In working this question employ a decimal scale of $\frac{1}{2}$ inch to 1 unit.

Draw a circular arc, radius 10 units, centre O. Mark a chord AB of this arc 3·47 units long, and draw the radii OA, OB. Measure the angle AOB in degrees.

From B draw a perpendicular BM on OA, and at A draw a tangent to meet OB produced in N. Measure carefully BM, AN and the arc AB (on the above unit scale) and calculate the sine and tangent of the angle, and the angle in radians.

Give the correct answers for the degrees, sine, tangent, and radians, the numbers being taken directly from the tables. [B.E.]

9. On the base AB (Fig. 11) construct the polygon ABCDEF to the dimensions given. Then measure the side AF and the angles θ and ϕ.

[The answers obtained by calculation are: AF = 2·65 inches, $\theta = 84° 56'$, and $\phi = 72° 4'$.]

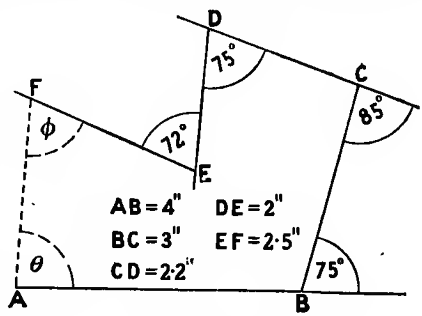


FIG. 11.

6. Some Terms used in Modern Geometry.—

Straight lines which pass through a point are said to be *concurrent*. Points which lie on a straight line are said to be *collinear*. Points which lie on the circumference of a circle are said to be *concyclic*. A straight line drawn across a system of lines is called a *transversal*.

The three straight lines which join the angular points of a triangle and the middle points of the opposite sides are called the *medians* of

the triangle, and the point at which the medians intersect is called the *centroid* of the triangle.

The three perpendiculars drawn from the angular points of a triangle to the opposite sides meet at a point called the *orthocentre* of the triangle, and the triangle formed by joining the feet of the perpendiculars is called the *pedal triangle*, or *orthocentric triangle* of the original triangle.

If a straight line *bc* be drawn to cut the sides AC and AB of a triangle ABC (Fig. 12) at *b* and *c* respectively, so that the angle A*cb* is equal to the angle ACB, then *bc* with reference to AB and AC is said to be *antiparallel* to BC. An antiparallel *bc* is parallel to the tangent AD to the circumscribing circle at A. The middle points of the antiparallels to one side of a triangle lie on a straight line which passes through the opposite angular point of the triangle, and this line is called a *symmedian* of the triangle. The three symmedians of a triangle intersect at a point called the *symmedian point* of the triangle.

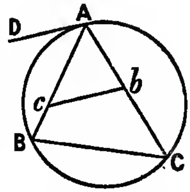


FIG. 12.

7. Equality of Two Triangles.—A triangle has six parts, three sides and three angles, and, except in two cases, a triangle is completely determined when three of its six parts are known. Another way of stating the above is that if there be two triangles having three parts in the one equal respectively to the corresponding parts in the other, then, except in two cases, the two triangles are equal in every respect. The different cases are as follows:—

Two triangles are equal in every respect when—

(1) The three sides of the one are equal to the three sides of the other, each to each.

(2) Two sides and the included angle in one are equal to two sides and the included angle in the other, each to each.

(3) Two angles and the side adjacent to them in one are equal to two angles and the side adjacent to them in the other, each to each.

(4) Two angles and the side opposite to one of them in one are equal to two angles and the side opposite to one of them in the other, each to each, the equal sides being opposite to equal angles.

These four cases are illustrated in Fig. 13; the corresponding equal parts in the two triangles have similar marks on them.

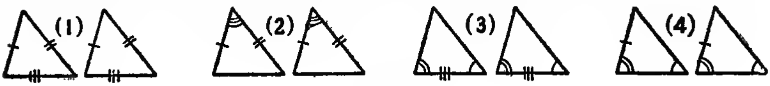


FIG. 13.

Of the two exceptional cases mentioned above, one is that in which the three angles of one triangle are equal to the three angles of the other. In this case, the two triangles have the same *shape*, but not necessarily the same *size*. The other case is that in which two sides and an angle opposite to one of them in one triangle are equal to

two sides and an angle opposite to one of them in the other, each to each, the equal angles being opposite to equal sides. In this case the triangles *may* be equal, but they are not necessarily so, as is shown in Fig. 14, where ABC_1 is one triangle and ABC_2 the other.

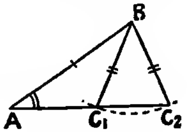


FIG. 14.

8. To bisect the angle between two straight lines.—First, let the two straight lines OA and OB intersect at a point O which is accessible (Fig. 15). With O as centre and any radius describe arcs to cut OA at C and OB at D. With centres C and D describe intersecting arcs of equal radii. The straight line OE, which joins O with the point of intersection of these arcs, will bisect the angle AOB.

An important property of the bisector of an angle is that the perpendiculars, such as FH and FK, drawn from any point on it to the lines containing the angle are of equal length.

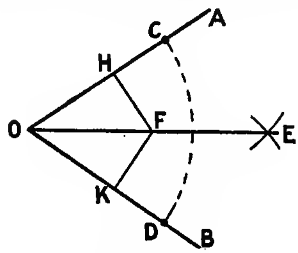


FIG. 15.

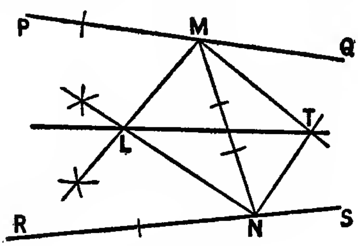


FIG. 16.

Next, let the given lines PQ and RS (Fig. 16) meet at a point which is inaccessible. Draw any straight line MN to cut PQ at M and RS at N. Bisect the angles PMN and RNM by straight lines which intersect at L. Through M and N draw perpendiculars to LM and LN respectively, and let them meet at T. The straight line LT will if produced meet PQ and RS produced at the same point, and will bisect the angle between PQ and RS. It is easy to see that MT and NT bisect the angles NMQ and MNS respectively, which suggests another way of drawing these lines.

9. To find a point on a given straight line such that the sum of its distances from two given points shall be the least possible.—Let LN (Fig. 17) be the given straight line and A and B the two given points. The given points are on the same side of LN. Draw ACD at right angles to LN, cutting LN at C. Make CD equal to AC. Join DB, cutting LN at E. E is the point required. Join AE.

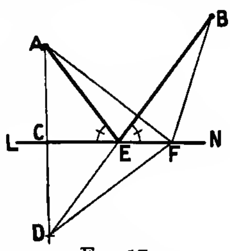


FIG. 17.

Comparing the triangles ACE and DCE, it is obvious that AE is equal to DE. Hence AE + BE = DB. If any other point F in LN

be taken and F be joined to A, B, and D, it is evident that AF is equal to DF, and AF + BF = DF + BF. But DF + BF is greater than DB, therefore AF + BF is greater than AE + BE.

It is evident that the angles AEL and BEN are equal, and the problem may be stated in another way, namely, to find a point E in a given line LN such that the lines joining E to two given points A and B shall be equally inclined to LN.

If LN represents a mirror, and a ray of light from A is reflected from the mirror and passes through B, then the incident ray will strike the mirror at the point E found by the above construction.

If the points A and B are on opposite sides of LN it is obvious that E will be the point where the straight line AB cuts LN. In this case, however, there is no corresponding case of reflection from a mirror.

Referring to Fig. 17, and considering LN to be a mirror, D is the *image* of the point A in the mirror.

10. Similar Rectilineal Figures.—Two rectilineal figures are said to be similar when they have their several angles equal, each to each, and the sides about their equal angles proportionals. For example, the figure ABCD (Fig. 18) is similar to the figure *abcd* when the angles A, B, C, and D of the one are equal to the angles *a*, *b*, *c*, and *d*, respectively, of the other, and the ratios $\frac{DA}{AB}, \frac{AB}{BC}, \frac{BC}{CD}$, and $\frac{CD}{DA}$ are equal to the ratios $\frac{da}{ab}, \frac{ab}{bc}, \frac{bc}{cd}$, and $\frac{cd}{da}$ respectively. Stated in another way, the similar figures have the *same shape* but have *different dimensions.*

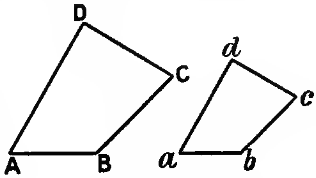


FIG. 18.

When the two figures are triangles, they are similar if the angles of the one are equal to the angles of the other, each to each. The equality of the ratios of the sides about the equal angles follows as a consequence of the equality of the angles.

It is important to remember that the ratio of the areas of two similar figures is equal to the ratio of the squares on their corresponding sides. For example, referring to Fig. 18,

area of ABCD : area of *abcd* : : $\overline{AB}^2 : \overline{ab}^2$.

11. Through a given point to draw a straight line which shall be concurrent with two given straight lines when the point of concurrence is inaccessible.—AB and CD (Fig. 19) are the given lines, and P is the given point. Draw a triangle PEF having one angular point at P and the other angular points one on AB and one on CD. Draw HK parallel to EF, meeting AB at H and CD at K. Through H and K draw parallels to EP and FP respectively, and let them

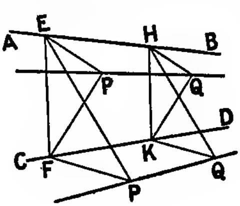


FIG. 19.

meet at Q. A straight line through P and Q will, if produced, meet AB and CD produced at the same point.

12. Graphic Arithmetic.—The solution of ordinary arithmetical problems by geometrical constructions is of some interest when considered as a section of practical geometry, but *graphic arithmetic*, when regarded as a means to an end, is in general a poor substitute for the ordinary method by calculation. The subject of graphic arithmetic will therefore be treated somewhat briefly here.

Representation of numbers by lines. If the line AB, Fig. 20 (a), be taken to represent the number *one*, then a line CD n times the length of AB will represent the number n. Again, if AB represents the number *one*, the number represented by the line EF will be the number of times that EF contains AB. In the above examples AB is called the *unit*.

Ordinary drawing scales may be used in marking off or measuring lengths which represent numbers. Scales which are decimally divided are the most convenient for this work.

Addition and subtraction. To add a series of numbers together. Draw a straight line OX, Fig. 20 (b), of indefinite length. Fix upon a unit, that is, decide what length shall represent the number *one*. Mark a definite point O on OX. Make OA = the first number, AB = the second number, BC = the third number, and so on. From O to the last point determined in this way will be the answer required.

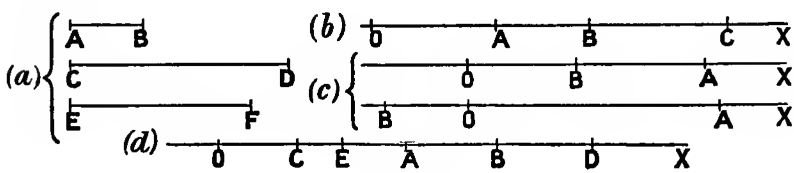


FIG. 20.

From one number to subtract another. Make OA, Fig. 20 (c), measured to the *right* of O, equal to the first number. Make AB, measured to the *left* of A, equal to the second number which is to be subtracted from the first. From O to B is the answer required. If OB is to the *right* of O the answer is *positive* (+), and if OB is to the *left* of O, the answer is *negative* (−).

To find the value of such an expression as $a + b - c + d - e$, where a, b, c, d, and e represent numbers, make (Fig. 20 (d)) OA = a, AB = b, BC = c, CD = d, and DE = e, then OE = $a + b - c + d - e$. Note that in *adding* the lengths are measured to the *right*, while in *subtracting* they are measured to the *left*.

Proportion. In Figs. 21 and 22, OX and OY are two straight lines making any convenient angle with one another. AB and CD are parallel lines meeting OX at A and C, and OY at B and D. The triangles AOB and COD are similar, and therefore OA : OB : : OC : OD,

and if any three of the terms of this proportion are known the figure can be drawn and the fourth term found.

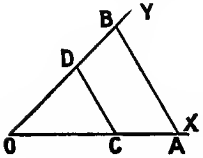


FIG. 21.

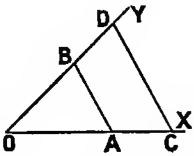


FIG. 22.

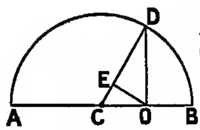


FIG. 23.

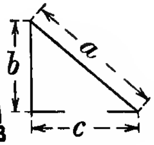


FIG. 24.

Arithmetical, Geometrical, and Harmonical means of two numbers. Let M and N be two numbers. On a straight line AB (Fig. 23) make AO = M and OB = N. Bisect AB at C. With centre C and radius CA describe the semicircle ADB. Draw OD at right angles to AB. Join CD. Draw OE at right angles to CD. Then

$$\text{CB} = \text{arithmetical mean of M and N} = \tfrac{1}{2}(\text{M} + \text{N}),$$

$$\text{OD} = \text{geometrical mean of M and N} = \sqrt{\text{M} \times \text{N}},$$

$$\text{DE} = \text{harmonic mean of M and N} = \frac{2\text{M} \times \text{N}}{\text{M} + \text{N}}.$$

OD, the geometrical mean of M and N, is also called a *mean proportional* between M and N. If $\text{OD} = \sqrt{\text{M} \times \text{N}}$, then $\text{OD}^2 = \text{M} \times \text{N}$, and M : OD : : OD : N.

Multiplication. In Figs. 21 and 22, since OA : OB : : OC : OD, it follows that OA × OD = OB × OC. Make OB equal to the unit, then OA × OD = 1 × OC = OC. Hence the construction for finding the product of two numbers. Make OA = one of the numbers, OD = the other number, and OB = the unit. Join AB, and draw DC parallel to BA, then OC is the answer.

If the given numbers are large, it is better to first divide them by some power of 10 so as to avoid using a very small unit. For example, let it be required to find 485 × 363. Taking the unit as 1 inch find the product x of 4·85 and 3·63, then 485 × 363 = 4·85 × 3·63 × 10,000 = 10,000x. After the value of x has been found, 10,000x is found by simply changing the position of the decimal point.

Division. In Figs. 21 and 22, since OA : OB : : OC : OD, it follows that $\frac{\text{OA}}{\text{OB}} = \frac{\text{OC}}{\text{OD}}$. Make OB = the unit, then $\frac{\text{OC}}{\text{OD}} = \frac{\text{OA}}{1} = \text{OA}$. Hence the construction for finding the quotient of one number divided by another. Make OC = the dividend, OD = the divisor, and OB = the unit. Join CD, and draw BA parallel to DC, then OA is the answer.

Square root. In Fig. 23, $\text{OD}^2 = \text{OA} \times \text{OB}$, therefore $\text{OD} = \sqrt{\text{OA} \times \text{OB}}$. Hence if OB is made equal to the unit, $\text{OD} = \sqrt{\text{OA}}$.

If the given number is a large one it is better to make OB = n times the unit, and OA = $\frac{1}{n}$ of the given number, where n is a convenient whole number. For example, to find the square root of 48,

make OB = 10 times the unit and OA = 4·8 times the unit, then $OD = \sqrt{4{\cdot}8 \times 10} = \sqrt{48}$.

If a be the hypotenuse of a right-angled triangle (Fig. 24) of which b and c are the sides, then $a^2 = b^2 + c^2$, therefore $a = \sqrt{b^2 + c^2}$, also $c^2 = a^2 - b^2$, therefore $c = \sqrt{a^2 - b^2}$.

Involution. The values of a^2, a^3, a^4, etc., may be obtained by multiplication as already explained. For $a^2 = a \times a = b$, and $a^3 = a^2 \times a = b \times a$. The following construction is, however, sometimes more convenient. Draw two axes XOX_1 and YOY_1 (Fig. 25) at right angles to one another. Make OA = the unit, and $OA_1 = a$, the given number. Join A_1A and draw A_1A_2 at right angles to A_1A. Then $OA_2 = a^2$.

Draw A_2A_3 at right angles to A_1A_2. Then $OA_3 = a^3$.

Draw A_3A_4 at right angles to A_2A_3. Then $OA_4 = a^4$, and so on.

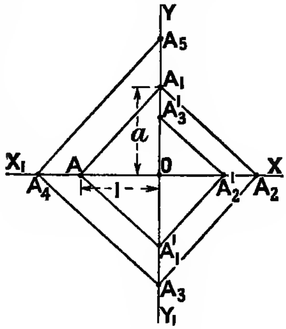


FIG. 25.

If AA'_1 be drawn at right angles to AA_1, and $A'_1A'_2$ be drawn at right angles to AA'_1, and so on, then $OA'_1 = \frac{1}{a}$, $OA'_2 = \frac{1}{a^2}$, $OA'_3 = \frac{1}{a^3}$, and so on.

13. Definitions relating to Curves.—*Tangent and Normal.* If a straight line PQ (Fig. 26) cuts a curve at two points P and Q. and if the straight line be turned about the point P so as to cause the point Q to approach nearer and nearer to P, then the ultimate position of the straight line is the *tangent* to the curve at P.

The *normal* to the curve at P is a straight line through P at right angles to the tangent to the curve at P.

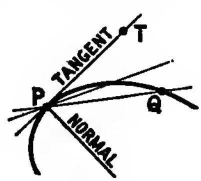


FIG. 26.

A good practical method of drawing a tangent to a curve from an external point T (Fig. 26) may be given here. Place a straight-edge on the paper and adjust it until the edge passes through T and touches the curve, then draw the tangent. This method would not be recognized by the mathematician, but as a practical method it is as good as any other, and much simpler. The exact point of contact must be obtained by some other construction depending on the nature of the curve.

Circle of Curvature. If a circle be drawn through three points Q, P, and R in a curve APB (Fig. 27), and if the points Q and R be moved so as to approach nearer and nearer to P, then in the limit the circle becomes the *circle of curvature* of the curve APB at P. The centre of the circle of curvature will evidently lie on

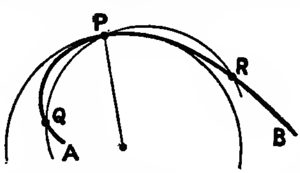


FIG. 27.

the normal to the curve at P. The centre of the circle of curvature is called the *centre of curvature*, and the radius of the circle of curvature is called the *radius of curvature.*

Evolute and Involute. The locus of the centre of curvature of a curve is called the *evolute* of the curve. A curve has only one evolute.

The locus of a point on a straight line which rolls without sliding on a curve is called an *involute* of the curve. A curve has any number of involutes.

In Fig. 28, ABC is a curve and *abc* is its evolute. Again, *abc* is a curve and ABC is an involute of *abc*. The curves PQR and STU are also involutes of *abc*.

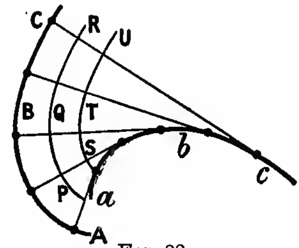


FIG. 28.

A curve is the evolute of each of its involutes, and a curve is an involute of its evolute.

The normals to a curve are tangents to its evolute, and the tangents to a curve are normals to its involutes.

Envelope. If a curve moves in a definite manner the curve which it always touches is the *envelope* of the moving curve. Fig. 29 shows the envelope of a circle which moves so that its centre traces the curve ABC.

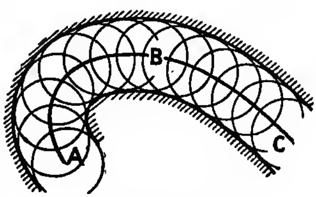


FIG. 29.

The evolute of a curve is also the envelope of the normal to the curve.

Cusp. If two branches BA and CA of a curve (Fig. 30) terminate at a point A on a common tangent AD, the terminating point A is called a *cusp*. There are two kinds of cusps; in the first kind the branches of the curve are on opposite sides of the common tangent, and in the second kind the two branches of the curve are on the same side of the common tangent.

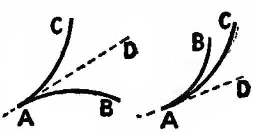


FIG. 30.

Node or Multiple Point. A point through which two or more branches of a curve pass is called a *node* or *multiple point.* The node shown at (*a*), Fig. 31, is a *double point*, and the node shown at (*b*) is a *triple point.* At the node (*a*) two tangents can be drawn to the curve. At the node (*b*) three tangents can be drawn to the curve.

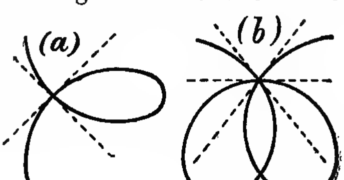


FIG. 31.

Point of Inflexion. If the tangent to a curve at a point also cuts or crosses the curve at that point (Fig. 32), then the point is a *point of inflexion.*

Asymptotes. When a curve approaches nearer and nearer to a

straight line, but never actually meets the line however far the line and curve be continued, the line and curve are said to be *asymptotic*, and the line is called an *asymptote* to the curve.

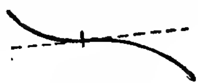


FIG. 32.

Curves which approach nearer and nearer to one another but never actually meet however far they are continued are said to be asymptotic, and each curve is an asymptote to the other.

Similar Curves. Two curves, PQ and P_1Q_1 (Fig. 33), are said to be *similar* when any radius vectors (or radii) OP, OQ in the one, drawn from a fixed point O, bear the same ratio to one another that radius vectors O_1P_1, O_1Q_1 drawn from a fixed point O_1, and including the same angle θ, bear to one another.

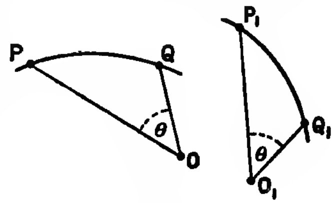


FIG. 33.

If, in addition to fulfilling the above conditions, OP and OQ are parallel to O_1P_1 and O_1Q_1 respectively, then the curves are said to be *similar*, and *similarly situated.*

The fixed points O and O_1 are called *centres of similarity*, and when these centres coincide the point is called a *centre of similitude* of the two curves.

All conics having the same eccentricity are similar curves.

Exercises Ib

1. Draw Fig. 34 full size to the dimensions given. Draw a straight line bisecting the angle between AB and CD, and through the points P and Q draw straight lines concurrent with AB and CD without using the point of concurrence.

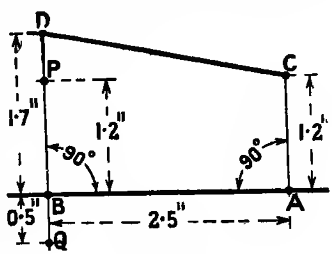


FIG. 34.

2. From the points P and Q (Fig. 34) draw two straight lines to meet on the line passing through A and the middle point of CD and be equally inclined to that line.

3. If a line $3\frac{7}{8}$ inches long represents 26, what does a line $1\frac{5}{16}$ inches long represent, and what is the unit? If a line $1\frac{9}{16}$ inches long represents $1\frac{3}{4}$, what is the unit?

4. Work the following exercises, taking for the unit in each case a line $\frac{3}{4}$ inch long:—

(*a*) $1\frac{1}{2} + \frac{5}{8} - \frac{7}{8} + \frac{3}{4} - 1\frac{1}{8} + 2\frac{1}{4}$. (*c*) $2{\cdot}4 + 0{\cdot}8 - 1{\cdot}9 - 2 + 4{\cdot}2 - 1{\cdot}6$.
(*b*) $1{\cdot}5 - 2 + 3 - 4 + 3{\cdot}5$. (*d*) $\frac{7}{12} + \frac{13}{12} - 1\frac{1}{2} + \frac{2}{3} - 2\frac{1}{2}$.

5. Taking a line $1\frac{1}{4}$ inches long to represent the area of a square of 1 inch side, determine a line which shall represent the area of a rectangle $2\frac{3}{4}$ inches long and $\frac{7}{8}$ inch broad.

6. A is a line $1\frac{7}{8}$ inches long, and B is a line $2\frac{1}{16}$ inches long. Determine a line which will represent the product of A and B, the unit being a line $1\frac{1}{4}$ inches long.

7. A is a line $2\frac{7}{16}$ inches long, B is a line $1\frac{5}{16}$ inches long, and C is a line $\frac{13}{16}$ inch long. If A represents the product of B and C, draw a line which will represent the product of A and C.

8. Find a line representing the volume of a rectangular solid whose dimensions are 3 inches $\times$ 1·75 inches $\times$ 1·25 inches. Unit = 0·5 inch.

9. The unit being a line 1 inch long, find a line to represent the cube of 1·25.

10. M and N are straight lines 1·25 inches and 1·5 inches long respectively. Find a line representing $\frac{M + N}{M \times N}$, the unit being 1 inch.

11. Draw lines representing $\sqrt{15}$ and $\frac{20}{\sqrt{15}}$. Unit = 0·25 inch.

12. Find a line to represent x^3 when $x = 2\sqrt{2} - \sqrt{3}$. Unit one inch.

13. A, B, C, and D are lines $3\frac{3}{16}$ inches, $2\frac{1}{8}$ inches, $2\frac{3}{4}$ inches, and $1\frac{9}{16}$ inches long respectively. If $\frac{A \times B}{C \times D} = C$, find the unit.

14. Using as unit a length of 1 inch, find a line to represent $\frac{1{\cdot}1}{2{\cdot}3}\sqrt{\frac{2{\cdot}3^2 + 1{\cdot}4^2}{1{\cdot}4^2 - 1{\cdot}1^2}}$.

CHAPTER II

THE CIRCLE

14. Properties of the Circle—Simple Problems.—

(1) *In a segment ABCD of a circle* (Fig. 35) *the angle AOD at the centre is double of the angle ABD at the circumference.*

(2) *Angles ABD and ACD in the same segment of a circle* (Fig. 35) *are equal to one another.*

(3) *The angle in a semicircle is a right angle.* The carpenter, in cutting a semicircular groove in a piece of wood, makes use of this property when he tests it with a square, as shown in Fig. 36.

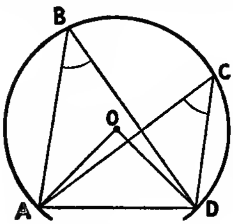


FIG. 35.

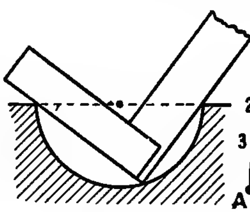


FIG. 36.

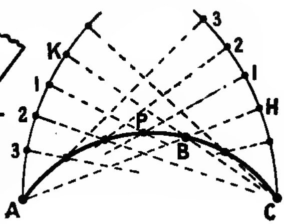


FIG. 37.

(4) *A straight line which bisects a chord at right angles passes through the centre of the circle.* This suggests a method of finding the centre of a circle to pass through three given points or to circumscribe a given triangle.

To describe an arc of a circle through three given points, A, B, and C, when the centre of the circle is inaccessible (Fig. 37). With centres A and C describe arcs CH3 and A3K. Join AB and CB, and produce these lines to meet the arcs at H and K. Mark off short equal arcs H1 and K1. The intersection P of the lines A1, C1, is another point on the arc required. In like manner other points may be found, and a fair curve drawn through all of them is the arc required.

The foregoing construction is based on the fact that the three angles of a triangle are together equal to 180°, and that the angles in the same segment of a circle are equal to one another; also that in equal circles equal arcs subtend equal angles at their centres. The student should have no difficulty in showing that the construction makes the angle APC equal to the angle ABC. For another method see Art. 59, p. 56.

(5) *A tangent to a circle is at right angles to the radius or diameter drawn to the point of contact.*

To draw a tangent to a circle from an external point. A good practical method is to place a straight-edge on the paper and adjust it so that the edge passes through the point and touches the circle, then the tangent may be drawn. The point of contact must, however, be obtained by dropping a perpendicular from the centre of the circle to the tangent.

The following construction is recommended when the one just given is objected to. With P (Fig. 38), the given external point as centre, describe the arc OSR, passing through O the centre of the given circle. With centre O and radius equal to the diameter of the circle describe an arc to cut the arc OSR at R. Draw OR cutting the circle at T. PT is the tangent required and T is the point of contact. The student should satisfy himself that this construction makes the angle OTP a right angle.

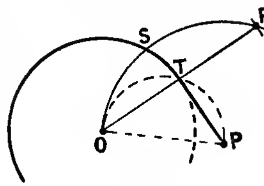


FIG. 38.

Another method is to describe a semicircle on OP as diameter, to cut the given circle at T, which will be the point of contact of the tangent required. In this case the angle OTP is obviously a right angle because it is the angle in a semicircle.

It is obvious that two tangents may be drawn to a circle from an external point, and that they are equal in length.

To draw a tangent to two given circles. This may be done very accurately by placing a straight-edge on the paper, adjusting it so that the edge touches the circles, and then drawing the tangent. The points of contact must however be obtained by dropping perpendiculars from the centres of the circles to the tangent.

When the preceding construction is objected to, the following may be used: A and B (Figs. 39 and 40) are the centres of the given circles. Join AB cutting the circles at C and D. Make DE equal to BC. Draw

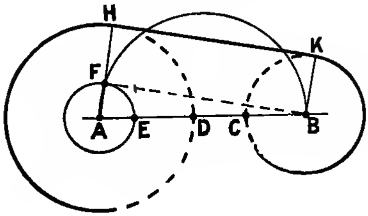


FIG. 39.

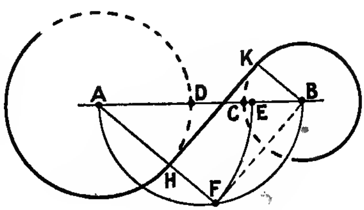


FIG. 40.

a circle with centre A and radius AE. Draw BF a tangent to this circle, F being the point of contact. Draw the line AF meeting the given circle, whose centre is A, at H. In Fig. 39, H is on AF produced. Draw BK parallel to AH, meeting the circle whose centre is B, at K,

The straight line joining H and K is a tangent to both the given circles, and H and K are the points of contact.

If the circles lie outside of one another four common tangents may be drawn to them.

(6) *The angle ABC* (Fig. 41) *between the chord AB and the tangent BC to the circle at B is equal to the angle ADB in the alternate segment.*

To inscribe in a circle ABD (Fig. 41) *a triangle equiangular to the triangle EFH.* Draw the tangent KBC. Draw the chord AB, making the angle ABC equal to the angle F. Draw the chord DB, making the angle DBK equal to the angle H. Join AD. The triangle ABD is the triangle required.

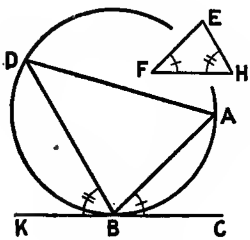


FIG. 41.

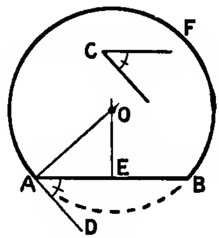


FIG. 42.

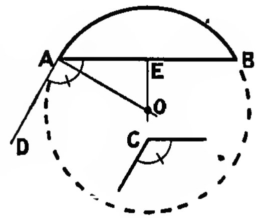


FIG. 43.

On a given line AB (Figs. 42 and 43) *to describe a segment of a circle which shall contain an angle equal to a given angle C.* Draw AD, making the angle DAB equal to the angle C. Draw AO at right angles to AD. Bisect AB at E. Draw EO at right angles to AB, meeting AO at O. O is the centre, and OA is the radius of the circle of which the segment AFB contains an angle equal to the angle C.

(7) *In a quadrilateral ABCD* (Fig. 44) *inscribed in a circle the opposite angles ABC and ADC are together equal to two right angles. Also the other pair of opposite angles are together equal to two right angles.*

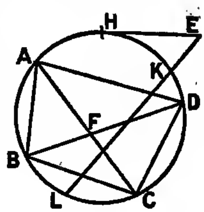


FIG. 44.

(8) In Fig. 44

$$\overline{AC} \times \overline{BD} = \overline{AB} \times \overline{CD} + \overline{AD} \times \overline{BC}.$$

(9) *AC and BD are any two chords of a circle intersecting at F.* $\overline{AF} \times \overline{CF} = \overline{BF} \times \overline{DF}$. In Fig. 44 the point F is within the circle, but it may be outside the circle.

(10) *EH is a tangent to the circle ABCD* (Fig. 44), *H being the point of contact. EKL is a line cutting the circle at K and L.* $\overline{EH}^2 = \overline{EK} \times \overline{EL}$.

To draw a tangent to a given arc of a circle, through a given point, without using the centre of the circle.

Fig. 45. The given point P is on the given arc APB but not near

either of its extremities. With centre P describe arcs of a circle to cut the given arc at A and B. A line through P parallel to the chord AB is the tangent required.

Let C be the middle point of the chord AB. If d is the diameter of the circle of which APB is an arc, then $\overline{PC}\,(d - PC) = \overline{AC} \times \overline{CB} = \overline{BC}^2$. Hence $d = \dfrac{\overline{BC}^2 + \overline{PC}^2}{PC} = \dfrac{\overline{PB}^2}{PC}$. The lengths of PB and PC may be measured and d found by arithmetic.

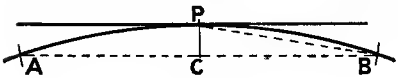


FIG. 45.

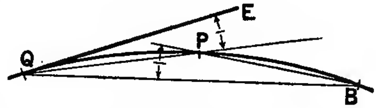


FIG. 46.

Fig. 46. The given point Q is on the given arc and near one of its extremities. Draw chords QP and QB. Join PB. Draw QE making the angle PQE equal to the angle PBQ. QE is the tangent required.

Fig. 47. The given point S is outside the given arc. Draw the line SAB cutting the arc at A and B. Find a length l such that $l^2 = SA \,.\, SB$ (see Art. 12, p. 11). With S as centre and radius equal to l, describe an arc of a circle to cut the given arc at T. A line ST is the tangent required and T is the point of contact.

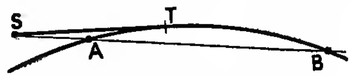


FIG. 47.

15. To construct a triangle having given the base, the vertical angle, and the length of the bisector of the vertical angle.—*First method* (Fig. 48). On AB the given base describe a

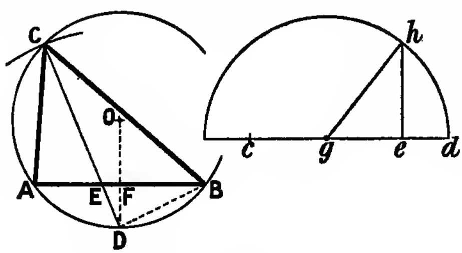


FIG. 48.

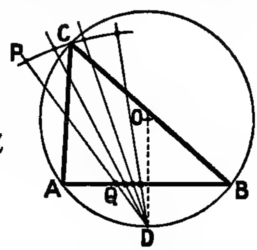


FIG. 49.

segment of a circle ACB containing an angle equal to the given vertical angle. From O the centre of this circle draw OF at right angles to AB and produce it to meet the circle at D. On any straight line make ce equal to the given length of the bisector of the vertical angle. Bisect ce at g. Draw eh at right angles to ce and make eh equal the chord DB. With centre g and radius gh describe an arc of a circle to cut ce produced at d. With centre D and radius equal to

dc describe an arc of a circle to cut the circle ACB at C. Join AC and BC. ACB is the triangle required.

The construction for finding the length of DC is based on the fact that $\overline{DC} \times \overline{DE} = \overline{DB}^2$.

Second method (Fig. 49). Determine the circle ACB and the point D as in the first solution. Through D draw a straight line DQP cutting AB at Q. Make QP equal to the given length of the bisector of the vertical angle. Repeat this construction several times so as to determine a sufficient number of points on the locus of P. The intersection of the locus of P and the circle ACB determines the vertex of the required triangle.

This second method is a very good illustration of the use of a locus. It is quicker and probably more accurate and certainly much easier to discover.

16. To find the locus of a point which moves so that the ratio of its distances from two given points shall be equal to a given ratio.—Let A and B (Fig. 50) be the two given points, and let P be one position of the moving point so that the ratio of AP to BP is equal to a given ratio. In Fig. 50 this ratio is 2 : 1.

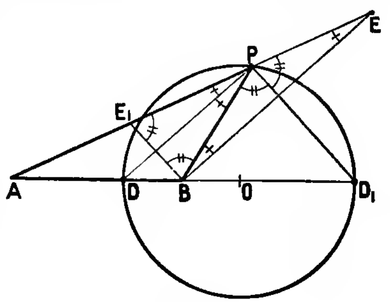


FIG. 50.

Draw PD bisecting the angle APB and meeting AB at D. Draw PD_1 bisecting the angle between AP produced and BP. Bisect DD_1 at O. A circle with centre O and radius OD will be the locus required.

Draw BE parallel to PD to meet AP produced at E. Draw BE_1 parallel to PD_1 to meet AP at E_1.

It is easy to show that $PE = PE_1 = PB$.

Also, AD : DB : : AP : PE, therefore AD : DB : : AP : PB, and D is a fixed point.

Again, $AD_1 : D_1B :: AP : PE_1$, therefore $AD_1 : D_1B :: AP : PB$, and D_1 is a fixed point.

The angle DPD_1 is obviously a right angle. Hence a circle described on DD_1 as diameter will pass through P.

If the figure be rotated about AD_1 as an axis, the circle will describe the surface of a sphere, and the surface of this sphere will evidently be the locus of a point moving in space so that the ratio of its distances from the two fixed points A and B is equal to a given ratio.

17. The Inscribed and Escribed Circles of a Triangle.—The inscribed circle of a triangle is the one which touches each of the three sides. An escribed circle touches one side and the other two sides produced. There are three escribed circles to a triangle. The constructions for drawing the inscribed and escribed circles are based on the fact, that when a circle touches each of two straight lines, its centre

lies on the line bisecting the angle between them. In Fig. 51, O is the centre of the inscribed circle of the triangle ABC, and O_1 is the centre of one of the escribed circles.

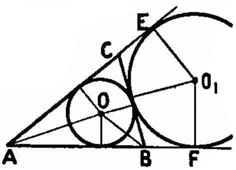


FIG. 51.

The following results are not difficult to prove :—

(1) *The sides of the triangle formed by joining the centres of the escribed circles pass through the angular points of the original triangle.*

(2) *The line joining the centre of one of the escribed circles and the opposite angle of the triangle passes through the centre of the inscribed circle and is perpendicular to the line joining the centres of the other two escribed circles.*

(3) Referring to Fig. 51, *AE = AF = half the perimeter of the triangle ABC.* This suggests the construction for the solution of the following problem.

Given the perimeter[1] *and one angle of a triangle, also the radius of the inscribed circle : to construct the triangle.* Make the angle EAF (Fig. 51) equal to the given angle. Draw a circle having a radius equal to the given radius to touch AE and AF. This will be the inscribed circle of the required triangle. Make AE and AF each equal to half the given perimeter. From E and F draw perpendiculars to AE and AF respectively to meet at O_1. A circle with O_1 as centre and O_1E or O_1F as radius will be an escribed circle of the triangle. A tangent BC to the two circles now drawn will complete the triangle required.

18. Circles in Contact.—In considering problems on circles in contact with one another two simple facts should be kept in view, viz.

(1) *When two circles touch one another the straight line which joins their centres or that line produced passes through the point of contact.*

(2) *When two circles touch one another the distance between their centres is equal to either the sum or difference of their radii.*

To draw a circle of given radius to touch two given circles. Three cases are shown in Figs. 52, 53, and 54. A and B are the centres of the given

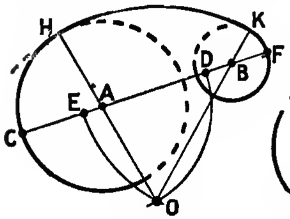


FIG. 52.

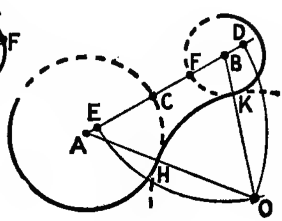


FIG. 53.

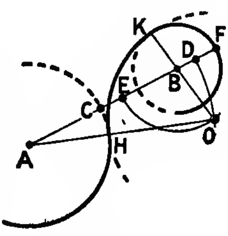


FIG. 54.

circles. AB or AB produced cuts the given circles at C and F. Make CD and FE each equal to the given radius. With centre A and radius AD draw the arc DO. With centre B and radius BE draw the arc

[1] The *perimeter* of a triangle is the sum of its sides.

EO, cutting the former arc at O. Join O with A and B, and produce these lines if necessary to meet the given circles at H and K. O is the centre of the required circle, and H and K are the points of contact.

19. To draw a series of circles to touch one another and two given lines.—AB and CD (Fig. 55) are the two given lines. Draw EF, bisecting the angle between AB and CD. Let E be the centre of one circle: its radius is EA, the perpendicular on AB from E. Draw HK perpendicular to EF. Make KL equal to KH. Draw LM perpendicular to AB to meet EF at M. M is the centre and ML is the radius of the next circle.

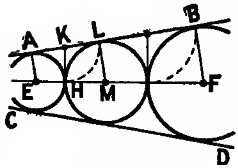


FIG. 55.

20. To draw a circle to pass through a given point and touch two given lines.—AB and AC (Fig. 56) are the two given lines and D is the given point. Draw AE bisecting the angle BAC. Join AD. Take any point F in AE. Draw FH perpendicular to AB. With F as centre and FH as radius describe a circle. This circle will touch the lines AB and AC. Let this circle cut AD at K. Draw DO parallel to KF, meeting AE at O. O is the centre and OD the radius of the circle required.

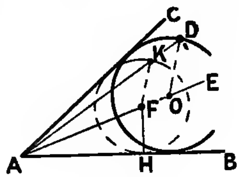


FIG. 56.

21. To draw a circle to touch two given lines and a given circle.—AB and CD (Fig. 57) are the given lines and EF is the given circle, N being its centre. Draw HK and LM parallel to AB and CD respectively, and at distances from them equal to the radius EN of the given circle. Draw, by preceding problem, a circle to pass through N, and touch the lines HK and LM. O the centre of this circle is the centre of the circle required, and OE is its radius, OEN being a straight line.

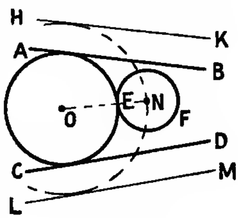


FIG. 57.

22. To draw a circle to touch a given line and pass through two given points.—AB (Fig. 58) is the given line, and C and D are the given points. Draw CD and produce it to meet AB at E. If the required circle touches AB at K, then $\overline{EK}^2 = \overline{ED} \times \overline{EC}$, or EK must be a mean proportional to EC and ED. Hence the following construction. Produce CE and make EF equal to ED. On CF describe a semicircle. Draw EH perpendicular to CF to meet the semicircle at H. Make EK equal to EH. Draw KO perpendicular to AB, and draw LO, bisecting CD at right angles to meet KO at O. O is the centre of the circle required.

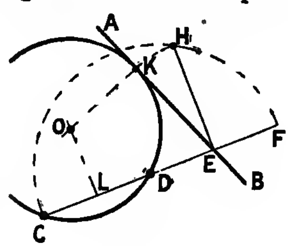


FIG. 58.

23. To draw a circle to touch a given circle and a given line at a given point in it.—ABC (Fig. 59) is the given circle, DE is the given line, and D the given point in it. Through F, the centre of the given circle, draw FE perpendicular to DE and produce it to meet the circle at C. Draw DO perpendicular to DE. Draw CD cutting the circle at B. Draw FB and produce it to meet DO at O. O is the centre and OD the radius of the circle required. There are two solutions, the second being obtained in the same way by joining A with D instead of joining C with D.

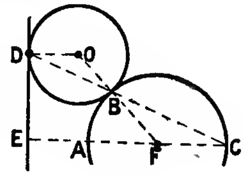


FIG. 59.

24. To draw a circle to pass through two given points and touch a given circle.—A and B (Fig. 60) are the given points and CDE is the given circle. Draw a circle CABE through A and B, cutting the circle CDE at C and E. Join CE, and produce it to meet AB produced at F. Draw FD, touching the given circle at D. Join H, the centre of the given circle, with D, and produce it to meet a line bisecting AB at right angles at O. O is the centre of the required circle. There are two solutions. The second is obtained by drawing the other tangent to the given circle from F and proceeding as before.

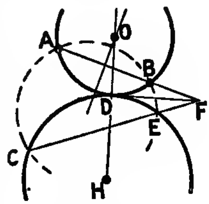


FIG. 60.

Considering a straight line to be a circle of infinite diameter, the student should endeavour to deduce from the foregoing solution the construction for drawing a circle to pass through two given points and touch a given line.

25. To draw the locus of the centre of a circle which touches two given circles.—A and B (Fig. 61) are the centres of the given circles. Join AB cutting the circles at C and D. Bisect CD at E. Mark off from E, above and below it, on AB a number of equal divisions. With centres A and B describe arcs of circles through these divisions as shown. A fair curve drawn through the intersections of these arcs as shown is the locus required. The curve is an hyperbola whose foci are the points A and B.

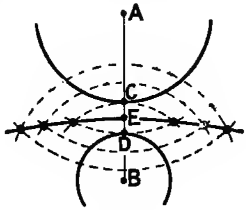


FIG. 61.

This locus may be used in solving problems on the drawing of a circle to touch two given circles and to fulfil some other condition.

When the given circles are external to one another, as in Fig. 61, four different loci may be drawn in the manner described above, because there are four positions of the point E. If AB produced cuts the upper circle again in F and the lower one again in H, then the remaining positions of the point E are, the middle point of FH, the middle point of FD, and the middle point of HC.

26. To draw the locus of the centre of a circle which touches a given line and a given circle.—AB (Fig. 62) is the given line and CDE is the given circle, O being its centre. Through O draw OH perpendicular to AB, meeting AB at H, and the given circle at D. Bisect DH at F. Mark off from F above and below it, on OH, a number of equal divisions. With centre O draw arcs through the divisions above F to meet parallels to AB through the corresponding divisions below F, as shown. A fair curve drawn through the intersections of the arcs and parallels, as shown, is the locus required. The curve is a parabola.

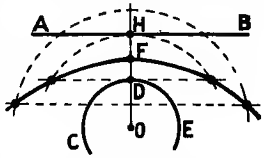


FIG. 62.

This locus may be used in solving problems on the drawing of a circle to touch a given circle and a given straight line, and to fulfil some other condition.

Two different loci may be drawn in the manner described above, because there are two positions of the point F. If the line OH produced cuts the circle again at K, then the second position of F is the middle point of HK.

27. Pole and Polar.—If through a given point P (Figs. 63 and 64) any straight line be drawn to cut a circle NQR at Q and R, tangents to the circle at Q and R will intersect on a fixed straight line LM. Conversely, the chord of contact of tangents from any point in LM, outside the circle, will pass through P. This fixed line LM is called the *polar* of the point P with respect to the circle, and the point P is called the *pole* of the line LM with respect to the circle.

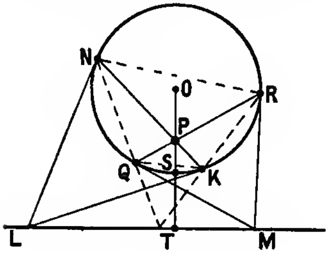


FIG. 63.

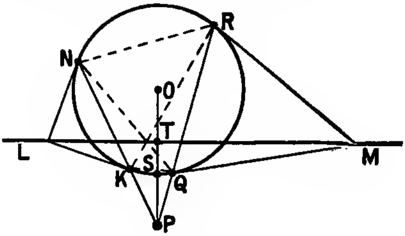


FIG. 64.

LM will be found to be perpendicular to OP, O being the centre of the circle, and if OP or OP produced cuts LM at T, and the circle at S, then $\overline{OP} \times \overline{OT} = \overline{OS}^2$.

When the pole P is outside the circle (Fig. 64) the polar is the chord of contact of the tangents from P to the circle.

If chords QR and NK of a circle meet (produced if necessary) at P, then the chords NQ and KR will intersect on the polar of P, also the chords NR and QK will intersect on the polar of P. This suggests a construction for drawing the polar of a point P by using a pencil and

straight-edge only ; and, having found the polar, lines from P to where the polar cuts the circle will be the tangents to the circle from P. Tangents to a circle from an external point may therefore be drawn, and their points of contact determined, by using a pencil and straight-edge only.

28. Centres of Similitude.—CD and EF (Figs. 65 and 66) are two parallel diameters of two circles whose centres are A and B. The

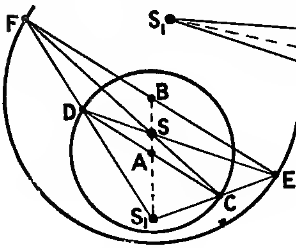


FIG. 65.

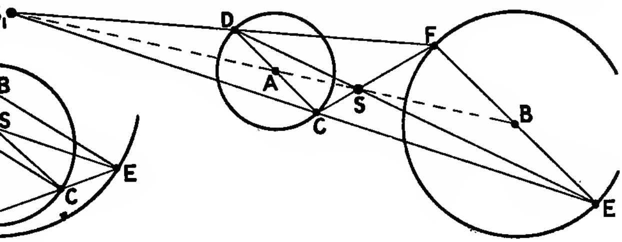


FIG. 66.

straight lines joining the extremities of these diameters intersect in pairs at fixed points S and S_1 on the line joining the centres of the circles. The points S and S_1 are called the *centres of similitude* of the two circles, and SA : SB :: AC : BE, also S_1A : S_1B :: AC : BE.

When the circles are external to one another, as in Fig. 66, one pair of the common tangents will intersect at S and the other pair will intersect at S_1.

When the two circles touch one another one of the centres of similitude will be at the point of contact of the circles.

29. The Radical Axis.—The locus of a point from which equal tangents can be drawn to two given circles is a straight line called the *radical axis* of the two circles.

If the circles touch one another the common tangent at their point of contact is their radical axis, and if the circles cut one another their common chord produced is their radical axis.

For any other case the following general construction may be used for finding the position of the radical axis of two circles. A and B (Figs. 67 and 68) are the centres of the given circles ECF and HDK. Describe any circle to cut the former circle at E and F and the latter at H and K. Draw the chords EF and HK and produce them to meet at P. A line PO at right angles to AB is the radical axis required. The proof is as follows : Since E, F, H, and K are on the same circle, $\overline{PE} \times \overline{PF} = \overline{PH} \times \overline{PK}$. But PC being a tangent to the circle ECF, $\overline{PC}^2 = \overline{PE} \times \overline{PF}$. Also PD being a tangent to the circle HDK, $\overline{PD}^2 = \overline{PH} \times \overline{PK}$; therefore $\overline{PC}^2 = \overline{PD}^2$ and PC = PD. Hence P is a point on the radical axis.

It can be proved that $\overline{AO}^2 - \overline{BO}^2 = \overline{AC}^2 - \overline{BD}^2$.

If a circle be described with its centre at any point P on the radical

axis of two given circles (Figs. 67 and 68) and having a radius equal to PC the tangent from P to one of the given circles, then this circle will cut the given circles *orthogonally* and will intersect AB at two fixed points L and M.

[Two curves are said to cut each other *orthogonally* when their

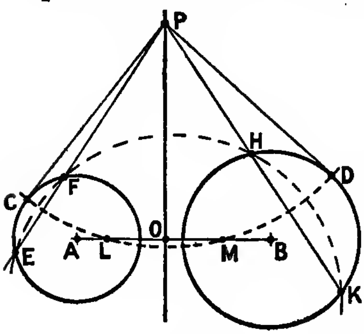


Fig. 67.

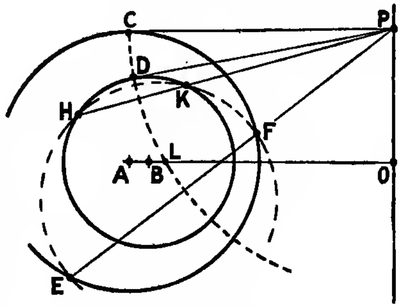


Fig. 68.

tangents at the point of intersection of the curves are at right angles to one another.]

If the radical axis of each pair of a system of three given circles be drawn, it will be found that the three radical axes will intersect at a point which is called the *radical centre* of the system.

30. To draw a circle to touch two given circles and pass through a given point.—A and B (Fig. 69) are the centres of the given circles and C is the given point. The line passing through A and B cuts the circles at D, E, H, and K as shown. Find S, one of the centres of similitude of the given circles (Art. 28). Draw a circle through the points D, E, and C, or through the points H, K, and C, cutting the line CS produced at F. Draw a circle to pass through the points C and F, and touch one of the given circles (Art. 24). This circle (centre O) will also touch the other given circle.

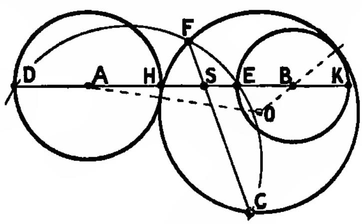


Fig. 69.

There may be as many as four solutions to this problem, two for each centre of similitude. In using the other centre of similitude (S_1) the construction is the same as described above, except that the point F is found at the intersection of S_1C and a circle through H, E and C, or through D, K and C.

Considering a straight line to be a circle of infinite diameter, the student should endeavour to deduce, from the foregoing solution, the construction for drawing a circle to touch a given circle and a given line and also pass through a given point.

31. To draw a circle to touch three given circles.—There may be as many as eight solutions to this problem. One solution is shown in Fig. 70. A, B, and C are the centres of the given circles. Let r_1, r_2, and r_3 be their radii respectively, and let r_1 be the radius which is not greater than r_2 or r_3. With centre B and radius equal to $r_2 - r_1$ describe the circle DE. With centre C and radius equal to $r_3 + r_1$ describe the circle FPH. Draw the circle APL which touches the circle DE with internal contact, and the circle FPH with external contact and passes through the point A (Art. 30). O, the centre of this circle, is the centre of the required circle, and ON is its radius. The points of contact with the given circles are M, K, and N. The construction obviously makes AM, PN, and LK equal to one another.

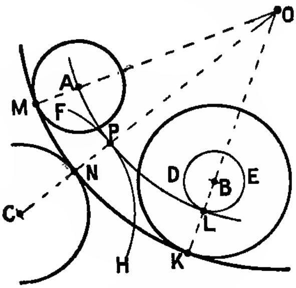


FIG. 70.

A second solution is obtained in the same way by making the circle APL touch the circle DE with external contact, and the circle FPH with internal contact.

Two solutions are obtained by making the radius of the circle DE equal to $r_2 + r_1$ and the radius of the circle FPH equal to $r_3 - r_1$.

Two solutions are obtained by making the radius of the circle DE equal to $r_2 - r_1$, and the radius of the circle FPH equal to $r_3 - r_1$; but for these solutions the circle APL must have either internal or external contact with both of the circles DE and FPH.

The remaining two solutions are obtained in the same way as the previous two except that the radii of the circles DE and FPH are $r_2 + r_1$ and $r_3 + r_1$ respectively.

Remembering that a straight line is a circle of infinite radius, the method illustrated by Fig. 70 may be easily modified to draw a circle to touch a given line and two given circles, or two given lines and a circle.

Exercises II

Note. *The points of contact of circles which touch one another, and the points of contact of tangents to circles, must be shown distinctly.*

1. ABC is a triangle. AC = 1·7 inches. BC = 1 inch. Angle C = 90°. Draw the circle which touches AC at A and passes through B.

2. A line OA, 1·5 inches long, is a radius of a circle whose centre is O. AB is a line 2·2 inches long making an angle of 120° with OA. Draw the circle which touches the given circle at A and passes through B.

3. On a straight line 3·5 inches long describe a segment of a circle to contain an angle of 150° without using the centre of the circle.

4. In a circle 3 inches in diameter inscribe a triangle whose angles are to one another as 2 : 3 : 4.

5. Draw an arc of a circle of 6 inches radius, the chord of the arc being 4 inches long. Then draw the arc of a circle concentric with the first and of 7 inches radius without using the centre. The two arcs are to subtend the same angle at the common centre.

6. OA and OL are two straight lines. Angle AOL = 80°. OA = 2·5 inches. OL is of indefinite length. B is a point in OA. OB = 1 inch. Find a point P in OL such that the angle APB is the greatest possible.

7. The centres of two circles are 1·75 inches apart. The radius of one is 1·2 inches, and the radius of the other is 0·9 inch. P is one of the points of intersection of the circles. It is required to draw two straight lines APB and CPD, 3 inches long, and terminated by the circles. [*Hint.* If Q is the middle point of AP, and R is the middle point of BP, then QR is half of AB.]

8. From a point 2·2 inches distant from the centre of a circle of 1 inch radius draw a straight line to cut the circle so that the part of it within the circle shall be 1 inch long.

9. Two straight lines include an angle of 60°. Draw a circle 3 inches in diameter, cutting one of the lines at points 1·7 inches apart, and the other at points 2·3 inches apart.

10. The centres of two circles are 3 inches apart. The radius of one is 0·6 inch, and the radius of the other is 1·4 inches. Draw the four common tangents to these circles.

11. In a circle 3 inches in diameter draw a chord dividing the circle into two segments, one of which shall contain an angle of 65°. In the smaller segment inscribe a circle ¾ inch in diameter, and in the larger inscribe a circle which shall pass through the centre of the original circle.

12. ABC is a triangle. AB = 2 inches. BC = 1·75 inches. CA = 1·5 inches. Draw the inscribed and escribed circles of this triangle.

13. Draw a sector of a circle. Radius of circle, 2 inches. Angle of sector, 60°. In this sector inscribe a circle.

14. In a circle 4 inches in diameter draw eight equal circles, each one to touch the original circle and two of the others.

15. Draw an equilateral triangle of 3 inches side and in it place three equal circles, each one touching one side of the triangle and the other two circles.

16. The vertical angle of a triangle is 50°, its altitude is 1·5 inches, and its perimeter is 6·75 inches. Construct the triangle.

17. AB (Fig. 71) is an arc of a circle of 2 inches radius. BC is an arc of a circle of 1·75 inches radius. The centres of these circles are 1·25 inches apart. ADC is an arc of a circle of 0·625 inch radius which touches the arcs AB and BC. Draw the figure ABCD.

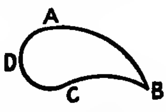


FIG. 71.

18. Draw a triangle ABC. AB = 3 inches. BC = 2·5 inches. CA = 2 inches. Draw a circle of 1 inch radius to touch the side AB and pass through the point C.

19. Draw two circles having their centres 3 inches apart. The diameter of one circle (A) to be 2 inches, and the diameter of the other (B) to be 3 inches. Draw a circle (C), 4 inches in diameter, to touch the circles A and B so that A is inside, and B outside C.

20. Within a circle 3 inches in diameter draw another, 1·8 inches in diameter, touching it. Next draw a third circle, 1 inch in diameter, inside the first circle, outside the second, and touching both. Then draw a circle to pass through the centre of the second circle, touch the first circle internally, and the third externally.

21. The section of a hand-rail is shown in Fig. 72. Draw this figure to the given dimensions, which are in millimetres. Show the constructions by which the centres of the circular arcs are determined and mark the junctions of the arcs. [B.E.]

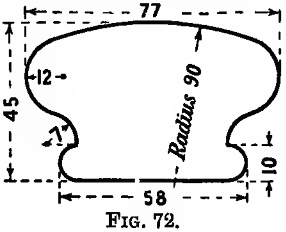


FIG. 72.

22. ABC is a triangle. AB = 1·5 inches. BC = 1·4 inches. CA = 1 inch. C is the centre of a circle of 1·6 inches radius. Draw two circles to touch this circle and pass through the points A and B.

23. Draw the locus of the centre of a circle which touches a fixed circle of

1 inch radius and passes through a fixed point 2 inches distant from the centre of the fixed circle.

24. AB, 2 inches long, is a diameter of the circle ACB (Fig. 73). BDE is an arc of a circle of 4 inches radius which touches the circle ACB at B. AE is at right angles to AB. Draw the figure ACBDE and add the circle F which touches the circle ACB, the arc BDE and the straight line AE.

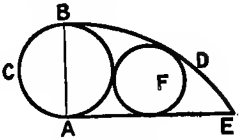


FIG. 73.

25. Draw a circle 2·5 inches in diameter, and take a point P at a distance of 2·25 inches from its centre. From P draw a tangent to the circle and determine the exact point of contact, using a pencil and straight-edge only.

26. LM is a straight line of indefinite length, A and B are two points which are 2 inches apart and are on the same side of LM. The perpendicular distances of A and B from LM are 1 inch and 1·5 inches respectively. B is the centre of a circle of 0·75 inch radius. Draw the two circles which have their centres on LM, pass through A, and touch the circle whose centre is B.

27. The polar of a point P with respect to a circle 2·75 inches in diameter is a straight line at a perpendicular distance of 0·75 inch from its centre. Determine the position of the point P, using a pencil and straight-edge only.

28. Draw three circles and find the centres of similitude of each pair. Then show by using a straight-edge that the line joining any two of the six centres of similitude either passes through a third or through the centres of two of the circles.

29. ABC is a triangle. AB = 3 inches. BC = 4 inches. CA = 3·5 inches. A, B, and C are the centres of three circles whose radii are 1 inch, 1·25 inches, and 2 inches respectively. Find the radical centre of the three circles, and draw the circle which cuts the three circles orthogonally.

30. Draw all the circles which touch each of the three circles given in the preceding exercise.

CHAPTER III

CONIC SECTIONS

32. The Conic Sections.—The curves known as the *conic sections* or *conics* are, the *ellipse*, the *hyperbola*, and the *parabola*. They are plane curves, and they may be defined with reference to their properties as plane figures, or they may be defined with reference to the cone of which they are plane sections.

33. Conics defined without Reference to the Cone.—If F (Fig. 74) is a fixed point and XM a fixed straight line, and if a point P move in the plane containing F and XM in such a manner that the distance FP always bears the same ratio to the perpendicular PM to the fixed line, then the curve traced out by the point P is called a *conic section* or *conic*.

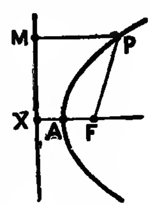


Fig. 74.

The fixed point F is called the *focus*, and the fixed straight line XM is called the *directrix* of the conic.

A straight line through the focus at right angles to the directrix is called the *axis*, and the point A where the axis cuts the curve is called the *vertex* of the conic.

P being any point on the curve, the constant ratio of FP to PM is called the *eccentricity* of the conic.

When FP is less than PM the conic is an *ellipse*.

When FP is equal to PM the conic is a *parabola*.

When FP is greater than PM the conic is an *hyperbola*.

A conic is therefore an ellipse, a parabola, or an hyperbola according as the eccentricity is less than, equal to, or greater than unity.

34. To construct a Conic having given the Focus, Directrix, and Eccentricity.—In Fig. 75, F is the given focus and XM the given directrix. Take a point D on the axis, and draw DE perpendicular to XD, and of such a length that DE : XD is equal to the given eccentricity. For example, if the eccentricity is 2 : 3 or $\frac{2}{3}$, make XD equal to 3 to any scale, and make DE equal to 2 to the same scale. Join XE. Draw any straight line N*n* parallel to the given directrix XM, cutting XE at *n*. With centre F and radius equal to N*n* describe arcs of a circle to cut *n*N and *n*N produced at P and P. P, P are points on the required conic. In like manner any number of

points may be determined, and a fair curve drawn through them is the conic required.

It is obvious that the ratio of Nn to XN is the same as the ratio of DE to XD, and therefore the ratio of FP to PM is the same as the ratio of DE to XD which was made equal to the given eccentricity.

The line XE will touch the conic at a point R obtained by drawing FR perpendicular to the axis to meet XE at R.

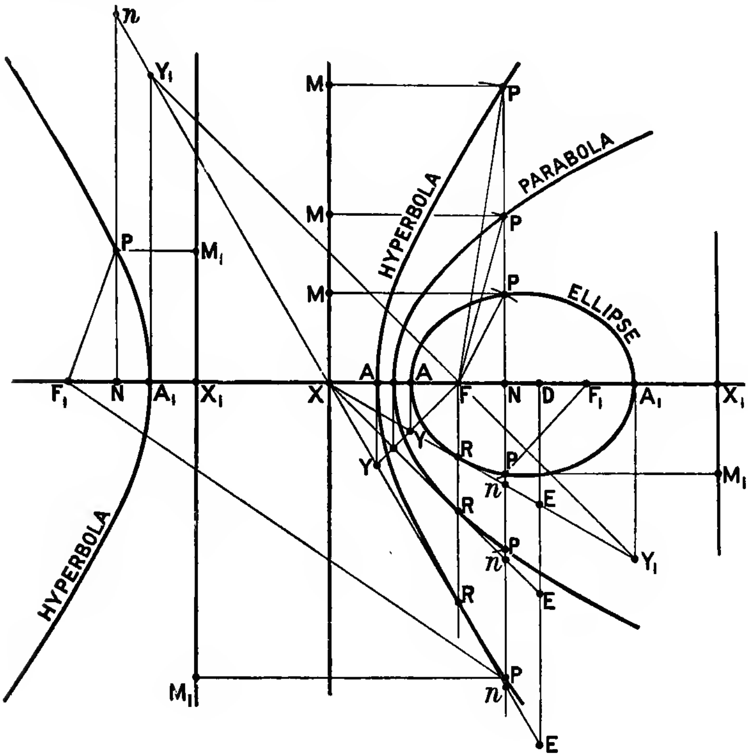


Fig. 75.

In Fig. 75, three different cases are shown. In the ellipse the eccentricity is $1 : \sqrt{3}$, in the parabola the eccentricity is of course $1 : 1$, and in the hyperbola the eccentricity is $\sqrt{3} : 1$. These eccentricities make the angle DXE equal to $30°$ for the ellipse, $45°$ for the parabola, and $60°$ for the hyperbola.

It is obviously unnecessary to draw the line XE in the case of the parabola. The radius for the arc through P may be taken at once from XN in this case, but it is instructive to notice that the construction given applies to all the conics.

If lines FY and FY$_1$ be drawn making $45°$ with the axis and

meeting XE at Y and Y_1, then perpendiculars from Y and Y_1 to the axis determine the points A and A_1 where the conic cuts the axis. In the case of the parabola there is only one such point, or to put it in another way, the point A_1 for the parabola is at an infinite distance from A.

It will be seen that the ellipse is a closed curve with two vertices, and that the hyperbola has two separate branches each with its own vertex.

If A_1F_1, measured to the left of A_1, be made equal to AF, and if A_1X_1 measured to the right of A_1 be made equal to AX, and if X_1M_1 be drawn perpendicular to the axis, then the ellipse may be constructed from the focus F_1 and directrix X_1M_1, both to the right of the figure, in the same way as from the focus F and directrix XM, using the same eccentricity. Also the hyperbola may be constructed from the focus F_1 and directrix X_1M_1, both to the left of the figure, in the same way as from the focus F and directrix XM, using the same eccentricity.

Referring to Fig. 75, it will be seen that as the eccentricity increases or diminishes the angle DXE increases or diminishes, and therefore as the eccentricity of the ellipse increases and approaches to unity the ellipse approaches to the parabola, and as the eccentricity of the hyperbola diminishes and approaches to unity the hyperbola also approaches to the parabola. A parabola is therefore the limiting form of an ellipse or an hyperbola. Remembering this fact, many of the properties of the parabola may be deduced at once from those of the ellipse or hyperbola. For example, the tangent to an hyperbola at any point P on the curve bisects the angle between the focal distances FP and F_1P. Now in the parabola the focus F_1 is at an infinite distance from the focus F, therefore F_1P is parallel to the axis, and the tangent to a parabola at any point P on the curve bisects the angle between the focal distance FP and the perpendicular PM on the directrix.

35. Conics defined with Reference to the Cone.—In Figs. 76, 77, and 78, *vtu* is the projection of a right circular cone on a plane parallel to its axis, *vs* being the projection of the axis of the cone. In each Fig. xx_1 represents a plane which cuts the cone and is perpendicular to the plane of projection.

If xx_1 cuts *vt* and *vu* below the vertex of the cone (Fig. 76), the section is an ellipse. If xx_1 cuts *vt* above and *vu* below the vertex (Fig. 78), the section is an hyperbola. If xx_1 is parallel to *vt* (Fig. 77), the section is a parabola.

In each Fig. the true shape of the section is shown, and is obtained by the rules of solid geometry. The axis XX_1 of the true shape of the section is drawn parallel to xx_1, and any point P on the curve is determined as follows:—Through any point *n* within the projection of the cone and on xx_1 draw *n*N perpendicular to xx_1 meeting XX_1 at N. Through *n* draw a line perpendicular to *vs* and terminated by *vt* and *vu*. On this line as diameter describe a semicircle. Through *n* draw *np* parallel to *vs* to meet the semicircle at *p*. On the line *n*N make NP equal to *np*. P is a point on the true shape of the section. By repeating this construction any number of points may be obtained,

and a fair curve drawn through them is the curve required. The theory of the above construction will be understood after Art. 226, Chap. XVIII, has been studied.

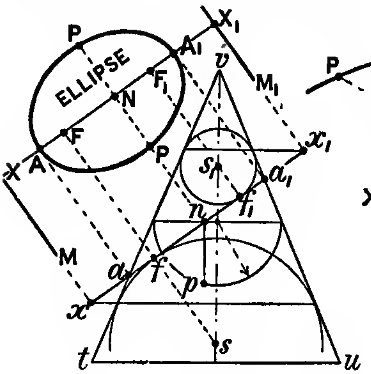


FIG. 76.

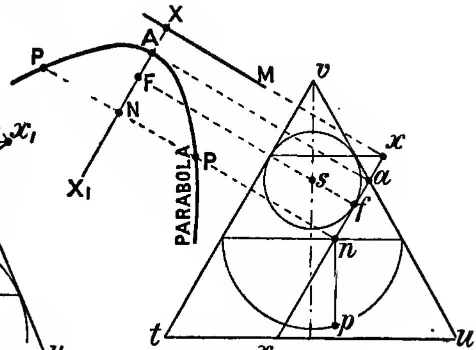


FIG. 77.

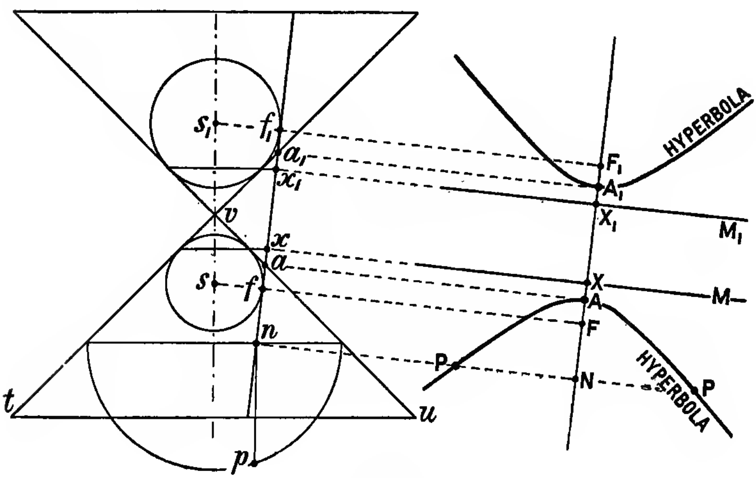


FIG. 78.

To determine the positions of the directrices and foci of the conics, draw spheres inscribed in the cone and touching the plane of section. These spheres are represented by the circles whose centres are at s and s_1 on the projection of the axis of the cone. These spheres will touch the cone in circles whose projections are the chords of contact of the circles which are the projections of the spheres and the lines vt and vu. The planes of these circles of contact intersect the plane of section in lines which are the directrices of the sections, and the projections of these lines are the points x and x_1. (In the case of the parabola the

point x_1 is at an infinite distance from x.) Hence if perpendiculars be drawn from x and x_1 to XX_1 the directrices XM and X_1M_1 of the true shape of the section are obtained.

The inscribed spheres touch the plane of section at the foci of the section, and the projections of these points are at f and f_1. Hence perpendiculars from f and f_1 to xx_1 to meet XX_1 determine the foci F and F_1 of the true shape of the section.

A reference to the sketch shown in Fig. 79 will perhaps make the meaning of the foregoing statements a little clearer.

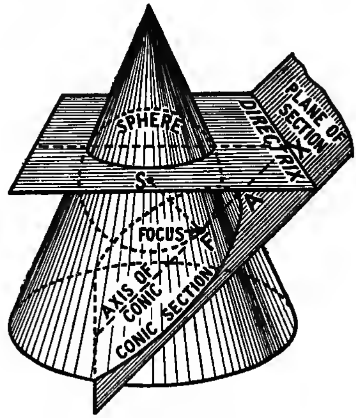


Fig. 79.

As in Art. 34, it is instructive to observe that the parabola is the limiting form of an ellipse or an hyperbola, as the plane of section (Figs. 76 and 78) is turned round so as to come nearer and nearer to a position (Fig. 77) in which it is parallel to vt. Also, when the hyperbolic section is taken through the vertex of the cone, the hyperbola becomes two straight lines, and when the plane which gives the elliptic section is turned round so as to be perpendicular to the axis of the cone the ellipse becomes a circle. Again, if the plane of the parabolic section be moved parallel to itself nearer and nearer to vt, the ultimate form of the parabola will be a straight line. Studying Figs. 76, 77, and 78 still further, it will be seen that when the hyperbola becomes two straight lines, the directrices will coincide, and the foci will coincide at the point where the axis of the conic cuts the directrix, and where the two straight lines which form the hyperbola intersect. Again, when the parabola becomes a straight line, that line will be the axis of the conic, and the focus will be on the directrix. Lastly, when the ellipse becomes a circle the foci will coincide at the centre of the circle, and the directrices will move off to infinity.

36. Additional Definitions relating to Conics.—A perpendicular PN (Fig. 80) from a point P on a conic to the axis is called the *ordinate* of the point P, and if PN be produced to cut the conic again at P′, the line PP′ is called a *double ordinate* of the conic.

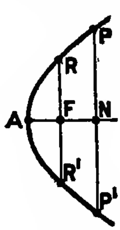


Fig. 80.

RFR′, the double ordinate through the focus, is called the *latus rectum* of the conic.

In the ellipse and hyperbola the point which is midway between A and A_1, the points where the conic cuts the axis, is called the *centre* of the conic, and the ellipse and hyperbola are called *central conics*.

A straight line joining two points on a conic is called a *chord* of the conic.

37. General Properties of Conics.—Following from the general definition of a conic (Art. 33), there are many properties possessed by all conics which may be demonstrated. A few of the more important of these general properties will now be given.

(1) *If a straight line cuts the directrix at D and the conic at P and Q* (Figs. 81 *and* 82), *then, F being the focus, the straight line DF will bisect, either the exterior or the interior angle between PF and QF.* Only in the case of the hyperbola, and only when P and Q are on different branches of the curve (Fig. 82) is it the interior angle PFQ which is bisected by DF.

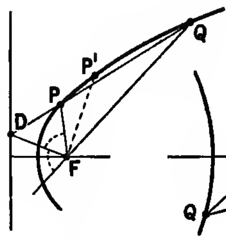


FIG. 81.

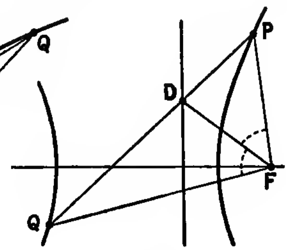


FIG. 82.

If the straight line FP′ be drawn bisecting the angle PFQ (Fig. 81) and meeting the conic at P′, then the angle DFP′ is evidently a right angle, and if the line DPQ be turned about D so as to make P and Q approach nearer and nearer to one another, P′ will always lie between P and Q, and in the limit when P and Q coincide they will coincide at P′ and a straight line through D and P′ will be a tangent to the conic at P′. Hence the next general property of conics.

(2) *The portion of a tangent intercepted between its point of contact and the directrix subtends a right angle at the focus.*

(3) *Tangents at the extremities of a focal chord intersect on the directrix.* This follows at once from the preceding property. It is evident, conversely, that *if tangents be drawn to a conic from a point on the directrix, the chord of contact passes through the focus.*

(4) *If PFP′ and QFQ′* (Fig. 83) *be two focal chords, the straight lines P′Q and Q′P intersect at a point on the directrix; also the straight lines PQ and Q′P′ intersect at a point on the directrix, and the portion of the directrix DD′ between the points of intersection subtends a right angle at the focus.* The above follows very easily from the first property given in this article.

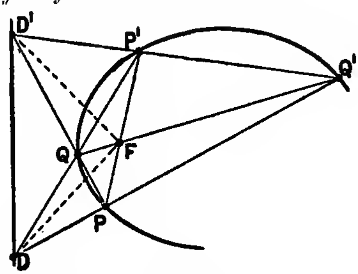


FIG. 83.

(5) *Tangents TP and TQ* (Fig. 84) *from any point T subtend equal angles at the focus F.*

(6) *If the normal at P* (Fig. 84) *meet the axis at G, the ratio of FG to FP is equal to the eccentricity of the conic.*

(7) *PL* (Fig. 84) *the projection of PG on FP is equal to the semi-latus rectum.*

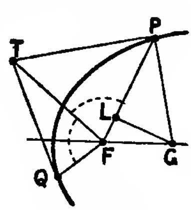


FIG. 84.

(8) *The locus of the middle points of a system of parallel chords is a straight line.* This straight line is called a *diameter* of the conic, and the point where a diameter cuts the conic is called the *vertex* of the diameter.

(9) *All diameters of a central conic pass through the centre of the conic, and all diameters of a parabola are parallel to the axis.*

(10) *Tangents at the extremities of a chord PP′* (Fig. 85) *intersect at a point T on the diameter TW which bisects the chord.*

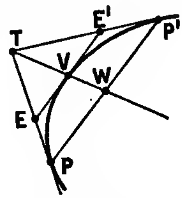


FIG. 85.

(11) *The tangent at the vertex of a diameter is parallel to the system of chords bisected by that diameter.*

(12) *TVW* (Fig. 85) *being a diameter, and TP and TP′ being tangents to the conic, EE′, the portion of the tangent at V intercepted between TP and TP′, is bisected at V the vertex of the diameter.*

38. Properties of the Parabola.—A knowledge of the following properties of the parabola will enable the student to solve a considerable number of problems.

(1) *If from the ends of a focal chord* PP_1 (Fig. 86) *perpendiculars be drawn to the directrix* KXK_1, *then* KK_1 *subtends a right angle at the focus F.*

(2) *The tangent PT* (Fig. 87) *at any point P bisects the angle between the focal distance FP and the perpendicular PK to the directrix.* From this it follows that the tangent is equally inclined to the focal distance of the point of contact and the axis. Also FT = FP

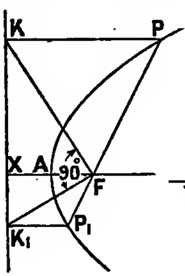


FIG. 86.

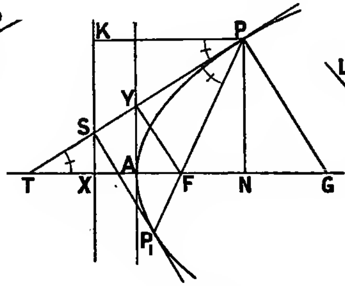


FIG. 87.

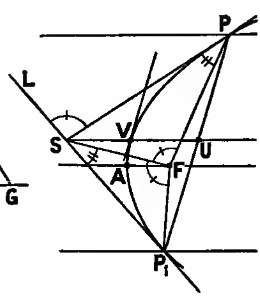


FIG. 88.

(3) *If the tangent at any point P* (Fig. 87) *meets the axis at T, and PN be the ordinate of P, then AT = AN.* This follows easily from the preceding property thus: FT = FP = PK = XN. But AF = AX, therefore AT = AN.

(4) *Y* (Fig. 87), *the foot of the perpendicular from the focus F on the tangent at any point P, lies on AY the tangent at the vertex.*

(5) $PN^2 = 4AF \cdot AN$ (Fig. 87).

(6) *Tangents PS and* P_1S *at the ends of a focal chord* PP_1 *intersect at right angles at a point S on the directrix* (Fig. 87).

Definition. If PG (Fig. 87) the normal at P meets the axis at G then NG is called the *subnormal.*

(7) *The subnormal is constant and equal to 2AF.* For, since PT bisects the angle FPK and PG is perpendicular to PT it follows that PG bisects the angle FPL.[1] Hence, angle FPG = angle LPG = angle FGP, therefore FG = FP = PK = NX, and NG = FX = 2AF.

(8) *The angles SFP and* SFP_1 (Fig. 88) *subtended at the focus F by tangents SP and* SP_1 *are equal to one another and to the angle LSP.*

(9) *The triangles FPS and* FSP_1 (Fig. 88) *are similar and* $FS^2 = FP \cdot FP_1$.

(10) The focus F (Fig. 89) and the points of intersection S_1, S_2, and S_3 of three tangents to a parabola lie on the same circle.

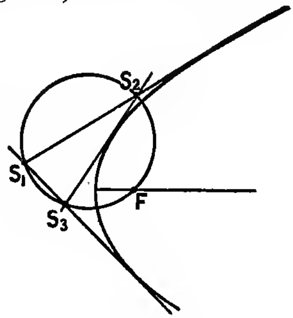


Fig. 89.

Definition. A straight line VU (Fig. 88) drawn parallel to the axis AF through any point V of a parabola is called a *diameter* and the point V is called the *vertex of the diameter* VU.

(11) *S* (Fig. 88), *the point of intersection of two tangents SP and* SP_1, *is equidistant from the diameters through the points of contact P and* P_1.

(12) *The diameter SVU* (Fig. 88) *through the point of intersection of tangents SP and* SP_1 *bisects the chord of contact* PP_1. *Also SV = VU, and* PP_1 *is parallel to the tangent at V.*

(13) *A diameter bisects all chords parallel to the tangent at its vertex.*

Definition. The focal chord parallel to the tangent at the vertex of a diameter is called the *parameter* of that diameter.

(14) *The parameter of any diameter is four times the focal distance of the vertex of that diameter.*

(15) *The area enclosed by the parabola AQP* (Fig. 90), *the ordinate PN, and the axis AN, is two-thirds of the area of the rectangle ANPL.*

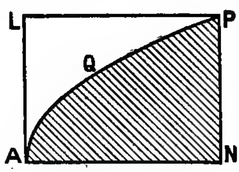


Fig. 90.

It follows from this that the area of the figure AQPL is one-third of the area of the rectangle ANPL.

39. To draw the Tangents to a Parabola from an external point.— Let A (Fig. 91) be the vertex and F the focus of the parabola and let S be the external point from which the tangents are to be drawn.

Join SF and on SF as diameter describe a circle. Draw YY_1 the tangent at the vertex cutting the circle at Y and Y_1.

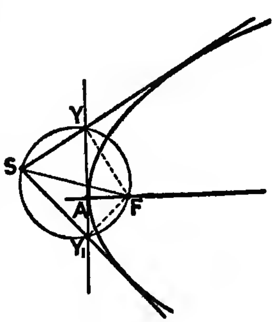


Fig. 91.

[1] PL is KP produced (Fig. 87).

Join S to Y and Y_1 and produce them. These are the tangents required.

Y and Y_1 are evidently the feet of the perpendiculars from F on SY and SY_1 and these are on the tangent at the vertex. Hence by Art. 38 (4) SY and SY_1 are tangents from S.

40. To construct a Parabola, having given the Vertex, the Axis, and another point on the Curve.—Let A (Fig. 92) be the vertex, AN the axis, and P another point on the parabola. This is a problem of very frequent occurrence and the following construction is the most convenient.

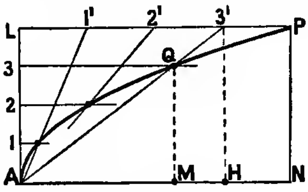


FIG. 92.

Draw the ordinate PN and complete the rectangle ANPL. Divide AL into any convenient number of equal parts, and number the points of division 1, 2, 3, etc. from A to L. (In Fig. 92, four equal parts have been taken.) Divide LP into the same number of equal parts, and number the points of division, 1′, 2′, 3′, etc. from L to P. From the points 1, 2, 3, etc. on AL draw lines parallel to AN. Join A to the points 1′, 2′, 3′, etc. on LP. The lines having the same numbers at their ends intersect at points on the parabola.

Consider one of these points, Q, where A3′ cuts the line through 3 parallel to AN. Draw QM and 3′H perpendicular to AN. The ordinate PN is $\frac{4}{3}$ of the ordinate QM. Again AN is $\frac{4}{3}$ of AH and AH is $\frac{4}{3}$ of AM. Therefore $AN = \frac{4}{3} \times \frac{4}{3}AM = \frac{16}{9}AM$.

But [Art. 38, (5)] $PN^2 = 4AF \,.\, AN$

therefore $(\frac{4}{3}QM)^2 = 4AF \times \frac{16}{9}AM$, and $QM^2 = 4AF \cdot AM$, which shows that Q is a point on the parabola.

41. Definitions relating to Central Conics.—A central conic cuts the axis at two points A and A_1, and there are two foci F and F_1. There are also two directrices, one belonging to each focus. In referring to the focus and directrix of a conic it is understood that if the conic is a central conic the focus and directrix belong or correspond the one to the other.

In a central conic the line AA_1 terminated by the two vertices of the curve is called the *transverse axis*.

In the ellipse, Fig. 93, the straight line BCB_1 drawn through the centre at right angles to AA_1 and terminated by the curve is called the *conjugate axis* but more generally the *minor axis*, and the transverse axis is generally called the *major axis*.

In the hyperbola, Fig. 94, there is also a conjugate axis BCB_1 passing through the centre at right angles to AA_1, but it does not meet the curve. The length of the conjugate axis of the hyperbola is obtained by the following construction. With centre A and radius equal to CF describe an arc of a circle to cut the line through C perpendicular to AA_1 at B and B_1. Then BB_1 is the length of the conjugate axis. The conjugate axis of the hyperbola is not necessarily

a minor axis, as it may be either less than, equal to, or greater than the transverse axis.

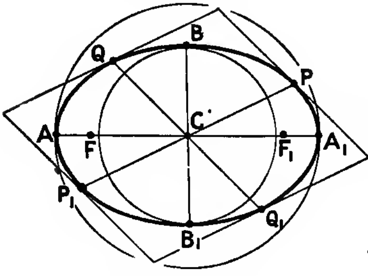


FIG. 93.

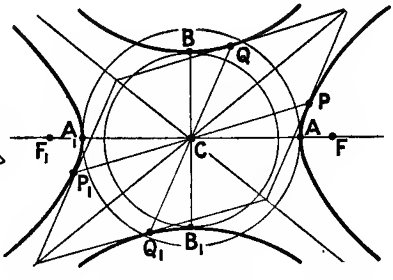


FIG. 94.

The circle described with the transverse axis AA_1 as a diameter is called the *auxiliary circle.*

The circle described on the minor axis of an ellipse as a diameter is sometimes called the *minor auxiliary circle.*

If an hyperbola be described having BCB_1 for its transverse axis, and ACA_1 for its conjugate axis, this hyperbola is called the *conjugate hyperbola.*

Diameters PCP_1 and QCQ_1 are called *conjugate diameters* when QQ_1 is parallel to the tangents at P and P_1, or when PP_1 is parallel to the tangents at Q and Q_1.

42. General Properties of Central Conics.—The ellipse and hyperbola have many properties in common, and there are other properties possessed by one which when stated in a slightly modified form are also possessed by the other. A number of these properties will now be given.

(1) *The sum or difference of the focal distances of any point P* (Figs. 95 and 96) *on the curve is equal to the transverse axis, or* $PF \pm PF_1 = AA_1$. In the ellipse it is the *sum* and in the hyperbola it is the *difference* which must be taken.

(2) *The tangent and normal at any point P* (Figs. 95 and 96) *on the curve bisect the angles between the focal distances of the point.* In the ellipse it is the exterior angle between PF and PF_1 which is bisected by the tangent, while in the hyperbola it is the interior angle between these lines which is bisected by the tangent.

(3) *A circle drawn through a point P* (Figs. 95 and 96) *on the curve and the foci F and F_1 cuts the conjugate axis at the points t and g, where the tangent and normal to the curve at P cut that axis.* (In Fig. 96 the point g falls outside the limits of the figure.)

(4) *Perpendiculars FY and F_1Y_1* (Figs. 95 and 96) *on the tangent at any point P meet that tangent on the auxiliary circle, and BC, the semi-conjugate axis, is a mean proportional between them or* $BC^2 = FY \cdot F_1Y_1$.

(5) *Tangents OP and OQ* (Figs. 95 and 96) *are equally inclined to the focal distances OF and OF_1 of the point O.*

(6) *The locus of the intersection O* (Figs. 95 and 96) *of pairs of tangents PO and QO, which are at right angles to one another, is a circle whose centre is C, and whose radius is equal to* $\sqrt{AC^2 \pm BC^2}$. This circle is called the *director circle.* In determining the radius the *plus* sign must be taken in the case of the ellipse and the *minus* sign in the case of the hyperbola. In the case of the hyperbola when BC = AC, the radius of the director circle is zero, and when BC is greater than

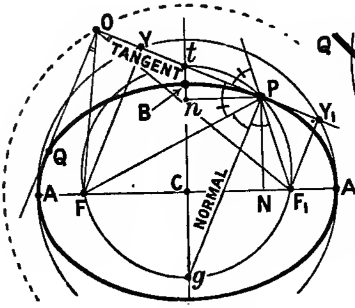


FIG. 95.

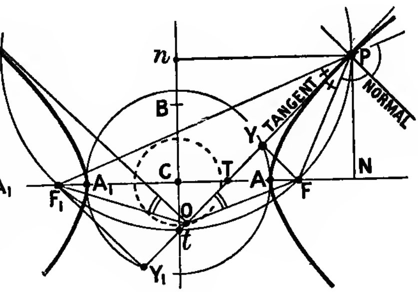


FIG. 96.

AC there is no director circle, and no pairs of tangents can be drawn which are perpendicular to one another.

In Fig. 96 the tangents OP and OQ are shown touching different branches of the hyperbola, but for certain positions of O they will touch one branch only.

(7) *If the tangent at P meet the transverse axis at T* (Figs. 95 and 96) *and the conjugate axis at t, and if N and n be the feet of the perpendiculars from P on the transverse and conjugate axes respectively,* then $CN \cdot CT = AC^2$, and $Cn \cdot Ct = BC^2$.

(8) $PN^2 : AN \cdot A_1N :: BC^2 : AC^2$.

(9) CP and CQ (Figs. 93 and 94) being conjugate semi-diameters, $CP^2 \pm CQ^2 = CA^2 \pm CB^2$. The *plus* sign is taken for the ellipse and the *minus* sign for the hyperbola.

(10) *The area of the parallelogram formed by the tangents at the ends of conjugate diameters is equal to the rectangle contained by the transverse and conjugate axes.*

43. To construct a Central Conic having given the Foci and the Transverse Axis.—F and F_1 (Figs. 97 and 98) are the given foci and AA_1 is the given transverse axis. For the ellipse take any point a in AA_1 and for the hyperbola take any point a in A_1A produced. With centres F and F_1 and radius equal to Aa describe arcs of circles. With centres F and F_1 and radius equal to A_1a describe arcs of circles to cut the former arcs at P, Q, P_1 and Q_1. P, Q, P_1, and Q_1 are points on the conic required, and by taking other positions of the point a and repeating the above construction any number of points on the conic may be determined and a fair curve drawn through them will complete the construction.

It will be observed that the property of central conics given in Art. 42 (1) is made use of in the above construction.

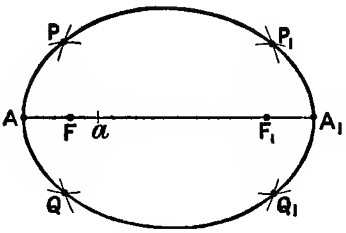


FIG. 97.

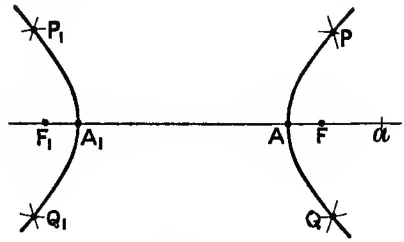


FIG. 98.

44. To construct an Ellipse from the Auxiliary Circles.—It has already been given [Art. 42, (8)] that $PN^2 : AN \cdot A_1N :: BC^2 : AC^2$. Let the ordinate PN (Fig. 99) be produced to meet the auxiliary circle at a, then by a property of the circle, $aN^2 = AN \cdot A_1N$, therefore $PN^2 : aN^2 :: BC^2 : AC^2$, or $PN : aN :: BC : AC$. Draw aC cutting the minor auxiliary circle at b, then, since $aC = AC$ and $bC = BC$, $PN : aN :: bC : aC$, therefore Pb is parallel to AC. Hence the following construction for finding points on the ellipse. Draw a radius aC of the auxiliary circle cutting the minor auxiliary circle at b. Through a draw aN parallel to the minor axis, and through b draw bP parallel to the major axis to meet aN at P. P is a point on the ellipse. In like manner any number of points on the ellipse may be found, and a fair curve drawn through them completes the construction.

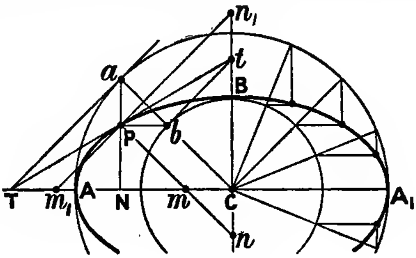


FIG. 99.

It may be noted here that the tangent to the auxiliary circle at a, and the tangent to the ellipse at P meet at a point T on the major axis produced. Also the tangent to the minor auxiliary circle at b and the tangent to the ellipse at P meet at a point t on the minor axis produced.

45. The Trammel Method of Drawing an Ellipse.—Referring to Fig. 99, if Pn be drawn parallel to aC, meeting AC at m and BC produced at n, then $Pn = aC = AC$, and $PN : aN :: Pm : aC$; but it has been shown that $PN : aN :: BC : AC$, therefore $Pm : aC :: BC : AC$, and consequently $Pm = BC$. Hence if a straight line be drawn across an ellipse, cutting the curve at P, the major axis at m, and the minor axis at n, and if $Pn = AC$, then $Pm = BC$. Conversely it follows that if a straight line Pmn in which $Pn = AC$ and $Pm = BC$ be placed so that n is on the minor axis and m is on the major axis, then P will lie on the ellipse.

On the straight edge of a strip of paper, Fig. 100, mark points P, m, and n, such that Pn is equal to the semi-major axis, and Pm is equal to the semi-minor axis of the ellipse. Place this paper trammel on the paper so that m is on the major axis and n is on the minor axis, a dot made on the paper at P will be on the ellipse. By moving the trammel into a number of different positions a sufficient number of points on the ellipse may be obtained and a fair curve is then drawn through them.

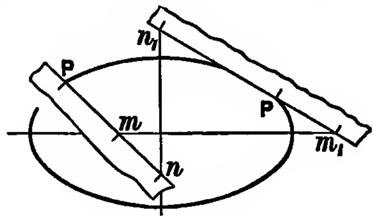


FIG. 100.

Referring again to Fig. 99, if m_1Pn_1 be drawn making the angle Pn_1n equal to the angle Pnn_1, then Pn_1 will be equal to Pn, and Pm_1 will be equal to Pm. Hence the points m and n on the trammel may be on opposite sides of P as shown on the trammel m_1Pn_1, Fig. 100. The latter form of the trammel should be used when the difference between the major and minor axes of the ellipse is small.

The trammel method of drawing an ellipse is most convenient as it keeps the paper free of construction lines other than the axes.

46. Given a Pair of Conjugate Diameters of an Ellipse, to find the Axes.—Let PCP_1 and QCQ_1 (Fig. 101) be the given conjugate diameters. Draw PD perpendicular to CQ. Make PH and PH_1 each equal to CQ. Join CH and CH_1. The major axis ACA_1 bisects the angle HCH_1, and the minor axis BCB_1 is of course perpendicular to ACA_1. Join P to the middle point of CH, cutting the major axis at m and the minor axis at n. Pn is the length of the semi-major axis, and Pm is the length of the semi-minor axis.

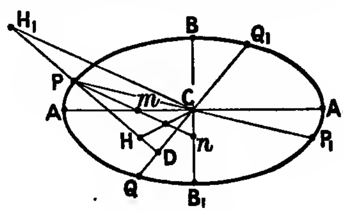


FIG. 101.

47. To construct an Ellipse having given a Pair of Conjugate Diameters.—Let PCP_1 and QCQ_1 (Figs. 102 and 103) be the given conjugate diameters.

First Method. Through P, Q, P_1, and Q_1 (Fig. 102) draw parallels to the given diameters forming the parallelogram RT. Divide CP into a number of equal parts, and divide RP into the same number of equal parts. Join Q to $1'$ the first point of division on RP. A line joining Q_1 to 1 the first point of division on CP will when produced cut $Q1'$ at a point on the ellipse required. Repeating this construction

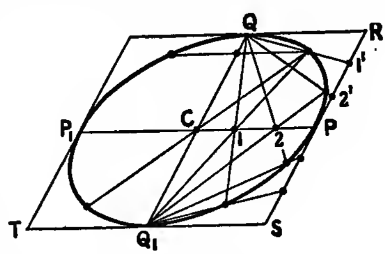


FIG. 102.

with the other points of division on RP and CP other points on the arc PQ are obtained. In like manner points on the arc PQ_1 are obtained. Points on the other half of the ellipse may be found by the same method, or by a construction depending on the fact that all diameters are bisected at the centre C, or by making use of the fact that chords parallel to a diameter are bisected by its conjugate.

Second Method. Find the axes by the construction described in the preceding article and then use a paper trammel as explained in Art. 45.

Third Method. By a triangular trammel. From P, Fig. 103, draw Pm perpendicular to CQ. From Q draw QD perpendicular to CP and make Pn equal to QD. Join mn. If the triangle Pmn be drawn on a strip of paper, mn being on one edge and P on the opposite edge, and if this strip be moved into different positions, m being on QQ_1 and n on PP_1, then P will be on the ellipse. A second position of the trammel is shown to the right of the figure. Instead of using a strip of ordinary paper, the points m, n, and P may be marked on a piece of tracing paper, a needle hole being made in the tracing paper at P, through which points on the ellipse may be marked by a sharp round-pointed pencil.

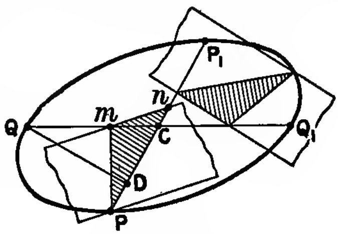


FIG. 103.

48. The Ellipse as the Projection of a Circle.—Many of the properties of the ellipse and many of the constructions connected with it may be readily demonstrated by considering the ellipse as the projection of a circle. A few of these will be considered here, it being assumed that the student has already studied some of the later chapters in this work relating to projection.

Referring to Fig. 104, ABA_1B_1 is a circle lying in a horizontal plane, AA_1 and BB_1 being diameters at right angles to one another. Imagine the circle ABA_1B_1 to revolve about AA_1 as an axis until its plane is inclined at an angle θ to the horizontal, and let it then be projected on to the horizontal plane containing AA_1. The point P on

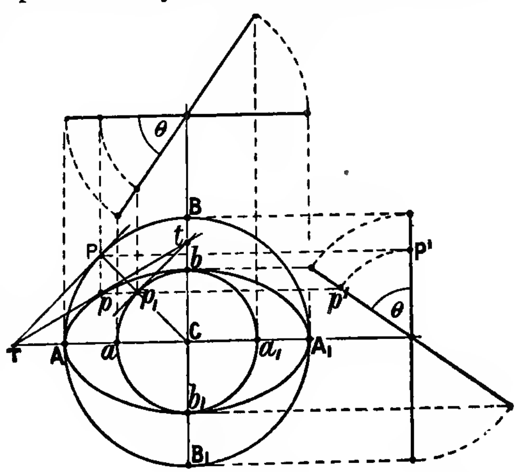


FIG. 104.

the circle in its new position will have the point p for its plan, the point p being determined by the construction shown. Other points on the circle may be treated in the same way, and all points such as p being joined by a fair curve the ellipse ApA_1b_1 is determined.

On bb_1 as diameter describe the circle aba_1b_1. This circle will have its centre at C, the centre of the other circle. Now imagine the ellipse, which is horizontal, to revolve about bb_1 as an axis until its plane is inclined at the angle θ to the horizontal and let it then be projected on to the horizontal plane containing bb_1. The point p on the ellipse in its new position will have the point p_1 for its plan, and it is easy to show that the point p_1 is on the circle aba_1b_1, also that the points P, p_1, and C are in the same straight line. It will be seen that the circles ABA_1B_1 and aba_1b_1 are the auxiliary circles of the ellipse AbA_1b_1 and the theory of the construction given in Art. 44 is further explained.

It is easy to establish the following theorem in projection:—
"*The projection of the tangent to a curve at any point is the tangent to the projection of the curve at the projection of the point.*"

Referring again to Fig. 104, TP is a tangent to the circle ABA_1B_1 at P, and when the circle is turned about AA_1 as an axis and projected into the ellipse as already described, the point T which is on that axis is stationary and therefore Tp will evidently be the tangent to the ellipse at p. Let Tp produced meet BB_1 at t. Then when the ellipse is turned about BB_1 as an axis and projected into the circle aba_1b_1 as already described, the point t which is on that axis is stationary and tp_1 will evidently be the tangent to the circle aba_1b_1 at p_1.

Another theorem in projection which is easily proved is as follows:—
The projections on a plane of two intersecting straight lines will intersect at a point which is the projection of the point of intersection of the original lines, and the point of intersection of the projections will divide the projections into segments which are to one another as the corresponding segments of the original lines.

This enables a simple demonstration to be given of the construction for drawing an ellipse having given a pair of conjugate diameters which has been described in Art. 47 and illustrated by Fig. 102.

Referring to Fig. 105, PP_1 and QQ_1 are two diameters of a circle at right angles to one another. The circle is supposed to be lying on the horizontal plane. A square RSTU is shown circumscribing the circle, the points of contact being P, Q_1, P_1, and Q. The radius CP is divided into any number of equal parts, in this case three, at the points 1 and 2, and RP is divided into the same number of equal parts at the points 1′ and 2′. If the lines joining Q_1 to the points 1 and 2 be produced it is easy to show that they will intersect the lines joining Q to 1′ and 2′ respectively at points on the circle as shown.

Now imagine the circle, with all the lines connected with it, to revolve about an axis MN in its plane into an inclined position, and let a projection of the whole be then made on the horizontal plane as shown. The square RSTU projects into the parallelogram $rstu$, and

the circle projects into an ellipse which touches the sides of the parallelogram at p, q_1, p_1, and q. Also pp_1 and qq_1 will evidently be conjugate diameters of the ellipse. Lastly, the lines through Q_1 and Q which intersect on the circle project into lines which intersect on the

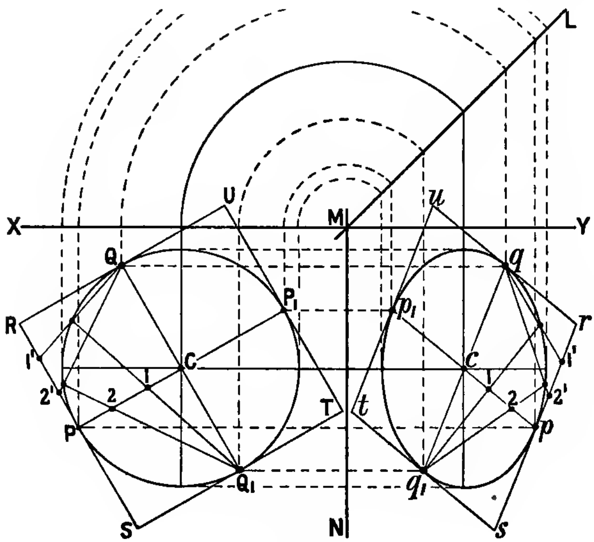


FIG. 105.

ellipse and divide cp and rp into three equal parts. Hence the construction for drawing the ellipse having given two conjugate diameters.

49. Properties characteristic of the Hyperbola.—It has already been pointed out that the hyperbola has two separate branches and that there is a conjugate hyperbola also having two separate branches. Also the transverse and conjugate axes of the hyperbola are the conjugate and transverse axes respectively of the conjugate hyperbola.

Referring to Fig. 106, F and F_1 are the foci and AA_1 is the transverse axis of the hyperbola whose branches are Q and Q_1. BB_1 is the conjugate axis and Q' and Q'_1 are the branches of the conjugate hyperbola. A construction has already been given (Art. 41, p. 38) for finding the conjugate axis, and the following construction will evidently give the same result. With centre C, the middle point of AA_1, and radius CF describe a circle. Draw the tangent to the hyperbola at A to cut this circle at L and L_1. Through C draw BCB_1 at right angles to AA_1. Parallels to AA_1 through L and L_1 will cut BCB_1 at B and B_1 the extremities of the conjugate axis. The circle whose centre is C and radius CF cuts BB_1 produced at F' and F'_1 the foci of the conjugate hyperbola.

The lines CL and CL_1 produced both ways are the *asymptotes* of the hyperbola. It will be seen that the asymptotes are the diagonals of the rectangle formed by the tangents at the vertices of the hyperbola and its conjugate. The asymptotes are tangents to the hyperbola and its conjugate at an infinite distance from the centre C.

The following properties should be specially noted :—

(1) *Perpendiculars from the foci to the asymptotes are tangents to one or other of the auxiliary circles.* Referring to Fig. 106, FD is a perpendicular to the asymptote CL and FD is a tangent to the auxiliary circle described on AA_1 as diameter and D is the point of contact. F′D′ is a perpendicular to CL and F′D′ is a tangent to the auxiliary circle described on BB_1 as diameter and D′ is the point of contact.

(2) *The auxiliary circles intersect the asymptotes at points on the directrices.* DX (Fig. 106) is a directrix of the hyperbola whose transverse axis is AA_1 and D′X′ is a directrix of the conjugate hyperbola.

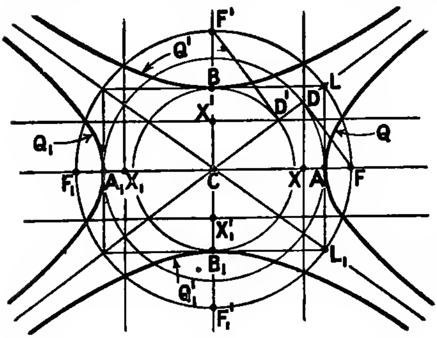


Fig. 106.

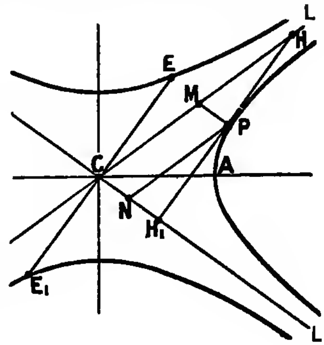


Fig. 107.

(3) *The portion HH_1* (Fig. 107) *of a tangent lying between the asymptotes is bisected at the point of contact P, and HH_1 is equal to the diameter ECE_1 which is parallel to HH_1.*

This suggests a method of drawing the tangent to the curve at a point P on it. Draw PM parallel to CL_1 to meet CL at M. Make MH equal to CM. Join HP and this will be the tangent required.

(4) *If from any point P on the curve* (Fig. 107) *PM and PN be drawn parallel to the asymptotes, meeting them at M and N respectively, then the product PM . PN is constant.*

This last property may be used to construct an hyperbola when the asymptotes and a point on the curve are given as follows. Referring to Fig. 108, CL and CL_1 are the given asymptotes and P is a given point on the hyperbola. Draw PM and PN parallel to CL_1 and CL respectively and produce PM and PN both ways. Through C draw a number of radial lines to cut PM at points 1, 2, 3, etc., and PN at points 1′, 2′, 3′, etc. Parallels to CL through the points 1, 2, 3, etc.

will intersect parallels to CL_1 through 1′, 2′, 3′, etc. respectively at points on the hyperbola as shown.

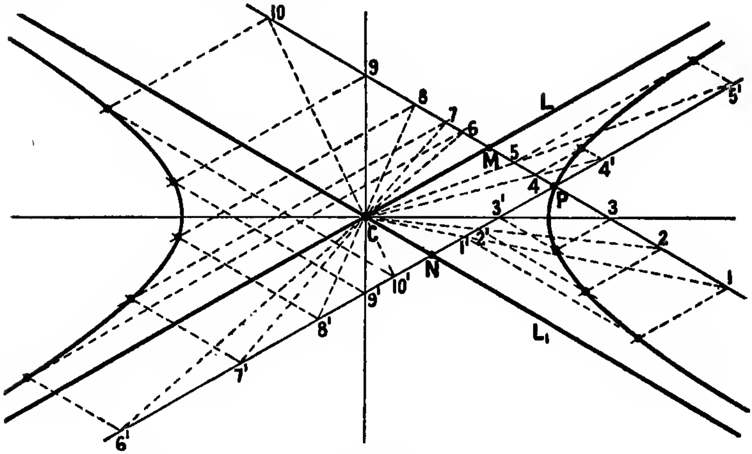


FIG. 108.

50. The Rectangular Hyperbola.—When the transverse and conjugate axes of an hyperbola are equal the asymptotes are at right angles to one another, and the hyperbola is then said to be *equilateral* or *rectangular.*

In Fig. 109, OX and OY, at right angles to one another, are the asymptotes of a rectangular hyperbola and P is a point on the curve. Other points on the curve are obtained by the construction already given and illustrated by Fig. 108. PM and PN being parallel to OX and OY respectively, it will be seen that for the rectangular hyperbola (Fig. 109) PMON is a rectangle.

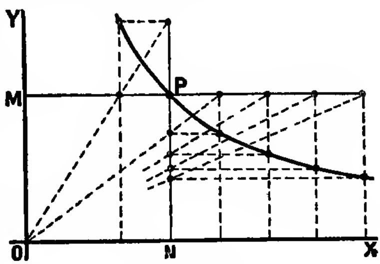


FIG. 109.

If distances from OY parallel to OX represent to scale the volume v of a given weight of a gas, and if distances from OX parallel to OY represent to scale the corresponding pressure p of the gas, then if the gas is expanded or compressed and the pressure is inversely as the volume, the product pv is constant and the co-ordinates of points on a rectangular hyperbola will show the relation between the pressure and volume of the gas as it is expanded or compressed. The hyperbola is then the *expansion curve* or the *compression curve* for the gas.

51. Centre of Curvature of a Conic.—The following construction is applicable for finding the centre of curvature of any conic.

P (Fig. 110) is a point on the curve, F the focus, FG the direction of the axis, and PG the normal to the curve at P. Draw GH at right angles to PG meeting PF or PF produced at H. Draw HS at right angles to PH meeting PG produced at S. S is the centre of curvature of the conic at P.

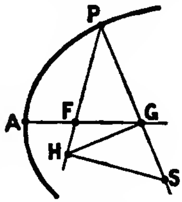


FIG. 110.

In the case of the parabola the point H may be obtained by making FH, on PF produced, equal to PF.

The above construction fails when the point P is at A the vertex of the conic. It will be observed that as the point P approaches nearer and nearer to A, the points G, H, and S approach nearer and nearer to one another and in the limit they will coincide at a point on the axis. Now in the parabola FH is equal to PF; hence the centre of curvature of the parabola at A, its vertex, is on the axis at a distance from A equal to 2AF.

In the ellipse and hyperbola $PF : PF_1 :: FG : F_1G$, and when P coincides with A, this becomes $AF : AF_1 :: FS : F_1S$. Hence the following construction for the centre of curvature at A. Draw F_1D (Figs. 111 and 112) inclined to F_1F. Make F_1D equal to AF_1, and

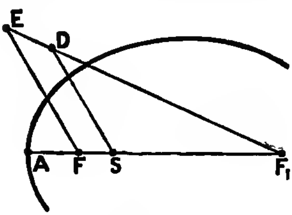


FIG. 111.

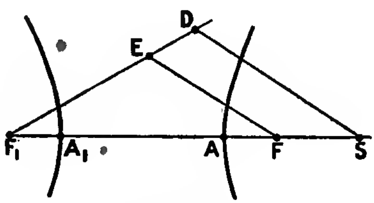


FIG. 112.

DE equal to AF. Join EF, and draw DS parallel to EF meeting the axis at S. S is the point required.

Another construction which gives the centre of curvature of a central conic at an extremity of the transverse axis, and at the same time gives the centre of curvature at an extremity of the conjugate axis, is shown in Figs. 113 and 114.

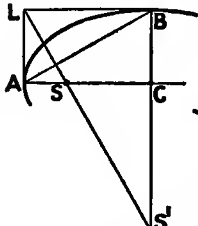


FIG. 113.

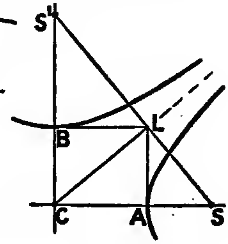


FIG. 114.

Referring to the ellipse (Fig. 113) CA is the semi-major axis and CB is the semi-minor axis. Complete the rectangle ACBL. Join AB. Draw LSS' at right angles to AB to cut AC at S and BC produced at S'; then S and S' are the centres of curvature of the ellipse at A and B respectively.

The ellipse may be quickly drawn with sufficient accuracy for many purposes by describing circular arcs through the extremities of the axes from the corresponding centres of curvature and then joining these arcs with fair curves to please the eye.

Referring to the hyperbola (Fig. 114) CA is the semi-transverse axis and CB is the semi-conjugate axis. Complete the rectangle ACBL. Join CL. Draw SLS′ at right angles to CL to cut CA produced at S and CB produced at S′. S is the centre of curvature of the hyperbola at A, and S′ is the centre of curvature of the conjugate hyperbola at B. Note that CL is an asymptote of the hyperbola.

52. Evolute of a Conic.—The evolute being the locus of the centre of curvature, if the construction of the preceding article be applied to a sufficient number of points on the conic, a fair curve drawn through the centres obtained will be the evolute of the conic. Fig. 115 shows the evolute of a parabola, and Fig. 116 shows the

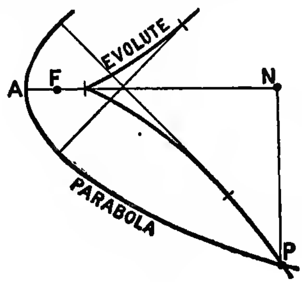


FIG. 115.

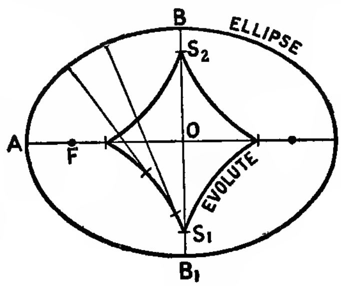


FIG. 116.

evolute of an ellipse. In the parabola the evolute cuts the conic at two points and if the ordinate PN of one of them be drawn, then AN $=8$AF. In the ellipse when OB the semi-minor axis is equal to OF the points S_1 and S_2 will coincide with B_1 and B respectively. When OB is greater than OF, S_1 and S_2 will lie within the ellipse as in Fig. 116, and when OB is less than OF the points S_1 and S_2 will lie outside the ellipse.

53. Pole and Polar.—If through a given point P (Figs. 117 and 118) any straight line be drawn to cut a conic at Q and R, tangents to the conic at Q and R will intersect on a fixed straight line LM. Conversely, the chord of contact of tangents to the conic from any point in LM will pass through P.

The fixed line LM is called the *polar* of the point P with respect to the conic, and the point P is called the *pole* of the line LM with respect to the conic. The point P may be within or without the conic.

In the case of a central conic (Fig. 117) if C the centre be joined to P and produced, if necessary, to cut the conic at S, then LM is parallel to the tangent to the conic at S. In the case of the parabola (Fig. 118) if PS be drawn parallel to the axis to meet the curve at S, then LM is parallel to the tangent to the parabola at S.

If CP or CP produced cuts the conic at S and LM at T, then $\overline{CP} \times \overline{CT} = \overline{CS}^2$.

If chords QR and KN of a conic (produced if necessary) intersect at P, then the chords QK and NR (produced if necessary) will intersect on LM the polar of P. Also the chords QN and KR (produced if necessary) will intersect on LM the polar of P. This suggests the simplest construction for finding the polar of a given point P.

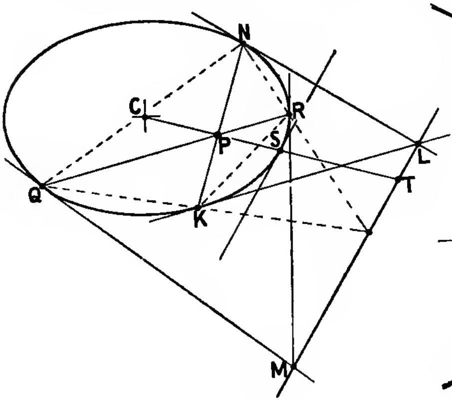


FIG. 117.

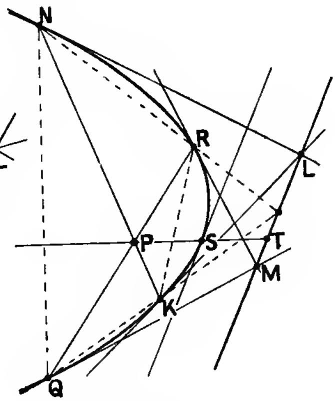


FIG. 118.

A more general way of stating the preceding property is as follows. If a quadrilateral QNRK be inscribed in a conic, the opposite sides and diagonals will (produced if necessary) intersect in three points such that each is the pole of the line joining the other two.

Since the circle is a particular form of a conic it follows that all that has been said about the pole and polar with respect to a conic will be true for the pole and polar with respect to a circle. The student should therefore compare this article with Art. 27, p. 24.

Exercises III

1. Draw an ellipse, a parabola, and an hyperbola as in Fig. 75, p. 31, having given, FX = 1·5 inches, eccentricity of ellipse = $\frac{5}{9}$, and eccentricity of hyperbola = $\frac{8}{5}$.

2. Using the curves of the preceding exercise, work out on each, the following:—

(*a*) P is any point on the conic. PL is the tangent at P, L being on the directrix. G is the point where the normal at P cuts the axis, and F is the focus. PG and LF are joined and produced to meet at *p*. Construct the locus of *p*.

(*b*) PQ is a chord of the conic subtending an angle of say 60° at the focus F. Determine the locus of the intersection of the tangents at P and Q, and also the envelope of the chord PQ. [It will be found that the curves required are conics having the same focus and directrix as the given conic.]

3. Using the drawing of exercise 1, take a point T on the directrix XM and 0·5 inch from X, and from T draw all the possible tangents to each of the three conics, by the method given in Art. 13, p. 12; then determine the points of contact by Art. 37, (2).

4. Construct Figs. 76, 77, and 78, p. 33, as explained in Art. 35, to the following dimensions :—

For Fig. 76, vertical angle of cone $tvu = 50°$, $va = 2·5$ inches, $va_1 = 1·25$ inches.

For Fig. 77, vertical angle of cone $tva = 60°$, $va = 1$ inch, xx_1 parallel to vt.

For Fig. 78, vertical angle of cone $tvu = 80°$, $va = 1$ inch, $va_1 = 0·75$ inch.

The eccentricity of the conic being the ratio of AF to AX, construct each conic separately on another part of the paper by the method of Art. 34 and Fig. 75. Make a tracing of each conic obtained in this way and test whether it agrees with the conic determined as a section of the cone.

5. PTQ is a triangle, PT = 2·75 inches, TQ = 1·75 inches, and PQ = 2 inches. R is a point in PT 1 inch from P. RF is perpendicular to PT, and RF = 1·25 inches. F and Q are on the same side of PT.

PT is a tangent to a conic, P being the point of contact. F is the focus, and Q is another point on the curve. Find the directrix and draw the conic. [Art. 37, (1) and (2).]

6. PFNQ is a quadrilateral. The angles at F and N are right angles. PF = 1·3 inches, FN = NQ = 1·6 inches. P and Q are points on a conic of which F is the focus and FN the direction of the axis. Construct the conic.

7. F is the focus and S is any fixed point on the axis of a conic. From S a perpendicular is drawn to the tangent at a point P on the curve meeting FP at Q. Show by actual drawing that the locus of Q is a circle.

8. A focal chord of a parabola is 2·3 inches long and it is inclined at 30° to the directrix. The middle point of the chord is at a perpendicular distance of 1·2 inches from the directrix. Draw the parabola.

9. AP, a chord of a parabola, is 1·8 inches long and is inclined at 50° to the axis. The point A being the vertex of the parabola, draw the curve.

10. TP, a tangent to a parabola from a point T on the axis, is inclined at 30° to the axis. P is the point of contact, and TP is 3 inches long. Draw as much of the parabola as lies between the vertex and a double ordinate whose distance from the vertex is 2·2 inches.

11. Construct a triangle FPQ. FP = 1·5 inches, PQ = 3·4 inches, and QF = 2·6 inches. Draw a parabola whose focus is F and which passes through P and Q.

12. Make a careful tracing of the parabola of the preceding exercise without any lines other than the curve; then determine the axis, focus, and directrix of the curve by constructions on the tracing.

13. Draw a line FS 1·3 inches long and a line SP making the angle FSP 45°. SP is a tangent to a parabola of which F is the focus and S a point on the directrix. Find the point of contact of the tangent and draw the parabola.

14. PNP_1, a double ordinate of a parabola, is 3·6 inches long. A being the vertex of the parabola, the area bounded by the curve PAP_1 and the double ordinate PNP_1 is 4·8 square inches. Draw the curve PAP_1.

15. Construct a triangle RST; RS = 1·7 inches, ST = TR = 2·6 inches. ST and SR, both produced, are tangents to a parabola and TR is parallel to their chord of contact. TR contains the focus. Draw the parabola.

16. ABC is a triangle. AB = 2·2 inches, BC = 2·3 inches, and CA = 1·1 inches. D is a point in BC 0·9 inch from B. Draw the parabola which touches BC at D and AB and AC produced.

17. The normal PG to a parabola at a point P on the curve is 1·9 inches long, the point G being on the axis. The parameter of the diameter through P is 4·8 inches long. Construct the parabola.

18. The major and minor axes of an ellipse are 4 inches and 3 inches long respectively. Construct the curve by the trammel method and find the foci.

19. The major axis of an ellipse is 3·5 inches long and the distance between the foci is 2·5 inches. Draw the ellipse.

20. ABC is a triangle. AB = 2·3 inches, BC = 1·1 inches, and CA = 2·2 inches. Draw an ellipse whose foci are A and B and which passes through C.

21. The minor axis of an ellipse is 2·2 inches long and the distance between

the foci is 2 inches. Draw the ellipse, and construct the locus of the middle points of all the chords through one focus.

22. The major and minor axes of an ellipse are 3 inches and 2 inches long respectively. Draw a half of the ellipse which lies on one side of the major axis. Divide the curve into twelve parts whose chords are equal, and from the points of division draw normals to the ellipse, each normal to project 0·5 inch outside the ellipse. Lastly, draw a fair curve through the outer extremities of the normals.

23. The distance between the foci of an ellipse is 2 inches. A tangent to the ellipse is inclined to the major axis at an angle of 30° and cuts that axis produced at a point 2·5 inches from the centre of the ellipse. Draw the ellipse.

24. SABT is a quadrilateral. The angles at A and B are right angles. SA = 0·75 inch, AB = 3 inches, and BT = 2·25 inches. The points S and T are on the same side of AB. ST is a tangent to an ellipse of which AB is the major axis. Construct the ellipse.

25. Two conjugate diameters of an ellipse are 3 inches and 3·5 inches long, and the angle between them is 60°. Draw the ellipse by each of the three methods described in Art. 47, p. 42.

26. CO is a straight link, 2 inches long, which revolves about a fixed axis at C. PON is another straight link, 4 inches long, jointed at its middle point O to the outer end of CO. N is constrained to move in a straight line which passes through C. Draw the loci of the middle points of OP and ON and also the locus of P.

27. Draw a quadrilateral FPOQ. FP = 3·2 inches, angle PFQ = 70°, angle FPO = 56°, FQ = 1·2 inches, and PO = 1·6 inches. Draw an ellipse touching OP at P and OQ at Q, and having F for one focus.

28. The transverse axis of an hyperbola is 2 inches long and the distance between the foci is 2·7 inches. Draw the hyperbola.

29. The transverse and conjugate axes of an hyperbola are 2·2 inches and 1·7 inches long respectively. Draw the hyperbola and the conjugate hyperbola.

30. The foci of an hyperbola are 2·3 inches apart. One point on the curve is 2·9 inches from one focus and 1·1 inches from the other. Draw the hyperbola.

31. The distance between the foci of an hyperbola is 2·9 inches. A tangent to the hyperbola is inclined to the transverse axis at 48° and cuts that axis at a point 0·6 inch from its centre. Draw the hyperbola.

32. The transverse axis of an hyperbola is 2 inches long. A tangent to the hyperbola is inclined at 58° to the transverse axis and cuts that axis at a point 0·8 inch from its centre. Draw the hyperbola.

33. PMM_1P_1 is a quadrilateral. MM_1 = 2·5 inches, the angles at M and M_1 are right angles, PM = 0·9 inch, P_1M_1 = 1·2 inches. P and P_1 are points on an hyperbola of which MM_1 (produced both ways) is a directrix, and whose eccentricity is $\frac{4}{3}$. Find the foci and draw the hyperbola.

34. The asymptotes of an hyperbola are at right angles to one another and one point on the curve is 1 inch from each asymptote. Construct the two branches of the curve by the method illustrated by Fig. 108, p. 47.

35. One cubic foot of air at a pressure of 100 lbs. per square inch expands until its volume is 10 cubic feet. The relation between the pressure p and volume v is given by the formula $pv = 100$. Construct the expansion curve. Pressure scale, 1 inch to 20 lbs. per square inch; volume scale, 1 inch to 2 cubic feet. Draw the tangent and normal to the curve at the point where the volume is 3 cubic feet.

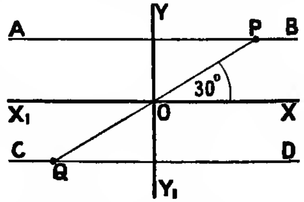


FIG. 119.

36. AB and CD (Fig. 119) are two straight lines of unlimited length. AB revolves with uniform angular velocity about the centre P, and CD revolves with the same angular velocity, but in the opposite direction, about the centre Q. PQ = 2·5 inches. O is the middle point of PQ. X_1OX and YOY_1 are two lines at right angles to one another, the angle POX being 30°. The initial positions

of AB and CD are parallel to X_1OX. Show by actual drawing that the locus of the point of intersection of AB and CD is a rectangular hyperbola of which X_1OX and YOY_1 are the asymptotes and P and Q points on the curve.

37. Draw the complete evolute of an ellipse whose major and minor axes are 4 inches and 2·75 inches long respectively.

38. The focal distance of the vertex of a parabola is 0·3 inch. Draw that part of the evolute of the parabola which lies between the vertex and a double ordinate whose distance from the vertex is 5 inches.

39. The angle between the asymptotes of an hyperbola is 60° and the vertex is at a distance of 1 inch from their intersection. Draw that part of the evolute of one branch of the hyperbola which lies between the vertex and a double ordinate whose distance from the vertex is 5 inches.

40. Draw an ellipse, major axis 3 inches, minor axis 2 inches. Take a point P within the ellipse 0·8 inch from the centre C and lying on a line through C inclined at 30° to the major axis. Through P draw a number of chords of the ellipse and at their extremities draw tangents to the ellipse. Find the point of intersection of each pair of tangents and see whether the points thus obtained are in one straight line. [By a pair of tangents is meant the tangents at the ends of a chord.] Repeat the construction for a point Q lying on CP produced. $CQ = 2$ inches.

CHAPTER IV

TRACING PAPER PROBLEMS

54. Use of Tracing Paper in Practical Geometry.—Frequently draughtsmen have to make geometrical constructions on complicated drawings in order to determine some point, line, or figure, and in such cases the fewer the construction lines the better. By using a piece of tracing paper in the manner explained in this chapter the desired result may be obtained very accurately in many cases without making any construction lines whatever on the drawing paper, and some problems can be easily solved by this method which would be impossible by ordinary geometrical methods, or which would otherwise involve very complicated constructions.

55. To find the Length of a Given Curved Line.—Let ABCD (Fig. 120) be the given curved line. On a piece of tracing paper TP draw a straight line 1 1. Mark a point A on this line and place the tracing paper on the drawing paper so that this point coincides with one end A of the curved line to be measured. Put a needle point through the tracing paper and into the drawing paper at A. Now turn the tracing paper round until the line 1 1 cuts the curve at a point B not far from A. Remove the needle point from A

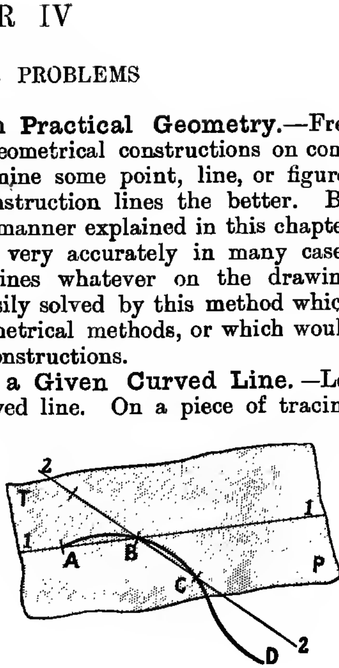


FIG. 120.

to B, taking care that the tracing paper does not change its position during the operation. Next turn the tracing paper round until the line on it takes up the position 2 2, cutting the curve at a point C not far from B. The needle point must then be moved to C and the operations continued until a point on the straight line coincides with the last point on the curve. The last point obtained on the straight line must be marked distinctly. The distance between the first and last points marked on the straight line will be approximately equal to the length of the curved line. The approximation will be closer the shorter the steps AB, BC, etc. When the curve has a larger radius of curvature the steps such as AB and BC may be longer than when the radius of curvature is smaller. In Fig. 120 the steps, for the

sake of clearness, are of greater length than would be adopted in practice.

The foregoing method is equivalent to stepping off the length of the curve with the dividers, but the tracing paper method has the advantage that the lengths of the different steps may be made to suit the variations of curvature when the curve is not an arc of a circle.

It is obvious that this method may also be used to mark off a portion of a given curved line which shall be of a given length.

56. To draw an Involute of a Given Curved Line.—Let ABCD (Fig. 121) be the given curved line. On a piece of tracing paper TP draw a straight line 1 1. Mark a point A on this line. This point will be called the tracing point. Place the tracing paper on the drawing paper so that the tracing point coincides with the point A on the curve from which the involute is to start. Put a needle point through the tracing paper and into the drawing paper at A. Now turn the tracing paper round until the straight line 1 1 cuts the curve at a point B not far from A. Remove the needle point from A to B, taking care that the tracing paper does not change its position during the operation. Next turn the tracing paper round until the straight line on it takes up the position 2 2, *touching* the curve at B, and with a sharp round-pointed pencil make a mark on the drawing paper through the needle hole at the tracing point. If these operations be continued, a number of points are obtained and a fair curve drawn through them will be an approximation to the involute required. The approximation will be closer the shorter the steps AB, BC, etc.

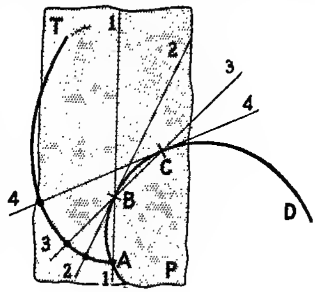


Fig. 121.

57. To draw a Straight Line to pass through a Given Point and cut two Given Lines so that the Portion intercepted between them shall have a Given Length.—Let AB and AC (Fig. 122) be the given lines and D the given point. On a piece of tracing paper TP draw a straight line EF, and mark two points H and K on this line such that HK is equal to the given length. Move the tracing paper into a number of different positions on the drawing paper, the point K being on the line AC and the line EF passing through D. A position will quickly be reached in which the point H is also on the line AB. Now make a mark on the drawing paper at F; a line joining this mark with D will be the line required.

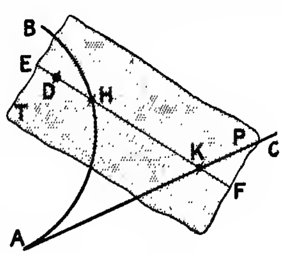


Fig. 122.

Instead of using tracing paper for this problem, the points H and

K may be marked on the straight edge of a strip of paper. This strip of paper may then be moved on the drawing paper until a position is found where H and K lie, one on AB and the other on AC. An ordinary drawing scale may be used in the same way.

58. To draw the Path traced by one Angular Point of a Given Triangle while the other Angular Points move, one on each of two Given Lines.—Let ABC (Fig. 123) be the given triangle. Let A be the tracing point, and let B move on the given line DE while C moves on the given line FH. Draw the triangle ABC on a piece of tracing paper TP. Make a small hole in the tracing paper at A with a needle. Place the tracing paper on the drawing paper so that B is on DE and C on FH. With a sharp round-pointed pencil make a mark on the drawing paper through the needle hole in the tracing paper at A. This will be one point in the path required. By moving the tracing paper into other positions other points may be obtained, and a fair curve KAL drawn through them will be the required path.

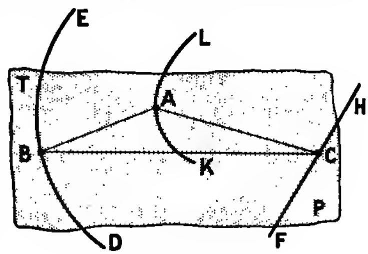


FIG. 123.

59. To draw an Arc of a Circle through Three Given Points without using the Centre of the Circle.—Let A, B, and C (Fig. 124) be the given points. Place a piece of tracing paper TP on the drawing paper, and draw on the former two straight lines AD and AE passing through B and C respectively. Make a small hole in the tracing paper at A with a needle. Move the tracing paper round into different positions so that the lines AD and AE always pass through B and C respectively. For each position of the tracing paper make a mark on the drawing paper with a sharp round-pointed pencil through the hole at A. A fair curve drawn through the points obtained in this way will be the arc required. This construction is based on the fact that all angles in the same segment of a circle are of the same magnitude. Another form of this problem is—To describe on a given line BC a segment of a circle which shall contain an angle equal to a given angle BAC.

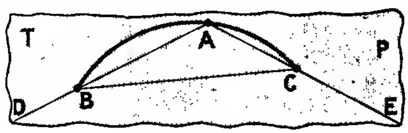


FIG. 124.

60. To draw any Roulette.—Let AHB (Fig. 125) be the directing line or base, and let CDE be the rolling curve. (In Fig. 125 the base is a straight line, but it may be any curved line; and the rolling curve is a circle, but it may also be any curved line.) The base is drawn on the drawing paper and the rolling curve is drawn on a piece of tracing paper. The tracing point P is also marked on the

tracing paper by two lines at right angles, and by a small needle hole. Place the tracing paper on the drawing paper so that the rolling curve CDE touches the base, say at C. Make a mark on the drawing paper through the needle hole in the tracing paper at P. Place a needle through the tracing paper and into the drawing paper at C. Turn the tracing paper round about the needle at C until the rolling curve cuts the base at a near point F.

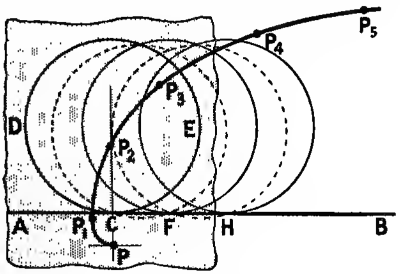


FIG. 125.

Transfer the needle from C to F and turn the tracing paper until the rolling curve touches the base at F. The tracing point will now have moved from P to P_1. Mark the drawing paper at P_1. Again turn the tracing paper until the rolling curve cuts the base at another near point H. Transfer the needle from F to H and turn the tracing paper until the rolling curve touches the base at H. The tracing point will now have moved to P_2. Mark the drawing paper at P_2, and continue the process, obtaining the points P_3, P_4, etc. A fair curve drawn through the points P, P_1, P_2, P_3, etc., will be the roulette required. In Fig. 125 the roulette is a trochoid.

61. To inscribe in a Given Figure a Figure similar to another Given Figure.—Let ABCD (Fig. 126) be the given figure in which it is required to inscribe a figure similar to a second given figure EFHK. Draw the figure EFHK on a piece of tracing paper. Take a point on the tracing paper as a pole (preferably one angular point of the figure EFHK, say E) and join this point to each of the angular points of the figure on the tracing paper. Graduate these lines, the divisions being proportional to the distances of the pole from the angular points of the figure. In the example illustrated EF,

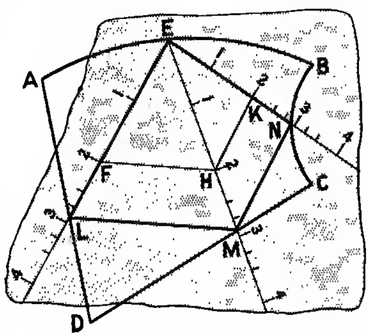


FIG. 126.

EH, and EK are each divided into two equal parts, and the graduations are extended on these lines produced. Place the tracing paper on the drawing paper and move the former about until the sides of the figure ABCD on the drawing paper cut the lines through the pole on the tracing paper at corresponding points L, M, and N; then prick the drawing paper through the tracing paper at these corresponding points and the positions of the angular points of the figure required are obtained.

It may happen that the corresponding points mentioned are not at points of graduation on the lines through the pole. In that case the graduations must be made finer in the neighbourhoods where they appear to be required. If the corresponding points required are still not at points of graduation, it may be possible, without further subdivision, to judge by the eye whether the points where the polar lines cut the sides of the first given figure are corresponding points.

62. Drawing Symmetrical Curves.—When a curve is symmetrical about an axis only the part on one side of that axis need be constructed, the part on the other side may be quickly and accurately drawn by means of a piece of tracing paper as follows.

Referring to Fig. 127, at (*a*) is shown one half of a curve which is symmetrical about the axis YY. A piece of tracing paper TP is

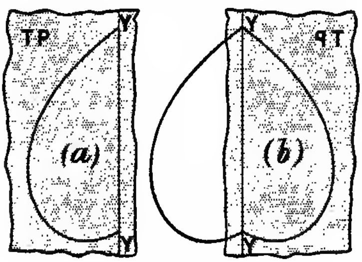


Fig. 127.

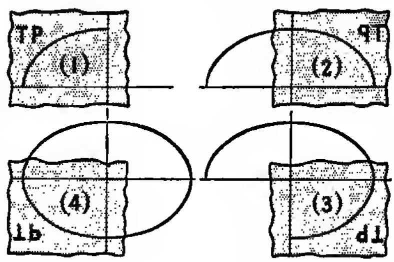


Fig. 128.

placed over this figure and the curve and the axis YY are traced *in pencil.* The tracing paper is then turned over and placed as shown at (*b*) and the curve on the tracing paper is traced with the pencil. It will then be found that the second half of the curve has appeared on the drawing paper in pencil, the lead for this curve having come from the lead which was put on the tracing paper when it was in the position (*a*). After removing the tracing paper the curve traced should be lined in to make it more distinct.

When a curve is symmetrical about two axes at right angles to one another only one quarter of the curve need be constructed, the other three quarters may be drawn, one quarter at a time, by means of a piece of tracing paper as just described.

Fig. 128 shows the method applied to an ellipse. At (1) is shown one quarter of the ellipse constructed, say, by the trammel method. A piece of tracing paper TP is placed over this and on it are traced the semi-axes and the curve lying between them. Turning the tracing paper over and placing it as shown at (2) the second quarter of the curve is transferred to the drawing paper, the lead coming from the first tracing. Turning the tracing paper over again and placing it as shown at (3) the third quarter of the curve is transferred to the drawing paper, the lead coming from the second tracing. Turning the

tracing paper over once more and placing it as shown at (4) the last quarter of the curve is obtained.

Exercises IV

The following exercises are intended to be worked out by the tracing paper method

1. Find the circumference of a circle 3 inches in diameter, and compare the result with that got by calculation.

2. The sides of a figure ABC are arcs of circles. Radius of AB = $1\frac{1}{2}$ inches, radius of BC = 2 inches, radius of CA = $2\frac{1}{4}$ inches. The arcs touch one another in pairs at A, B, and C. Find the perimeter of the figure ABC.

3. Draw the involute of a circle 2 inches in diameter. Also draw a tangent to the circle to meet the involute, the length of this tangent between the involute and the point of contact with the circle to be 5 inches. Find also the length of the involute between the starting point and the point where the forementioned tangent meets it.

4. Draw an ellipse, major axis 2 inches, minor axis $1\frac{1}{2}$ inches, and draw that involute of the ellipse which starts from one extremity of the major axis. Find the circumference of the ellipse.

5. ABC is an equilateral triangle of 3 inches side. D is a point in AB 1 inch from A. Draw through D a straight line to cut the side AC at E, and the side CB produced at F, such that EF = $4\frac{1}{2}$ inches.

6. Draw an ellipse, major axis 3 inches, minor axis 2 inches. Take a point P, 2 inches from one extremity of the minor axis, and $1\frac{1}{2}$ inches from one extremity of the major axis. Through P draw a straight line to cut the ellipse in two points which shall be 2 inches apart.

7. X'OX and Y'OY are two straight lines intersecting at O. Angle XOY = 60°. A straight line AB, 3 inches long moves with the end A on X'OX, and the end B on Y'OY. Draw the complete curve traced by a point P in AB which is $1\frac{1}{4}$ inches from A. Draw also the curve traced by the middle point of AB.

8. ABC is an equilateral triangle of 3 inches side. The triangle moves with the point A on the circumference of a circle $2\frac{1}{2}$ inches in diameter, and the point B moves on a diameter of that circle produced. Draw the path traced by the point C. Draw also the path traced by the middle point of AB.

9. On a straight line 3 inches long describe a segment of a circle containing an angle of 120°, without using the centre of the circle.

10. O is the centre and OP a radius of a circle 2 inches in diameter. Q is a point in OP $\frac{3}{4}$ inch from O, and R is a point in OP produced, and $1\frac{1}{2}$ inches from O. Draw the cycloid described by P, also the trochoids described by Q and R, as the circle rolls on a straight line. Draw as much of each curve as is obtained by a little more than one revolution of the circle.

11. Taking the same rolling circle and the same points Q and R as in the preceding exercise, draw the epicycloid described by P, and the epitrochoids described by Q and R as the circle rolls on the outside of a base circle 3 inches in diameter; draw also the hypocycloid described by P, and the hypotrochoids described by Q and R as the circle rolls on the inside of the same base circle. Draw as much of each curve as is obtained by a little more than one revolution of the rolling circle.

12. Draw the roulette described by one extremity of the major axis of an ellipse (major axis, $2\frac{1}{4}$ inches, minor axis, $1\frac{3}{4}$ inches) which rolls on the outside of another ellipse (major axis, 3 inches, minor axis, 2 inches). The roulette to start from one extremity of the minor axis of the fixed ellipse. Draw also, on the same figure, the roulette described by one extremity of the minor axis of the rolling ellipse while the first roulette is being described.

13. AB is a straight line 3 inches long. BC is an arc of a circle whose centre

is in AB and $2\frac{1}{2}$ inches from B, and whose chord is 4 inches long. AC is an arc of a oircle of 3 inches radius whose centre is on that side of AC which is remote from B. Draw the largest possible equilateral triangle which has its angular points, one on each of the sides of the figure ABC.

14. ABCD is a quadrilateral. AB = $2\frac{1}{4}$ inohes, BC = $1\frac{3}{4}$ inches, CD = $1\frac{1}{2}$ inches, DA = 2 inches, and AC = $2\frac{3}{4}$ inches. In this quadrilateral inscribe a square.

CHAPTER V

APPROXIMATE SOLUTIONS TO SOME UNSOLVED PROBLEMS

63. Rectification of Circular Arcs.—The ratio of the circumference of a circle to its diameter cannot be expressed exactly, in other words the two are incommensurable. The symbol π is always used to denote the ratio of the circumference to the diameter and its approximate value is 3·1416 or nearly $3\frac{1}{7}$.

The best geometrical constructions hitherto given for finding approximately the length of a circular arc, or for marking off an arc of given length, are those due to Rankine,[1] and are as follows :—

(*a*) *To draw a straight line approximately equal to a given circular arc AB* (Fig. 129). Join BA and produce it to D making AD

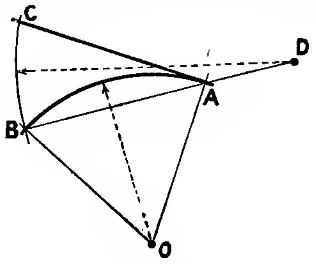


FIG. 129.

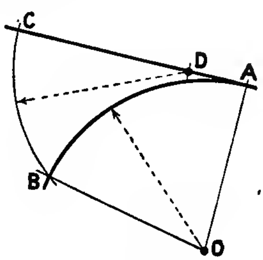


FIG. 130.

$= \frac{1}{2}$AB. With centre D and radius DB describe the arc BC cutting at C the tangent AC to the arc at A. AC is the straight line required.

The *error* varies as the fourth power of the angle AOB, where O is the centre of the circle of which AB is an arc. When the angle AOB is 30°, AC is less than the arc AB by about $\frac{1}{14400}$ of the length of the arc.

(*b*) *To mark off on a given circle an arc AB approximately equal to a given length* (Fig. 130). Draw a tangent AC to the circle at A, and make AC equal to the given length. Make AD $= \frac{1}{4}$AC. With centre D and radius DC describe the arc CB to cut the circle at B. AB is the arc required.

[1] *A Manual of Machinery and Millwork.*

The *error* in (*b*) as a fraction of the given length is the same as in (*a*), and follows the same law.

If in Fig. 129 the angle AOB is greater than one right angle and less than two right angles the length of one half of the arc AB should be determined by the construction and the result doubled. If the angle AOB is greater than two right angles the length of one quarter of the arc AB should be determined by the construction and the result quadrupled.

If in Fig. 130, AC is greater than one and a half times the radius OA the arc equal to one half of AC should be determined by the construction and this arc should then be doubled. If AC is greater than three times OA the arc equal to one quarter of AC should be determined by the construction and this arc should then be quadrupled.

The construction in (*b*) follows easily from that in (*a*), for if the construction in Fig. 129 be performed and CD be joined (Fig. 131), and the angle CDB be bisected by DE meeting AC at E, a circle with

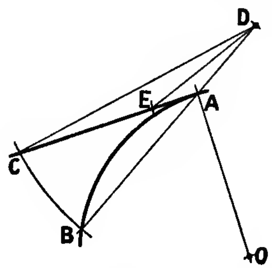


Fig. 131.

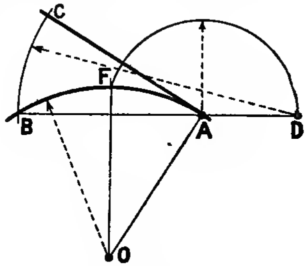


Fig. 132.

centre E and radius EC will pass through B. Since CD = DB = 3AD, and since DE bisects the angle CDB, it follows (Euclid VI, 3) that CE = 3AE or AE = $\frac{1}{4}$AC.

The following slight modification of Rankine's first construction (*a*) gives a more approximate result, and is to be preferred, especially when the angle AOB is greater than 60°. Instead of making AD equal to half the chord AB make it equal to the chord AF of half the arc AB (Fig. 132) and proceed as before.

For the case where the angle AOB is 90° the error in Rankine's construction is about 1 in 170 while in the modified construction the error is only about 1 in 2300.

64. To draw a Straight Line whose Length shall be approximately equal to the Circumference of a Given Circle.—Draw a straight line whose length shall be approximately equal to a quarter of the circumference by the modification of Rankine's construction explained in the preceding article; a line four times this in length will be the line required.

The following construction given by the late Mr. T. H. Eagles[1]

[1] *Constructive Geometry of Plane Curves*, p. 267.

gives a very close approximation. O (Fig. 133) is the centre and AOB a diameter of the given circle. Draw the tangent AC and make AC equal to three times AB. Draw a radius OD making the

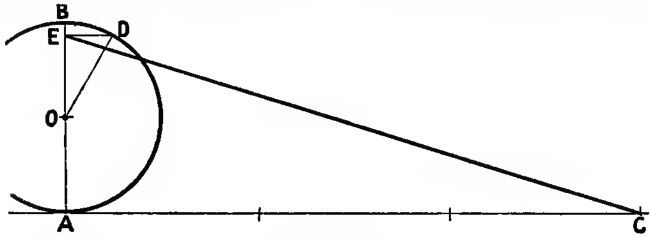


Fig. 133.

angle BOD = 30°. Draw DE at right angles to AB meeting the latter at E. Join EC. The length of EC is very nearly equal to the circumference of the circle. EC is a little longer than the true circumference, the error being about 1 in 21,700.

65. To draw a Straight Line whose Length shall represent approximately the Value of π.—The circumference of a circle of radius r is $2\pi r$, and therefore a quarter of the circumference is $\frac{\pi r}{2}$. If $r = 2$, then a quarter of the circumference is equal to π. Hence if a quarter of a circle be drawn with radius = 2, the length of the arc will be equal to π, and this length may be determined by one of the constructions of Art. 63. Instead of taking a quarter of a circle with a radius = 2, a sector whose angle is 60° and radius = 3 may be used, or generally a sector whose angle is $n°$, and radius $= \frac{180}{n}$ may be taken, and the length of its arc will be equal to π.

The circumference of a circle whose radius is 0·5 is equal to π and π may therefore be found by the construction given in the latter part of Art. 64, by making r equal to 0·5.

66. To find the Side of a Square whose Area shall be approximately equal to that of a Given Circle.—Solving this problem is known as "squaring the circle." O is the centre and AB a diameter of the given circle.

First Method (Fig. 134). Draw AC at right angles to AB and

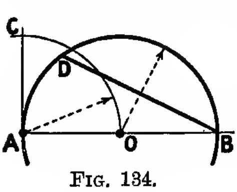


Fig. 134.

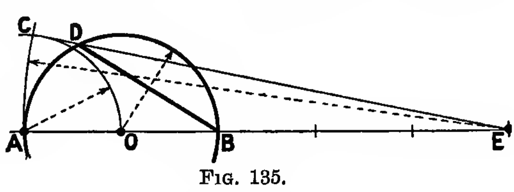


Fig. 135.

equal to AO. Draw BC cutting the circle at D. Join BD. BD is the line required. The error in this construction is that BD is too

long by an amount equal to $0{\cdot}0164r$, where r is the radius of the circle.

Second Method (Fig. 135). Produce AB to E, and make BE equal to three times BO. With centre A and radius AO describe the arc OC. With centre E and radius EA describe the arc AC to cut the former arc at C. Draw CE cutting the circle at D. Join BD. BD is the line required. The error in this construction is that BD is too short by an amount equal to $0{\cdot}0007r$, where r is the radius of the circle.

67. To draw a Straight Line whose Length shall represent approximately the Value of the Square Root of π.—The area of a circle whose radius is r is πr^2, and if s is the side of a square whose area is equal to that of the circle, then $s^2 = \pi r^2$, or $s = r\sqrt{\pi}$. If $r = 1$, then $s = \sqrt{\pi}$, and this may be found by one of the constructions in the preceding article.

68. To find the Side of a Square whose Area shall be approximately equal to that of a Given Ellipse.—If a and b are the semi-axes of an ellipse its area is πab. If r is the radius of a circle equal in area to the ellipse, then $\pi r^2 = \pi ab$, or $r^2 = ab$, and r is a mean proportional between a and b and may be found as in Art. 12, p. 11. The side of a square whose area is approximately equal to that of the circle may then be found by the construction of Art. 66.

69. To inscribe in a Given Circle a Regular Polygon having a Given Number of Sides.—AB (Fig. 136) is a diameter and O the centre of the given circle. With centre A and radius AB describe the arc BC. With centre B and radius BA describe the arc AC cutting the former arc at C. Divide the diameter AB into as many equal parts as there are sides in the polygon. D is the *second* point of division from A. Draw CD and produce it to cut the circle at E. The chord AE is one side of the polygon, and the others are obtained by stepping the chord AE round the circle.

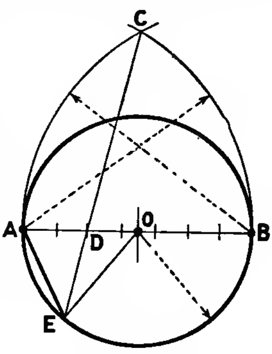


FIG. 136.

The above construction is exact for an equilateral triangle, a square, and a hexagon. For a pentagon the central angle AOE is too small, and when the chord AE is stepped round from A five times, the last point will fall short of A by an amount which subtends at O an angle of nearly a quarter of a degree. For a heptagon the central angle AOE is too large, and when the chord AE is stepped round from A seven times, the last point will be beyond A by an amount which subtends at O an angle of a little more than five-eighths of a degree.

Exercises V

1. Draw an arc of a circle of 2 inches radius subtending an angle of 60° at the centre of the circle; then draw by Rankine's construction, a straight line equal in length to the arc.

2. From the circumference of a circle of 2 inches radius cut off, by means of Rankine's construction, an arc equal in length to the radius.

3. Find, by construction, and by calculation, the circular measure of an angle of 45°.

4. Construct an angle whose circular measure is 1·2.

5. Find, by construction, and by calculation, the circumference of a circle whose diameter is 2·75 inches.

6. Find, by construction, and by calculation, the diameter of a circle whose circumference is 6 inches.

7. Find, in the simplest possible way, the diameter of a circle whose circumference is equal to the sum of the circumferences of two circles one of which is 1·75 inches, and the other 1·25 inches in diameter.

8. Draw a square whose area shall be equal to that of a circle whose radius is 1·5 inches.

9. Draw a circle having an area equal to that of a square of 2·25 inches side.

10. Construct a square having an area equal to that of an ellipse whose major and minor axes are 3·5 inches and 2·5 inches respectively.

11. In a circle of 2 inches radius inscribe a regular heptagon.

CHAPTER VI

ROULETTES AND GLISSETTES

70. Roulettes.—When one curve rolls without sliding on another curve, any point connected with the first curve describes on the plane of the second a curve called a *roulette*. The curve which rolls is called the *rolling curve* or *generating curve*, and the curve on which it rolls is called the *directing curve* or *base*. The directing curve is generally assumed to be fixed, and is sometimes called the *fixed curve*.

When the rolling and directing curves are circles the roulette becomes a *cycloidal curve*.

71. General Construction for Drawing a Roulette.—The best practical method of drawing any roulette is the tracing paper or transparent templet method which is described in Art. 60, p. 56.

The following is a general construction which may be used for drawing any roulette. ABCD (Fig. 137) is the base, A*bcd* is the rolling curve, and P is the tracing point. Take a number of points *b*, *c*, *d*, etc. on the rolling curve and determine points B, C, D, etc. on the base such that the arcs AB, BC, CD, etc. are equal to the arcs A*b*, *bc*, *cd*, etc. respectively. If the points are sufficiently near to one another the arcs may be assumed equal to their chords. Draw tangents to the base at B, C, D, etc. and tangents to the rolling curve at *b*, *c*, *d*, etc. From P draw P*m* perpendicular to the tangent at *b*. On the tangent at B make BM = *bm*, and draw MP_1 perpendicular to BM and equal to P*m*. P_1 will be the position of P when the rolling curve touches the base at B, and will therefore be a point in the roulette described by P. In like manner other points may be determined.

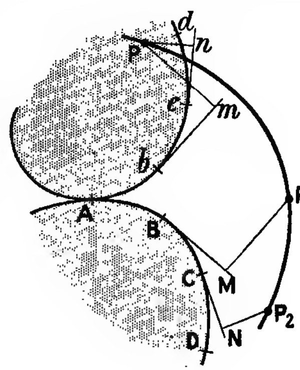


FIG. 137.

72. General Construction for the Centre of Curvature of a Roulette.—Three cases are illustrated in Figs. 138, 139, and 140. The description which follows applies to each.

O_1 is the centre of curvature of the base line AQB at Q. (If AQB

is an arc of a circle then O_1 is the centre of the circle, and if AQB is a straight line (Fig. 140), then O_1 is at an infinite distance from Q in a straight line through Q at right angles to AQB.)

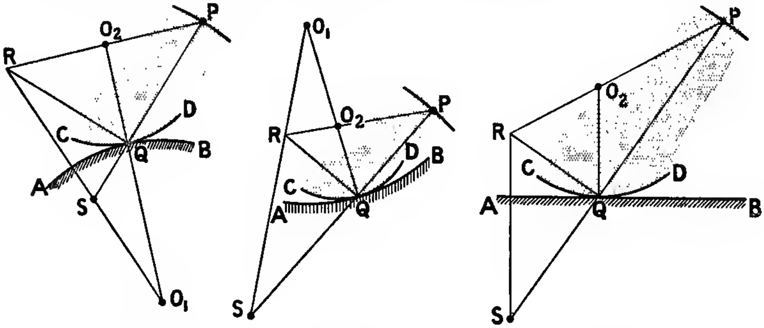


FIG. 138. FIG. 139. FIG. 140.

O_2 is the centre of curvature of the rolling curve CQD at Q. (If CQD is an arc of a circle, then O_2 is the centre of the circle.)

P is the position of the tracing point when the rolling curve is in the position shown.

Join PQ. Draw QR at right angles to PQ to meet PO_2 or PO_2 produced at R. Join RO_1 and produce it if necessary to meet PQ or PQ produced at S. S is the centre of curvature of the roulette at P.

When P, O_2, and Q are in a straight line the above construction fails. In this case, $SQ = \dfrac{\overline{QO_1} \times \overline{QO_2} \times \overline{PQ}}{\overline{QO_1} \times \overline{PO_2} \pm \overline{QO_2} \times \overline{PQ}}$, the plus sign being taken when O_1 and O_2 are on opposite sides of Q, and the minus sign being taken when O_1 and O_2 are on the same side of Q. $PQ = PO_2 \pm QO_2$.

If AB is a straight line then QO_1 is infinite and

$$SQ = \frac{\overline{QO_2} \times \overline{PQ}}{\overline{PO_2}}.$$

$$\text{If } PO_2 = QO_2, \text{ then } SQ = \frac{\overline{QO_1} \times \overline{PQ}}{\overline{QO_1} \pm \overline{PQ}}.$$

73. The Cycloid.—The cycloid is the curve described by a point on the circumference of a circle which rolls on a straight line, the circle and straight line remaining in the same plane. The ordinary geometrical construction for drawing the cycloid is shown in Fig. 141. $mo'e'$ is the straight line upon which the circle $Pbo'q$ rolls and P is the point which describes the cycloid. $Pbo'q$ is the rolling circle in its middle position, the diameter POo' being at right angles to $mo'e'$. Make $o'e'$ equal to half the circumference of the rolling circle. Divide $o'e'$ into a number of equal parts at a', b', c', etc. (preferably six, but for

the sake of a clearer figure $o'e'$ in Fig. 141 has been divided into five equal parts). Divide the semicircle Pbo' into the same number of equal parts at a, b, c, etc. Through O draw OE parallel to $o'e'$. From a', b', c', etc. draw perpendiculars to $o'e'$ to meet OE at A′, B′, C′, etc. With centres A′, B′, C′, etc. describe arcs of circles touching $o'e'$, and through a, b, c, etc. draw parallels to $o'e'$ to cut these arcs at A, B, C, etc. as shown. The points A, B, C, etc. are points on the half of the cycloid traced by the point P as the circle Pbo' makes half a revolution to the right. Points on the other half of the cycloid may be obtained in a similar manner, or, since the curve is symmetrical about Po' the part to the left of Po' may be copied from the part to the right.

If from a point q on the circle P$bo'q$ a line be drawn parallel to the base line mo' to meet the cycloid at Q then QR parallel to qo' is the normal and QT parallel to qP is the tangent to the cycloid at Q.

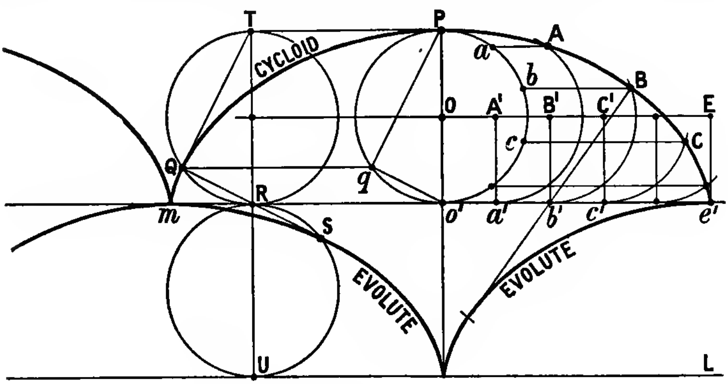


FIG. 141.

Again, if PT be drawn parallel to mo' to meet at T the tangent to the cycloid at Q, the length of the arc PQ of the cycloid will be twice the length of the tangent QT or twice the length of the chord qP. Hence the total length of the cycloid is four times the diameter of the rolling circle.

If QR be produced to S and RS is made equal to QR then S is the centre of curvature of the cycloid at Q. The locus of S is the evolute of the cycloid. If the rolling circle be drawn below the base line mo' and touching that line at R it is obvious that this circle will pass through S. Also if the rolling circle be drawn above the base line and touching it at R this circle will pass through Q and since the chord RS is equal to the chord QR the arc RS must be equal to the arc QR. But the arc QR is equal to mR, and mo' is equal to half the circumference of the rolling circle, therefore if RU is a diameter of the circle RSU, the arc SU is equal to Ro'. Hence if a line UL be drawn parallel to mo' and the circle RSU be made to roll on this line the

point S will describe a cycloid equal to the original cycloid. The evolute of a cycloid is therefore an equal cycloid.

74. The Trochoid.—When a circle rolls on a straight line and remains in the same plane, a point in the plane of the circle, connected to the circle but not on its circumference, describes a *trochoid.* If the describing point is outside the rolling circle the trochoid is called a *superior trochoid.* If the describing point is inside the rolling circle the trochoid is called an *inferior trochoid.* A superior trochoid is also called a *curtate cycloid* and an inferior trochoid is also called a *prolate cycloid.*

The geometrical construction for finding points on a trochoid is similar to that already given for the cycloid and is shown in Fig. 142. *mo'm'* is the base line. P is the position of the describing point when it is furthest from the base line and O is the corresponding position of

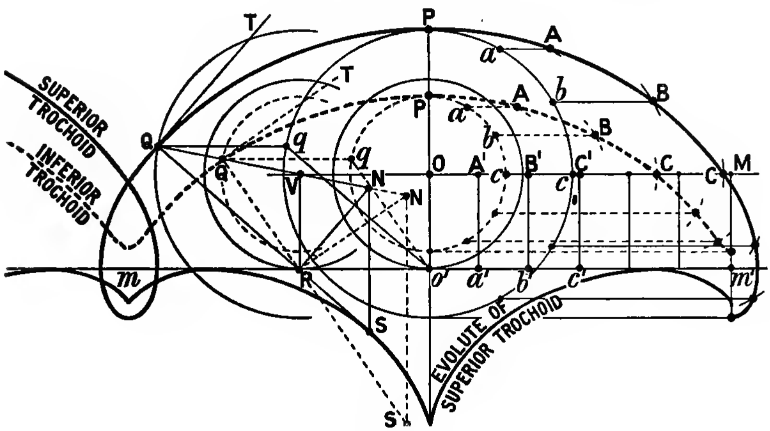


FIG. 142.

the centre of the rolling circle. *o'm'* is made equal to half the circumference of the rolling circle. The points *a'*, *b'*, *c'*, etc. and A', B', C', etc. are determined as for the cycloid.

With centre O and radius OP describe a circle, and divide the half of this circle which is to the right of OP into as many equal parts as *o'm'* is divided into. This determines the points *a*, *b*, *c*, etc. Through *a*, *b*, *c*, etc. draw parallels to the base line and with centres A', B', C', etc. and radius equal to OP describe arcs of circles to cut these parallels at A, B, C, etc. as shown. A, B, C, etc. are points on the half of the trochoid described by the point P as the rolling circle makes half a revolution to the right. Points on the other half of the trochoid may be obtained in a similar manner, or, since the curve is symmetrical about OP, the part to the left of OP may be copied from the part to the right.

If from a point *q* on the circle *q*P*b* a line be drawn parallel to the base line to meet the trochoid at Q, then QR parallel to *qo'* is the

normal and QT perpendicular to QR is the tangent to the trochoid at Q.

Draw RV perpendicular to the base line to meet the parallel through O to the base line at V. V is the position of the centre of the rolling circle when the describing point is at Q.

To find S the centre of curvature of the trochoid at Q, draw RN perpendicular to QR to meet QV produced at N and draw NS perpendicular to the base line to meet QR produced at S.

The evolute of the trochoid is the locus of the centre of curvature S. The evolute of the superior trochoid is shown in Fig. 142. The evolute of the inferior trochoid is partly on one side and partly on the other side of the curve, and the normals which have the least inclination to the base line are asymptotes of the evolute.

75. The Epicycloid.—When a circle rolls on the outside of a fixed circle, the two circles being in the same plane, a point on the circumference of the rolling circle describes an *epicycloid.*

The ordinary geometrical construction for finding points on an epicycloid is shown in Fig. 143. P is the position of the tracing point

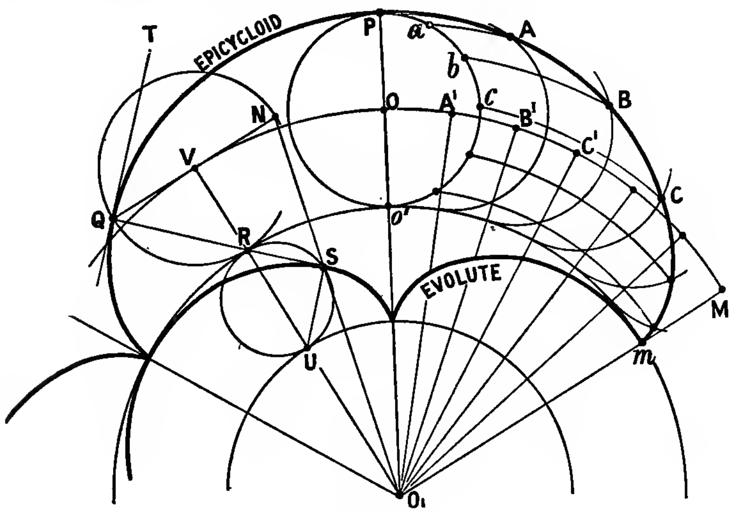


FIG. 143.

when it is furthest from the fixed or base circle, O is the corresponding position of the centre of the rolling circle and o' is the corresponding point of contact of the rolling and base circles. O_1 is the centre of the base circle. Make the arc $o'm$ equal to half the circumference of the rolling circle. Join O_1m and produce it to meet at M the circle through O concentric with the base circle. Divide the arc OM into a number of equal parts (preferably six or eight) at A′, B′, C′, etc. and divide the semicircle Pco' into the same number of equal parts at a, b, c, etc.

With centre O_1, describe arcs of circles through a, b, c, etc. and with centres A′, B′, C′, etc. and radius equal to OP describe arcs of circles to cut the former arcs at A, B, C, etc. as shown. A, B, C, etc. are points on the half of the epicycloid described by P as the rolling circle rolls from o' to m. Points on the other half of the epicycloid may be obtained in a similar manner, or, since the curve is symmetrical about O_1P, the part to the left of O_1P may be copied from the part to the right.

The centre of curvature of the epicycloid at any point Q is found as follows. Draw the rolling circle in the position which it occupies when the tracing point is at Q as shown. V is the centre of the rolling circle in this position and R is its point of contact with the base circle. Join QV and produce it to meet the rolling circle again at N. Join O_1N. Join QR and produce it to meet O_1N at S. QR is the normal and S is the centre of curvature of the epicycloid at Q. QT at right angles to QR is the tangent to the epicycloid at Q.

The locus of S is the evolute of the epicycloid.

Join O_1R and draw SU perpendicular to QS to meet O_1R at U. On UR as diameter describe the circle RSU, and with centre O_1 and radius O_1U describe another circle. If the circle RSU be made to roll on the outside of the circle whose centre is O_1 and radius O_1U, the point S will describe an epicycloid which will coincide with the evolute of the original epicycloid. This epicycloid which is the evolute of the original epicycloid is similar to it. Two epicycloids are similar when the ratio of the radii of their rolling circles to one another is the same as the ratio of the radii of their base circles.

It may be pointed out here that a given epicycloid on a given base circle may be described by a point on the circumference of either of two different rolling circles. Referring to Fig. 144, APB is an

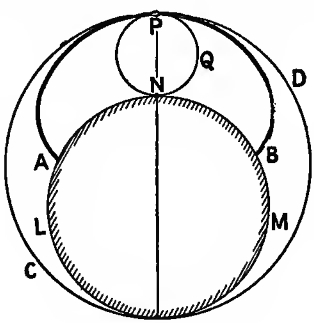


FIG. 144.

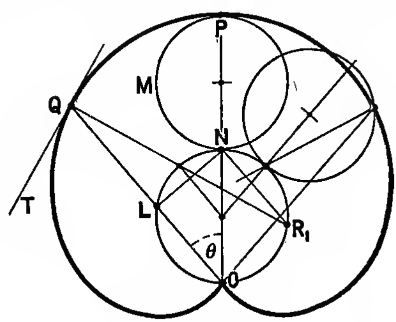


FIG. 145.

epicycloid on the base circle LMN. This epicycloid may be described by a point on the circumference of the rolling circle PQN or by a point on the circumference of the rolling circle OPD. The three circles are such that the diameter of the base circle is equal to the difference between the diameters of the two rolling circles. It will

be observed that while both rolling circles roll on the outside of the base circle the base circle is inside the larger rolling circle.

An interesting case of the epicycloid is that in which the rolling circle is equal to the base circle. This is shown in Fig. 145, where OLN is the base circle and OQP the epicycloid. PMN is the position of the rolling circle when the tracing point P is furthest from the base circle. The two ends of the epicycloid meet at O forming a cusp at that point.

If a straight line OLQ be drawn cutting the base circle at L and the epicycloid at Q then it is not difficult to prove that LQ is equal to ON the diameter of the base circle. This property suggests a simple method of finding points on this particular form of epicycloid. This form of the epicycloid is known as the *cardioid*. If the angle which OQ makes with OP be denoted by θ then $OQ = QL + LO = QL + ON \cos\theta$ or $r = d\,(1 + \cos\theta)$ where $r = OQ$ and d is the diameter of the circle OLN. This is the polar equation to the cardioid.

The normal to the curve at Q may be found by the construction already given or more simply as follows. Through N draw NR_1 parallel to OQ to meet the circle again at R_1. QR_1 is the required normal. The tangent QT at Q is of course perpendicular to QR_1.

76. The Epitrochoid.—When a circle rolls on the outside of a fixed circle, the two circles being in the same plane, a point in the plane of the rolling circle, connected to it but not on its circumference describes an *epitrochoid*. The epitrochoid is called a *superior epitrochoid* or an *inferior epitrochoid* according as the describing point is outside or inside the rolling circle.

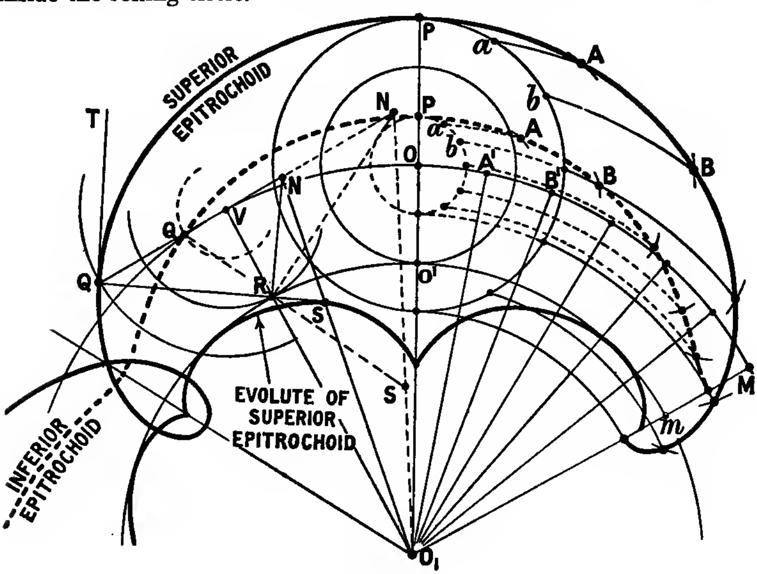


FIG. 146.

The geometrical construction for finding points on an epitrochoid is shown in Fig. 146. The construction is so like that already used for the cycloid, the trochoid, and the epicycloid that no detailed description of it need be given here.

The centre of curvature of the epitrochoid at any point Q is found as follows. Draw the rolling circle in the position which it occupies when the tracing point is at Q as shown. V is the centre of the rolling circle in this position and R is its point of contact with the base circle. Join QV and produce it. Join QR and produce it. Draw RN at right angles to QR to meet QV produced at N. Join O_1N cutting QR produced at S. QR is the normal and S is the centre of curvature of the epitrochoid at Q.

77. The Hypocycloid.—When a circle rolls on the inside of a fixed circle, the two circles being in the same plane, a point on the circumference of the rolling circle describes a *hypocycloid.*

The geometrical construction for finding points on a hypocycloid is shown in Fig. 147. After what has been done in preceding articles no detailed description of this construction is necessary.

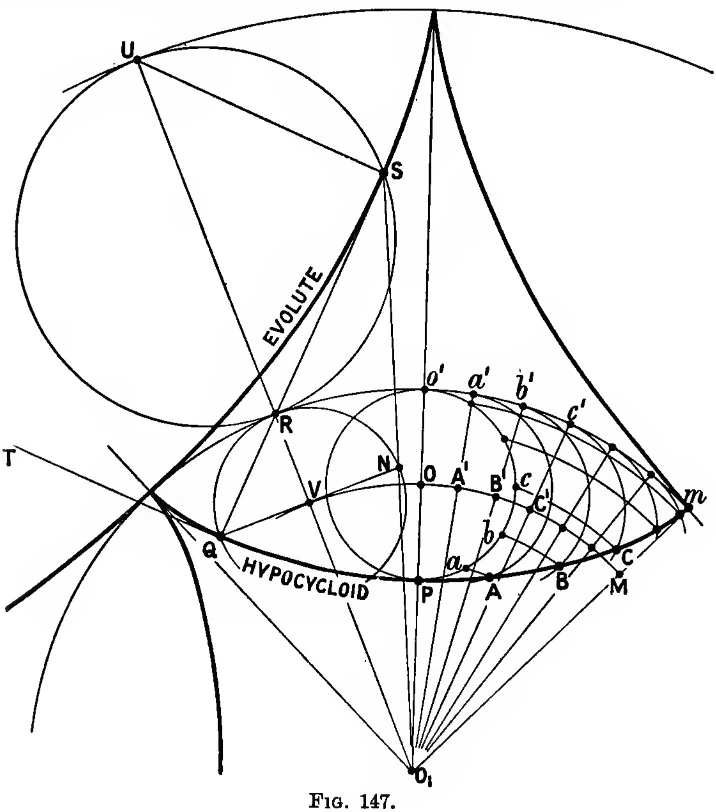


FIG. 147.

The construction for finding S, the centre of curvature of the hypocycloid at any point Q, is the same as that already given for the epicycloid except that S is on O_1N produced instead of on O_1N.

Join O_1R and draw SU perpendicular to QS to meet O_1R produced at U. On UR as diameter describe the circle RSU, and with centre O_1 and radius O_1U describe another circle. If the circle RSU be made to roll on the inside of the circle whose centre is O_1 and radius O_1U, the point S will describe a hypocycloid. This hypocycloid which is the evolute of the original hypocycloid is similar to it.

A given hypocycloid on a given base circle may be described by a point on the circumference of either of two different rolling circles. Referring to Fig. 148, ACBD is a base circle of which COD is a diameter. The hypocycloid APB may be described by a point on the circumference of the circle CP or by a point on the circumference of the circle DP. The three circles are such that the diameter of the base circle is equal to the sum of the diameters of the two rolling circles.

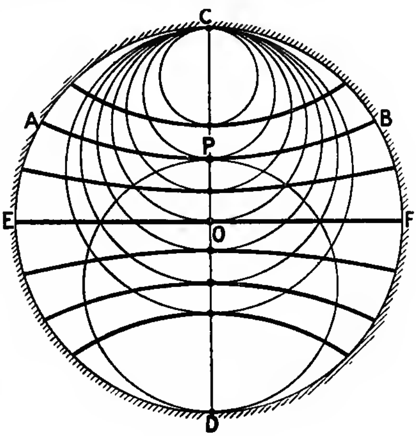


Fig. 148.

Fig. 148 also shows the variation in the form and size of the hypocycloid as the diameter of the rolling circle is altered.

A very important case of the hypocycloid is that in which the diameter of the rolling circle is equal to the radius of the base circle. In this case the hypocycloid becomes a straight line EOF (Fig. 148) which is a diameter of the base circle.

78. The Hypotrochoid.—When a circle rolls on the inside of a fixed circle, the two circles being in the same plane, a point in the plane of the rolling circle, connected to it but not on its circumference describes a *hypotrochoid*. The hypotrochoid is called a *superior hypotrochoid* or an *inferior hypotrochoid* according as the describing point is outside or inside the rolling circle.

The geometrical construction for finding points on a hypotrochoid is shown in Fig. 149.

The construction for finding the centre of curvature at any point is similar to that for the epitrochoid. The point S corresponding to the point Q on the inferior hypotrochoid (Fig. 149) falls outside the lower limit of the figure.

A special and important case of the hypotrochoid is that in which the diameter of the rolling circle is equal to the radius of the base circle. In this case the hypotrochoid is an ellipse. Referring to Fig. 150, AMLN is the base circle whose centre is O, and AHO is the initial position of the rolling circle. C is the centre of the circle

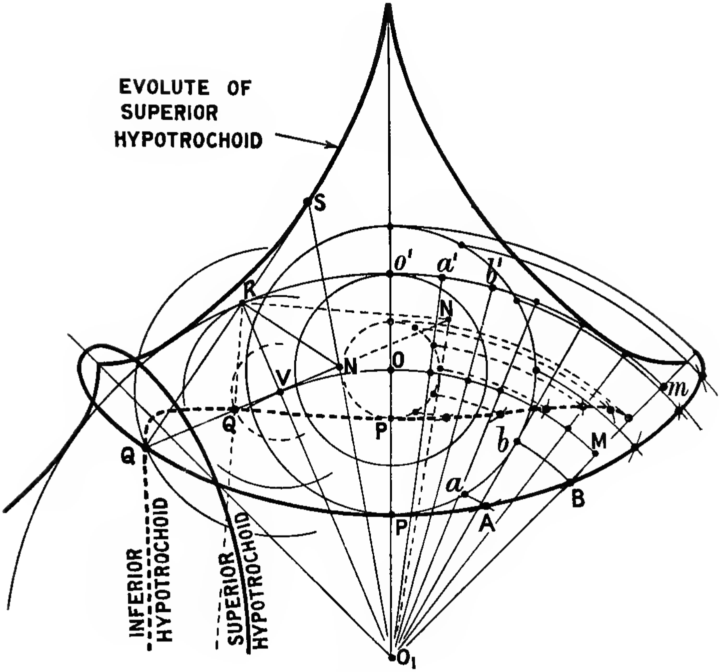


FIG. 149.

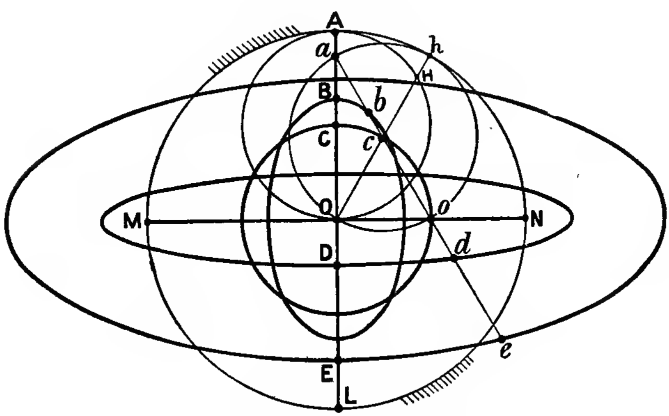


FIG. 150.

AHO, B is a point in AO and D and E are points in AO produced. *aho* is another position of the rolling circle. The points A and O are the initial positions of the points on the rolling circle which describe the straight lines AOL and MON respectively, AOL and MON being

diameters of the base circle at right angles to one another. The centre of the rolling circle will obviously describe a circle whose centre is O and radius OC. When the rolling circle has moved into the position *aho*, the line whose initial position is AOE will now be in the position *aoe*, and if *ab*, *ad*, and *ae* be made equal to AB, AD, and AE respectively the points *b*, *d*, and *e* will be points on the hypotrochoids described by the points whose initial positions are B, D, and E respectively.

The construction for finding the points *b*, *d*, and *e* is obviously the same as in the trammel method for drawing an ellipse described in Art. 45, p. 41.

79. The Involute of a Circle.—The *involute of a circle* is the roulette described by a point on a straight line which rolls on a fixed circle. The involute of a circle is therefore a special case of the epicycloid, being the epicycloid when the rolling circle is of infinite diameter.

The geometrical construction for finding points on the involute is shown in Fig. 151. *o*P is a diameter of the circle. Draw the tangent *om* and make *om* equal to half the circumference of the circle. Divide the semicircle P*co* into a number of equal parts, say six, at *a*, *b*, *c*, etc. continuing the divisions on to the other half of the circle if necessary, and divide *om* into the same number of equal parts at *a′*, *b′*, *c′*, etc. continuing the divisions beyond *m* if necessary. At the points *a*, *b*, c, etc. draw tangents *a*A, *b*B, *c*C, etc. to the circle and make *a*A, *b*B, *c*C, etc. equal to *oa′*, *ob′*, *oc′*, etc. respectively. P, A, B, C, etc. are points on the involute which starts at P and may be continued to any length. As the involute gets further and further from the circle the distance between the points as found by the above construction gets greater and greater, but intermediate points may be found by subdividing as shown for the points U and V.

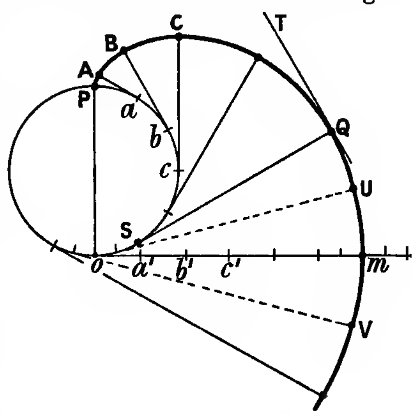


FIG. 151.

The normal to the involute at any point Q is the tangent QS to the circle and S the point of contact of this tangent and the circle is the centre of curvature of the involute at Q. The tangent QT to the involute at Q is perpendicular to QS.

80. The Catenary as a Roulette.—The curve known as the *catenary* will be referred to here as a roulette since it may be described by the focus of a parabola rolling on a fixed straight line.

Referring to Fig. 152, RAL is the fixed straight line, and BAC is the position of the parabola when its axis AY is perpendicular to

RAL. P is the focus and A the vertex of the parabola BAC. When the parabola has rolled into the position *bac* the focus has moved to

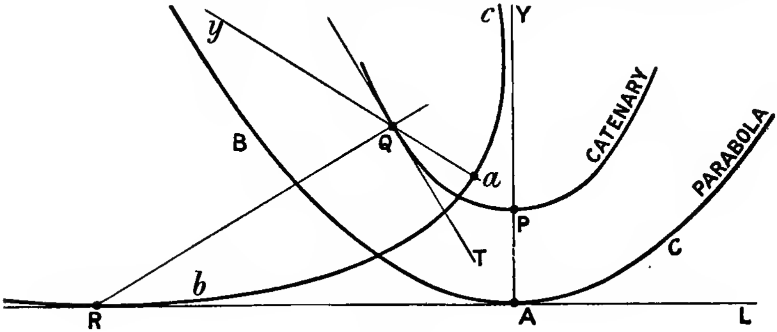


FIG. 152.

Q and has traced the arc PQ of the catenary. R being the point of contact of the parabola *bac* and the straight line RAL, the straight line QR is the normal and QT perpendicular to QR is the tangent to the catenary at Q.

81. Envelope Roulettes.—The roulettes so far considered have been curves traced by points and these may be called *point roulettes.* If the rolling curve carries with it a straight or a curved line the envelope of the carried line is called an *envelope roulette.*

An example of an envelope roulette is illustrated by Fig. 153. AB is a fixed circle on the outside of which another circle of the same diameter rolls. The rolling circle carries a tangent, the envelope of which is to be drawn. When the rolling circle is in its initial position CD the carried tangent is in the position EF being then furthest from the base circle. The rolling circle in successive positions may be drawn as in

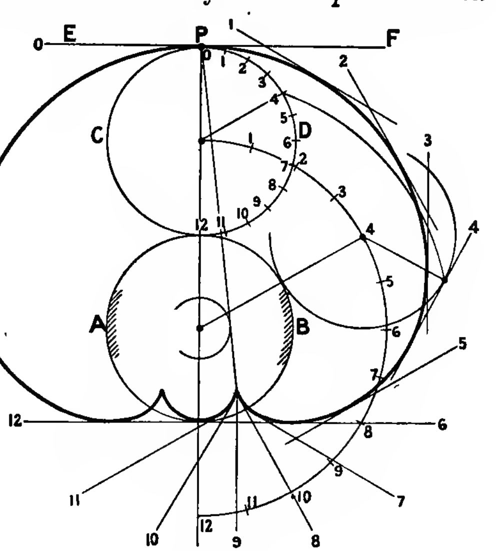


FIG. 153.

determining points on an epicycloid and the corresponding positions of the carried tangent can then be added. In Fig. 153 the carried tangent is shown in thirteen positions, numbered from 0 to 12. The construction lines for position 4 of the carried tangent are completely shown. The fair curve which touches the carried tangent in its successive positions is the envelope roulette of that tangent.

It will be observed that this roulette has two cusps. By mathematical analysis it can be shown that the tangents to the roulette at the cusps pass through P and touch a circle concentric with AB and having a radius equal to one-third of the radius of AB.

When the carried line is not a straight line or a circle or when the base line or the rolling curve is not a circle, the tracing paper method should be used. Successive positions of the carried line are then best found by placing a piece of carbonized paper between the tracing paper and the drawing paper, under the carried line, and going over the latter with a hard sharp-pointed pencil or a style.

82. Glissettes.—When a line is made to slide between two fixed points, or between a fixed point and a fixed line, or between two fixed lines, a point carried by the sliding line describes a *point glissette.* The envelope of the sliding line or of a line carried by it is an *envelope glissette.* The lines referred to may be either straight or curved.

Examples of glissettes are shown in Figs. 154 and 155. In Fig. 154 OX and OY are fixed straight lines at right angles to one another.

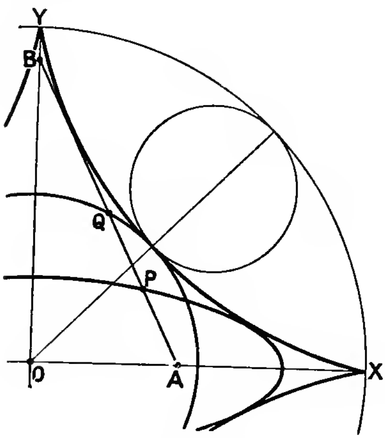


FIG. 154.

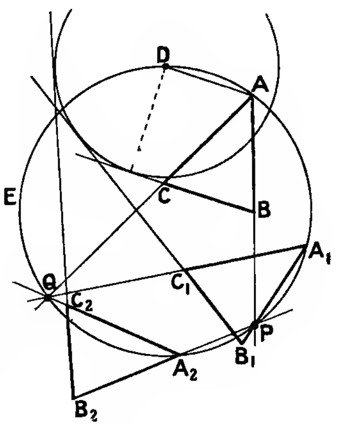


FIG. 155.

AB is a straight line of definite length which slides between OX and OY. The middle point Q of AB describes a circle whose centre is at O and whose radius is equal to half the length of AB. Any point P in AB describes an ellipse, whose centre is at O and whose axes lie on OX and OY. The semi-axis on OX is equal to BP and the semi-axis which is on OY is equal to AP.

The envelope of AB is a hypocycloid described by a point on the circumference of a circle, whose radius is equal to a quarter of AB, rolling inside a circle whose centre is at O and whose radius is equal to AB as shown.

In Fig. 155, P and Q are fixed points. ABC is a triangle whose sides AB and AC, or these sides produced, always pass through P and Q respectively. The locus of A is obviously a circle PQEA. Let ABC be one position of the triangle. Draw AD parallel to BC to meet the circle PQEA at D. Since the angle DAQ is equal to the angle ACB the arc DEQ is of constant length and D is a fixed point. The envelope glissette of BC or BC produced is a circle whose centre is D and whose radius is equal to the perpendicular distance of A from BC. The triangle is shown in three different positions.

Exercises VI

1. Draw the curve traced by a point on the circumference of a circle 2 inches in diameter as the circle rolls on a fixed straight line and makes one revolution. Draw also the evolute of the curve.

2. ABC is an equilateral triangle of 3 inches side. P is the middle point of AB. PQ is a circle 2 inches in diameter outside the triangle and touching AB at P. Draw the path traced by the point P on the circumference of the circle as the latter rolls round the triangle. Draw also the evolute of the path of P.

3. A circle PQ, 1·8 inches in diameter, touches a straight line PR at P. The length of PR is equal to the circumference of the circle PQ. The circle rolls with uniform velocity on the straight line from P to R while the straight line turns with uniform velocity about P through an angle of 270°. The angular motion of the line is in the same direction as the angular motion of the circle. Draw the curve traced by a point on the circumference of the circle, the initial position of the tracing point being P.

4. A circle 3 inches in diameter rolls on a straight line. P and Q are two points carried by the circle. P is 2 inches and Q is 1 inch from O, the centre of the circle, and O, P, and Q are in the same straight line. Draw the trochoids traced by the points P and Q. Draw also the evolutes of these trochoids.

5. Draw the epicycloids traced by a point on the circumference of a circle 2 inches in diameter which rolls on the outside, (1) of a circle 6 inches in diameter, (2) of a circle 4 inches in diameter, and (3) of a circle 2 inches in diameter. Draw also the evolutes of these epicycloids.

6. Draw the epicycloid traced by a point on the circumference of a circle 6 inches in diameter which rolls on the outside of a base circle 4 inches in diameter, the base circle being inside the rolling circle. Make a tracing of this epicycloid and apply it to the epicycloid (2) of the preceding exercise to show that the two curves are identical.

7. Take the rolling circle and carried points P and Q as in exercise 4 and draw the epitrochoids traced by P and Q when the base circle is 6 inches in diameter. Draw also the evolutes of these epitrochoids.

8. A rolling circle 3 inches in diameter carries a point P $2\frac{1}{2}$ inches from its centre. This circle rolls on the outside of a base circle $2\frac{1}{2}$ inches in diameter, the base circle being inside the rolling circle. Draw the epitrochoid traced by the point P. Draw also the evolute of this epitrochoid.

9. Draw the hypocycloid traced by a point on the circumference of a circle 2 inches in diameter which rolls on the inside of a circle 6 inches in diameter. Draw also the evolute of this hypocycloid.

10. Draw the hypocycloid traced by a point on the circumference of a circle 4 inches in diameter which rolls on the inside of a circle 6 inches in diameter. Make a tracing of this hypocycloid and apply it to the hypocycloid of the preceding exercise to show that the two curves are identical.

11. A rolling circle 2·5 inches in diameter carries two points P and Q which lie on a straight line passing through its centre O. OP = 2 inches, and OQ = 0·75 inch. Draw the hypotrochoids traced by the points P and Q, the base circle being 9 inches in diameter. Draw also the evolutes of these hypotrochoids.

12. The same as exercise 11 except that the base circle has a diameter twice that of the rolling circle.

13. A circle A (Fig. 156), of diameter EF = $3\frac{3}{8}$ inches, rolls on the line CD with uniform velocity from left to right, starting from E. Another circle B, whose diameter is half that of A, rolls inside the circumference of A, also with uniform velocity, but from right to left, starting at E when A begins to move. Circle B is in contact with circle A at F at the same time that F reaches the line CD. Draw the curve described by the centre of the circle B. [B.E.]

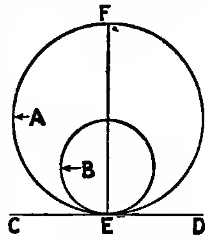


FIG. 156.

14. AB, 2 inches long, is a diameter of a circle. BC, 4 inches long, is a tangent to the circle. Draw that involute of the circle which passes through the point C.

15. Draw the roulette described by one extremity of the major axis of an ellipse (major axis $2\frac{1}{4}$ inches, minor axis $1\frac{3}{4}$ inches) which rolls on the outside of another ellipse (major axis 3 inches, minor axis 2 inches), the roulette to start from one extremity of the minor axis of the fixed ellipse. Draw also, on the same figure, the roulette described by one extremity of the minor axis of the rolling ellipse while the first roulette is being described.

16. Draw the roulette described by one focus of an ellipse (major axis $2\frac{1}{4}$ inches, minor axis $1\frac{3}{4}$ inches) while the ellipse rolls on a straight line.

17. A parabola, focal distance of vertex 1·2 inches, rolls on a straight line. Draw the roulette traced by the focus of the parabola.

18. APB is a circle 2 inches in diameter. PCD is an involute of this circle. Draw the curve traced by the point P on the circumference of the circle as the latter rolls on the outside of the involute PCD, the point P returning to PCD. Draw also the evolute of the path of P.

19. A circle 2 inches in diameter rolls on the outside of a fixed circle 4 inches in diameter. Draw the envelope roulette of a tangent carried by the rolling circle.

20. XOX′ and YOY′ are two fixed straight lines at right angles to one another. PQR is a straight line. PQ = QR = 1·5 inches. The line PQR moves so that Q is always on XOX′ and R is always on YOY′. Draw the complete path traced by the point P. Draw also the envelope glissette of the moving line.

21. Same as exercise 20 except that the angle between the fixed lines is 60° instead of 90°.

22. ABC (Fig. 157) is an equilateral triangle of 2 inches side. P is the centre of a circle of 0·5 inch radius. Q is the centre of a circle of 1 inch radius. PQ = 3 inches. The triangle ABC moves so that the sides AB and AC, or these sides produced, touch the circles whose centres are at P and Q as shown. Draw the glissette of the point A and the envelope glissette of the side BC of the triangle ABC.

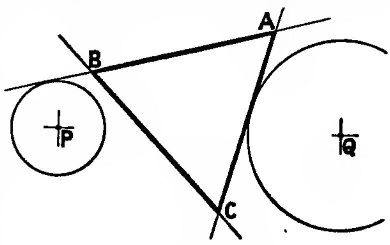


FIG. 157.

23. aPb (Figs. 158 and 159) is a moving variable triangle. The sides aP, Pb, and ba always pass through the fixed points A, B, and C respectively. The points

a and *b* move on the fixed lines YOY′ and X′OX respectively. The angle YOX is a right angle. The dimensions given are in inches. Draw the locus of the point

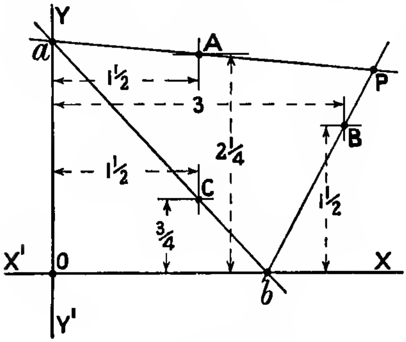


FIG. 158.

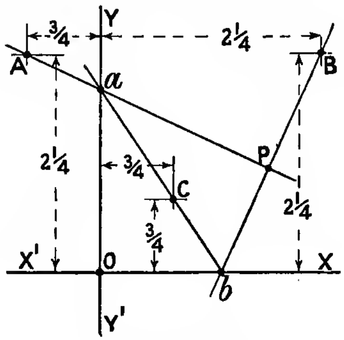


FIG. 159.

P In case I (Fig. 158) the locus is an ellipse. In case II (Fig. 159) the locus is an hyperbola.

CHAPTER VII

VECTOR GEOMETRY

83. Scalars, Vectors and Rotors or Locors.—A quantity which may be completely specified by stating the kind of quantity and its magnitude is called a *scalar quantity* or *scalar*. Thus the weight of a body is a scalar quantity. The weight of a body may be, say, 2 tons or 4480 pounds. The area of a plane figure is a scalar quantity. An area may be, say, 10 square feet or 1440 square inches.

In general the kind of quantity referred to is known from the name of the unit used in specifying its magnitude. Thus if a scalar quantity is 10 square feet it is known that the kind of quantity is area.

Time, temperature, volume, and energy are a few other examples of scalar quantities.

A scalar quantity is not associated with any definite direction in space but has magnitude only.

A scalar quantity may be represented by the length of a straight line drawn to scale, such line being drawn anywhere and in any direction. Thus an area of 30 square feet may be represented by a line 3 inches long. In this case the scale would be 1 inch to 10 square feet.

A quantity which, while having magnitude like a scalar quantity, is also associated with a definite direction is called a *vector quantity* or *vector*. Displacement, velocity, acceleration, and force are vector quantities because they have magnitude and direction. A displacement is referred to when it is stated that a body is moved 10 miles in the direction from west to east. A velocity is referred to when it is stated that a body is moving at the rate of 10 miles an hour in the direction from north to south.

All vector quantities have magnitude and direction but they may have other qualities which distinguish them from one another and it will presently be seen that there are important propositions relating to vectors apart from qualities other than magnitude and direction which they may possess.

A vector may be represented by a straight line AB drawn to scale in a definite direction. The length AB of the line represents the magnitude of the vector and the direction of the line represents the direction of the vector, provided it is made clear whether the direction is from A to B or from B to A. A line evidently has two directions, one being exactly opposite to the other and the distinction between

them is the *sense* of the direction. The sense of the direction of a vector which is represented by a line is best shown by an arrow head placed on the line. If the line which represents a vector is lettered at its extremities, it might be agreed to place the letters so that in reading them in the order in which they occur in the alphabet the sense of the direction would be given. For example if the letters at the extremities of the line are A and B then the sense of the direction would be from A to B.

A quantity which in addition to having magnitude and direction also has position is called a *rotor* or *locor*. A rotor or locor is therefore a *localized vector*. A locor has the qualities of a vector in addition to its quality of having position. A force acting at a definite point of a body is an example of a locor. A displacement of a body from one definite position to another is another example.

It has been seen that the vector part of a locor may be represented by a straight line and in order that this straight line may also represent the locor all that is necessary is to place the straight line in the proper position.

Absolute position in space cannot be defined and the position of a point or line can only be fixed in relation to other points or lines. In considering problems on locors it is only necessary to know their *relative positions*.

Since vectors have direction but not position, lines which represent vectors may be placed anywhere, provided that they have the proper directions.

The following is a convenient way of specifying a vector. Let OX (Fig. 160) be a fixed direction of reference, say from west to east, and let OA be a vector whose sense is from O to A and whose magnitude OA is equal to a. Also let the angle XOA, measured in the anti-clockwise direction, be denoted by θ. Then if the vector OA is referred to as the vector A it may be specified by the equation $A = a$. An extension of this method to locors is described in Art. 91, p. 91.

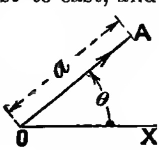


FIG. 160.

84. Addition of Parallel Vectors.—Let A, B, C, and D (Fig. 161) be parallel vectors, it is required to find a single vector which is the sum or resultant of A, B, C, and D.

Draw a straight line X_1X parallel to A, B, C, and D. Take a point O in X_1X. Mark off on X_1X a distance Oa equal to A. Observe that as the sense of A is from left to right Oa is measured to the right of O. Make ab equal to B, measuring to the right of a because the sense of B is from left to right. Make bc equal to C, measuring to the left of b because the sense of C is from right to left. Make cd equal to D, measuring to the right of c because the sense of D is from left to right. Then Od is a vector which is the sum or

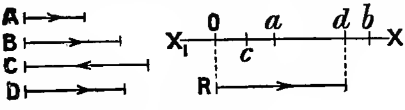


FIG. 161.

resultant of the vectors A, B, C, and D. If R is the resultant of the vectors A, B, C, and D, then R = A + B + C + D.

If a vector whose sense is from left to right is said to be positive, then a vector whose sense is from right to left would be said to be negative. Then the result of the above example would be written R = A + B − C + D. Opposite signs represent opposite senses.

Subtraction of parallel vectors is converted into addition by changing the signs of the vectors to be subtracted and then proceeding as in addition.

It should be observed that in the addition of vectors the order in which they are taken does not affect the result. For A + B + C + D = A + B + D + C = B + D + A + C.

85. Addition of Inclined Vectors. The Vector Polygon.—Let A, B, and C (Fig. 162) be three given vectors, it is required to find a single vector which is the sum or resultant of the vectors A, B, and C.

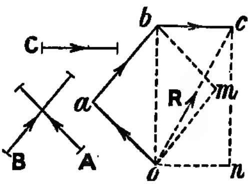


FIG. 162.

Take a point *o* and draw the vector *oa* parallel and equal to A. The sense of *oa* is from *o* to *a* the same as that of A. The vector A is now represented by *oa*. From *a* draw the vector *ab* parallel and equal to B. The sense of *ab* is from *a* to *b* the same as that of B. The vector B is now represented by *ab*. From *b* draw the vector *bc* parallel and equal to C. The sense of *bc* is from *b* to *c* the same as that of C. The vector C is now represented by *bc*. Join *oc*, then the vector *oc* whose sense is from *o* to *c* is the sum or resultant of the vectors A, B, and C. The polygon *oabc* is called a *vector polygon.*

The vector polygon is of very great importance in vector geometry and it has numerous applications in mechanics.

It will be useful to consider the vector polygon as applied (1) to displacements, (2) to velocities, (3) to accelerations, and (4) to forces.

(1) Suppose a man to be standing on the deck of a ship which is sailing through water which is itself moving in relation to the earth. Let the man walk across the deck along a straight line whose length and direction in relation to the ship are given by the vector A (Fig. 162). Let the vector B represent the direction and distance which the ship sails through the water in the time that the man walks the distance A. Lastly let the vector C represent the direction and distance which the water moves in the same time.

Let the vector polygon *oabc* (Fig. 162) be drawn. While the man is walking along *oa* that line is carried parallel to itself by the ship into the position *mb*. Consequently, neglecting for the moment the motion of the water, the man travels along the imaginary line *ob* over the water. But while this is happening the imaginary line *ob* is travelling parallel to itself with the water and reaches the position *nc* when the man has finished his walk. The actual displacement of the

man in relation to the earth is therefore represented by the line *oc*. That is, *oc* is the sum or resultant of the three displacements A, B, and C.

(2) Since velocity is displacement or distance moved in unit time it follows that if t is the time taken by the man in walking the distance A, t will also be the time of the displacement B of the ship and of the displacement C of the water, and if each displacement is divided by t the results are the several velocities. Hence the velocities are $\frac{oa}{t}$, $\frac{ab}{t}$, $\frac{bc}{t}$, and $\frac{oc}{t}$, and the polygon *oabc*, measured with a suitable scale, will be a polygon of velocities in which *oa* represents the velocity of the man's walking on the ship, *ab* represents the velocity of the ship through the water, *bc* represents the velocity of the water over the earth, and *oc* is the resultant velocity of the man or his velocity in relation to the earth.

(3) Let f denote the uniform acceleration or uniform rate of increase of the velocity of a moving body, then in time t the increase in velocity is ft. If A, B, and C (Fig. 162) represent three accelerations simultaneously impressed on a body these will also represent three corresponding increases in velocity and the resultant increase in velocity will be represented by *oc*, and *oc* will therefore represent the resultant acceleration. *oabc* is therefore an acceleration polygon.

(4) A force acting on a body causes it to move with a uniformly increasing velocity or acceleration and the magnitude of the acceleration is proportional to the magnitude of the force and takes place in the same direction as that of the force. Hence if *oabc* (Fig. 162) is an acceleration polygon it is also a force polygon.

If R is the sum or resultant of the vectors A, B, and C, then $R = A + B + C$ and the order in which the vectors are taken in performing the summation is immaterial. That is, if $R = A + B + C$, then, also, $R = B + A + C = C + A + B$. The addition is performed by drawing the vector polygon, three sides of which are A, B, and C, and the fourth or closing side is R.

Observe that if $R = A + B + C$, then $A + B + C - R = 0$, which shows that if the sense of R is reversed the sum of the four vectors A, B, C, and R is zero.

When the sum of a number of vectors is zero the vector polygon closes without the use of another vector.

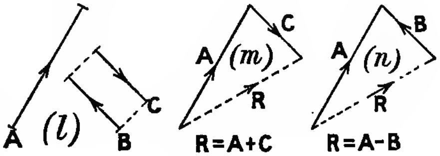


FIG. 163.

86. Subtraction of Vectors.—At (l) Fig. 163, three vectors A, B, and C are given, B and C being equal and parallel but the sense of C is opposite to that of B. Hence $C = -B$.

If $R = A - B$ then $R = A + C$.

The solution of $R = A + C$ is shown at (m) and the solution of $R = A - B$ is shown at (n).

The rule for subtraction is evidently, change the sign of the vector to be subtracted and then proceed as in addition.

87. Example.—A, B, C, and D are four vectors specified as follows. $A = 2{\cdot}4_{60^\circ}$, $B = 1{\cdot}5_{330^\circ}$, $C = 2{\cdot}0_{90^\circ}$, and $D = 2{\cdot}1_{180^\circ}$, the magnitudes being in inches. It is required to find the value of $R = A + B - C + D$. The magnitudes and directions of A, B, C, and D are shown to the right in Fig. 164. Take a point o. Draw oa parallel and equal to OA. From a draw ab parallel and equal to OB, then ob is equal to $A + B$. From b draw bc parallel to OC reversed and equal to OC, then oc is equal to $A + B - C$. From c draw cd parallel and equal to OD, then od is equal to $R = A + B - C + D = r_\theta$. R will be found to be equal to $0{\cdot}79_{301^\circ}$.

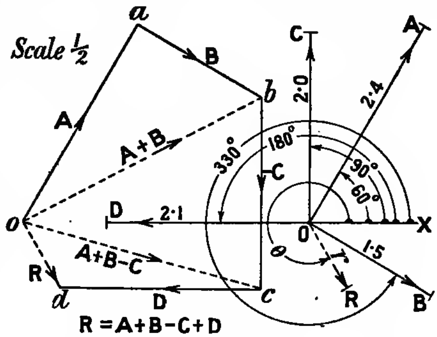


FIG. 164.

88. Resolution of Vectors.—If R is the sum or resultant of any number of vectors A, B, C, D, etc., then A, B, C, D, etc. are called the *components* of the vector R. The operation of finding R when A, B, C, D, etc., are given is called the *summation of vectors* or *composition of vectors.* The converse operation of breaking up a vector into a number of components is called the *resolution of a vector.* It is evident that a given vector may be resolved into any number of components by constructing a polygon on the given vector.

Generally when a vector has to be resolved into components only two components are required the directions of which are given. Let R or ab (Fig. 165) be a given vector and let OX and OY be two given directions; it is required to resolve R into two components P and Q whose directions shall be parallel to OX and OY respectively. Through a draw ac parallel to OX and through b draw bc parallel to OY to meet ac at c, then ac and cb will be the required components P and Q. A common case in practice is that in which the angle XOY is a right angle.

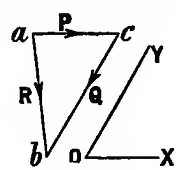


FIG. 165.

89. Example.—The thick curved line shown in Fig. 166 represents the section of a vane of a water wheel or turbine which revolves about an axis at O. A jet of water impinges on the vane at A with a velocity v_1. The jet is then deflected and flows over the vane leaving it at B. The wheel revolves in the direction of the arrow R. The radius of the wheel at A is r_1, and the linear velocity of the vane at A is c_1. The radius of the wheel at B is r_2 and the linear velocity of the vane at B is c_2. Obviously $c_2 : c_1 : : r_2 : r_1$.

Consider the water in the jet at A where it comes in contact with the vane. This water is beginning to slide along the vane with a velocity s in a direction tangential to the vane at A. This water is also carried round with the wheel with a velocity c_1 in the direction of the tangent to the wheel circle at A, and in order that there shall be no shock the resultant of these two velocities s and c_1 should be v_1. Of the three velocities v_1, c_1, and s, if one be known completely and the directions of the other two are given, their magnitudes are readily found. Or if two of the velocities be known completely, the magnitude and direction of the other is readily found.

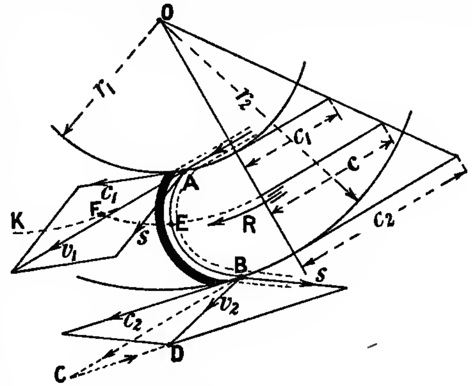


FIG. 166.

Assume that the water slides along the vane with a constant velocity s relatively to the vane and consider the water in the jet at B. This water has a velocity s in the direction of the tangent to the vane at B and also a velocity c_2 in the direction of the tangent to the wheel circle at B. Hence BD or v_2 the resultant or absolute velocity of the water at B is the resultant of the velocities s and c_2.

Draw BC parallel and equal to v_1 and join CD; then CD is the vector change $v_2 - v_1$ in the velocity of the water in passing over the vane.

The actual path of a particle of water in passing through the wheel may be found as follows. Make the arc AE equal to a definite fraction of s. With centre O and radius OE describe the arc EF. Make the arc EF equal to the same fraction of c, the velocity of the wheel at E, that the arc AE is of s. Then F is a point in the actual path AFK of a particle of water which enters the wheel at A.

Exercises VII

1. Three vectors, A, B, and C, acting in a horizontal plane are defined in the following table :—See also Fig. 167.

Vector.	Magnitude.	Direction
A	1·23 units	Eastwards.
B	1·95 ,,	33·2° northwards of east.
C	2·60 ,,	112° northwards of east.
A + B + C		

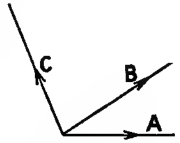


FIG. 167.

The angles of 33·2° and 112° are to be set off with the protractor, and not by copying the diagram.

Determine the resultant or vector sum A + B + C, using a scale of 1 inch to 1 unit. Measure and tabulate the results (thus completing the above table).

Theorem:—A vector sum is the same in whatever sequence the vectors are added. Verify this principle by actual drawing in the following case:—Show that A + B + C = A + C + B. [B.E.]

2. Three coplanar vectors A, B, C are given as follows:—

Vector.	Magnitude.	Direction.
A	37·2 units	23·6°
B	59·5 „	115·5°
C	88·0 „	238·0°
A + B + C		
A − B + C		

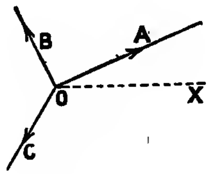


FIG. 168.

In defining direction, the vectors are supposed to act outwards from a point O (Fig. 168), and the angles are measured anti-clockwise from a fixed line OX. These angles must be set off with a protractor, and not copied from the diagram.

Find A + B + C and A − B + C. Measure and record the results (in the form required to complete the foregoing table).

Verify by drawing that A − (B − C) = A − B + C.

Use a scale of ½ inch to 10 units. [B.E.]

3. Find the vector sum A + B + C + D + E, having given $A = 2{\cdot}15_{0^\circ}$, $B = 1{\cdot}60_{0^\circ}$, $C = 2{\cdot}25_{180^\circ}$, $D = 2{\cdot}80_{0^\circ}$, and $E = 5\,35_{180^\circ}$, the unit of magnitude being 1 inch.

4. Four coplanar forces P, Q, R, and S, acting at a point, are specified as follows:—$P = 15_{0^\circ}$, $Q = 9_{213^\circ}$, $R = 10{\cdot}5_{315^\circ}$, and $S = 21_{165^\circ}$, the magnitudes of the forces being in pounds. Find the resultant of these forces, and specify it in the form $T = t_{\theta^\circ}$. Use a scale of 1 inch to 10 pounds.

5. A locomotive engine A represented by a point is approaching a level crossing C from the south at a speed of 15 miles an hour (22 feet per second), that is velocity of $A = 22_{90^\circ}$ f.s., directions being measured anti-clockwise from the east. An engine B is approaching from the W.S.W., its velocity being $44_{22\frac{1}{2}^\circ}$ f.s. They arrive together at C.

(*a*) Show their positions one second before the collision, scale 1 inch to 10 feet. Measure their distance apart and the direction from A to B.

(*b*) What is the relative speed of the two engines when the accident occurs? [B.E.]

6. An aeroplane is headed due west, and is propelled at 50 miles per hour relatively to a steady wind which is blowing at 20 miles per hour from the north-west. Find the actual direction and speed of flight as regards the earth.

If the pilot wishes to travel westward, in what direction must he apparently steer, and what will be his speed to the west? [B.E.]

7. A ship is sailing eastwards at 10 miles an hour. It carries an instrument for recording the apparent velocity of the wind, in both magnitude and direction.

(*a*) If the wind registered by the instrument is apparently one of 20 miles per hour from the north-east, what is the actual wind? Give the answer in miles per hour and degrees north of east of the quarter from which the wind comes.

(*b*) If a wind of 15 miles per hour from the north-east were actually blowing, what apparent wind would the instrument on the vessel register? State this answer in miles per hour and degrees north of east as before.

Use a scale of ¼ inch to 1 mile per hour. [B.E.]

8. A weight of 15 pounds is supported by two cords. One cord is inclined at 40° and the other at 55° to the horizontal. Determine the tensions in the cords.

9. A wheel weighing 100 lb. rolls at a certain speed on a horizontal rail. The wheel is out of balance to the extent that at the speed of rolling there is a radial centrifugal force of 20 lb. On a base 6 inches long, representing one revolution of the wheel, plot, for every 1–12th of a revolution, the pressure exerted by the wheel on the rail. Use a force scale of 1 inch to 20 lb.

10. A, B, C, D, E, and F are six forces acting in a plane at a point O. The magnitudes of the forces are 1·25, 1·10, 2·25, 1·75, 2·45, and 1·35 pounds respectively. The forces act outwards from O in lines inclined to a fixed line OX at the following angles, 30°, 60°, 135°, 180°, 225°, and 300° respectively. Find R the resultant of these forces. If these forces are balanced by a force P acting in the line OX and a force Q acting in a line at right angles to OX, determine P and Q.

11. A jet of water having a velocity of 35_{30° feet per second passes over a fixed vane as shown in Fig. 169, and is deflected, leaving the vane with a velocity of 35_{100° feet per second. If u_θ denote the change of velocity that has occurred, find u and θ; that is, solve the vector equation

$$u_\theta = 35_{100^\circ} - 35_{30^\circ}.$$

The mass m of water passing per second being 2·5 units (= 2·5 × 32·2 lb.), calculate mu_θ, the magnitude and direction of the change of momentum per second; find also $-mu_\theta$, the magnitude and direction of the force acting on the vane. [B.E.]

FIG. 169.

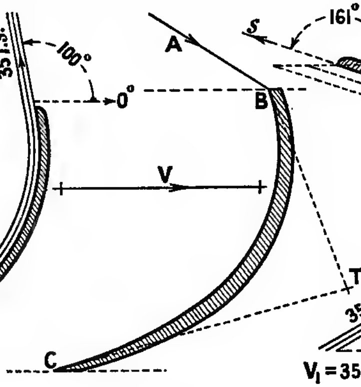


FIG. 170.

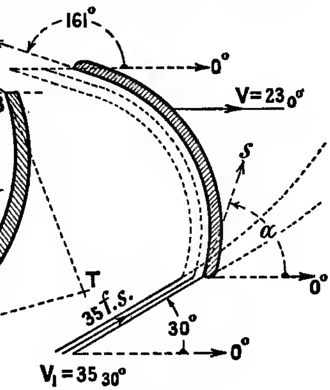


FIG. 171.

12. A stream of water flowing in the given direction AB (Fig. 170) impinges on a succession of moving vanes, one of which is shown. BT, CT are tangents to the curve of the vane at its ends. The vector V represents the velocity of the vane to a scale of 1 inch to 20 feet per second.

(*a*) Find and *measure* the speed of the water along AB in order that the water may come on to the vane at B tangentially, or without shock.

(*b*) Suppose the water to flow over the vane without change of relative speed, represent graphically to scale the absolute velocity of the water as it leaves the vane at C.

(*c*) Show graphically the vector change U of velocity of the water that has occurred owing to its passage over the vane. Find and *measure* the component of U in the direction of V.

(*d*) Determine, *either* the actual path of the water as it flows over the moving vane, *or* the line of the resultant force on the vane, due to the change of momentum of the water. [B.E.]

13. A jet of water having a velocity V_1 of 35_{30° feet per second passes over a succession of curved vanes (one of which is given in Fig. 171) moving with a velocity V of 23_{0° feet per second. Find $V_1 - V$, the velocity of the water relatively to the vane; that is, find s and α in the vector equation

$$s_\alpha = 35_{30^\circ} - 23_{0^\circ} = V_1 - V,$$

s being the speed along the vane, and α the direction of the vane at entrance, the water coming on tangentially.

The water leaves the vane with a velocity relatively to the latter of s_{161°. Find $U(= u_\theta)$, the change of velocity that has occurred.

The mass m of the water flowing per second being 2·5 units ($= 2{\cdot}5 \times 32{\cdot}2$ lb.), calculate mU, the magnitude and direction of the change of momentum per second. Find also the power developed, which is equal to the scalar product $- mUV$. [B.E.]

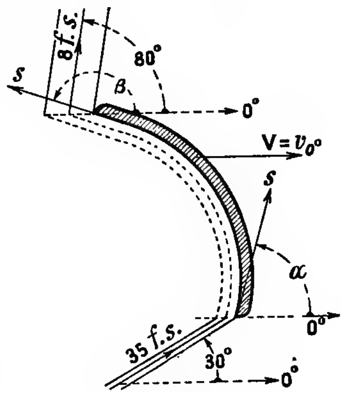


Fig. 172.

14. A jet of water passes over a succession of curved vanes (one of which is shown in Fig. 172), entering tangentially with a velocity V_1 of 35_{30° feet per second, and leaving with a velocity V_2 of 8_{80° feet per second. The vanes have a velocity V of v_{0° feet per second, and the speed s of the water along the vanes is assumed constant.

Find v, the speed of the vane. Find also α and β the directions of the vane at entrance and exit. The mass m of the water flowing per second being 2·5 units ($= 2{\cdot}5 \times 32{\cdot}2$ lb.), and U denoting the vector change of velocity of the water, find mU, the magnitude and direction of the change of momentum per second. Find also the power developed, which is equal to the scalar product $- mUV$. [B.E.]

CHAPTER VIII

GRAPHIC STATICS

90. Coplanar Forces.—Unless otherwise stated when a system of forces is considered it will be assumed that the lines of action of all the forces are in the same plane. In the great majority of problems in statics with which engineers have to deal the forces are in the same plane.

The composition and resolution of forces acting at the same point have been considered in Chapter VII. as particular cases of vectors.

91. Specification of Forces in a Plane.—A force is specified when its line of action, its sense, and its magnitude are known. Let OX (Fig. 173) be a fixed line or axis in the plane in which a force A acts, and let O be a fixed point in OX. Let the line of action of the force A cut OX at L, and let x denote the length of the intercept OL. Let α denote the angle, measured anti-clockwise, which the line

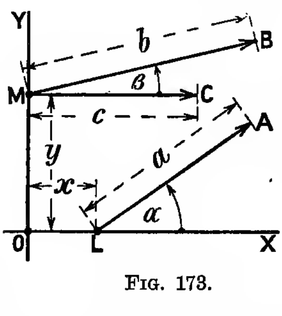


FIG. 173.

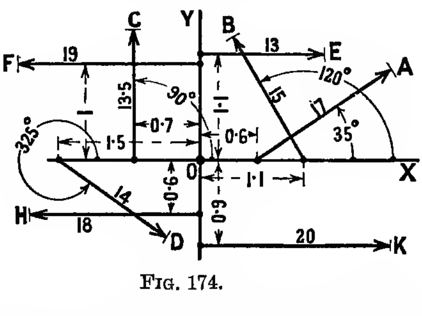


FIG. 174.

of action of the force makes with OX, the force being supposed to act outwards from L. Lastly let a denote the magnitude of the force. Then the force is specified by giving x, α, and a.

A convenient way of stating the quantities x, α, and a in specifying a force A is, $A = {}_x a_\alpha$. For example, if the magnitude of the force A is 19·5 units of force, and the intercept x is 0·7 unit of length and the angle α is 35 degrees, then $A = {}_{0\cdot7}19{\cdot}5_{35°}$.

This method of specifying a force fails when the line of action of the force is parallel to OX, but this difficulty may be overcome by giving the intercept OM $= y$ on an axis OY at right angles to OX. To show clearly that it is y and not x that is given when specifying

the force, the value of y may be placed above instead of below the level of the value of a. Thus the force C (Fig. 173) whose line of action is parallel to OX would be given by the equation $C = {}^{y}c_{0^\circ}$. The use of the value of y instead of the value of x is also desirable when the intercept x is large because the point L may be practically inaccessible although the line of action of the force may be at quite a convenient distance from O. The force B (Fig. 173) would therefore be specified by $B = {}^{y}b_{\beta}$.

The intercept x is positive when it is to the right of O and negative when to the left. Also, the intercept y is positive when it is above O and negative when below.

A number of forces are represented in Fig. 174. Without a drawing these forces would be specified as follows, the unit of force being the pound, and the unit of distance the inch.

$$A = {}_{0\cdot6}17_{35^\circ} \qquad B = {}_{1\cdot1}15_{120^\circ} \qquad C = {}_{-0\cdot7}13\cdot5_{90^\circ} \qquad D = {}_{-1\cdot5}14_{325^\circ}$$

$$E = {}^{1\cdot1}13_{0^\circ} \qquad F = {}^{1}19_{180^\circ} \qquad H = {}^{-0\cdot6}18_{180^\circ} \qquad K = {}^{-0\cdot9}20_{0^\circ}$$

92. Bow's Notation.—In Fig. 175 the diagram (m) shows the lines of action of a number of forces which are in equilibrium. The diagram (n) is the corresponding polygon of forces. In one system of lettering, each force is denoted by a single letter, as P. In *Bow's notation*, each force is denoted by two letters, which are placed on opposite sides of the line of action of the force in diagram (m), and at the angular points of the polygon in diagram (n). In Bow's notation the force P is referred to as the force AB. In like manner the force Q is referred to as the force BC. The diagram (m), which shows the lines of action of the forces, is called the *space diagram* or *frame diagram*, and the diagram (n) which shows the polygon of forces is called the *force diagram*.

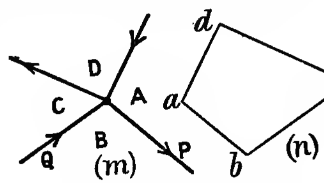


FIG. 175.

93. The Funicular Polygon or Link Polygon.—Let P, Q, R, and S (Fig. 176) be four forces, acting on a rigid body, but their lines of action do not meet at one point. The forces P, Q, R, and S are balanced by a fifth force T which is at present unknown. Draw the polygon of forces *abcde*, that is, draw a polygon whose sides are parallel to the lines of action of the forces and of lengths equal to the magnitudes of the forces as in the case of forces acting at a point. *ea* the closing side of the polygon will represent in magnitude and direction the fifth force T. It now remains to find a point in the line of action of the force T.

Take any point o and join it to a, b, c, d, and e. Take any point 2 in the line of action of P and draw the line 2B3 parallel to ob to meet the line of action of Q at 3. Draw 3C4 parallel to oc to meet the line of action of R at 4. Draw 4D5 parallel to od to meet the line of action of S at 5. Draw 5E1 parallel to oe to meet 2A1

parallel to *oa* at 1. Then 1 is a point in the line of action of T which may now be drawn parallel to *ea*.

Conceive that the lines A, B, C, D, and E represent bars jointed to one another at the points 1, 2, 3, 4, and 5. Then these bars may be supposed to take the place of the rigid body upon which the five forces P, Q, R, S, and T are supposed to act. In the case under consideration (Fig. 176) it is obvious that the bars A, B, C, D, and E are subjected to tension. Consider the point 2. Here there are three forces acting which balance one another, namely, the force P and the tensions in the bars A and B, and these three forces are represented in magnitude and direction by the three sides of the triangle *abo*. Again, the three forces acting at the point 3 are represented by the sides of the triangle *bco*, also the three forces acting at the point 4 are represented by the sides of the triangle *cdo*, and the three forces at 5 by the sides of the triangle *deo*. Now in order that the tensions in the bars E and A may be balanced by the force T, the force T

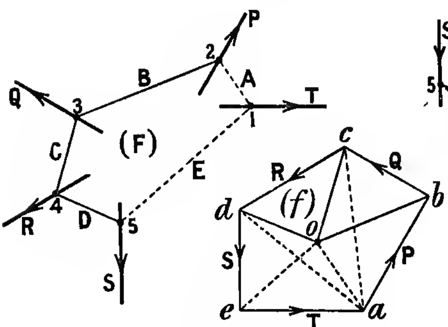


FIG. 176.

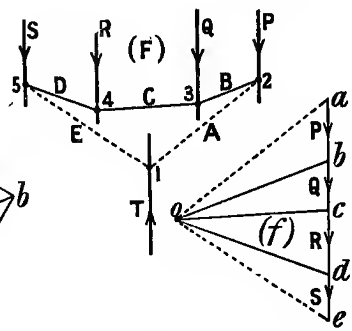


FIG. 177.

must act at the point of intersection of the bars E and A. The point 1 is therefore a point in the line of action of T.

The polygon 1 2 3 4 5 is called the *funicular polygon* or *link polygon* of the forces P, Q, R, S, and T with reference to the point *o*, which is called the *pole*.

Since the pole *o* may have an infinite number of positions, there is an infinite number of link polygons to any system of balanced forces.

If the diagrams (F) and (*f*) (Fig. 176) be compared it will be seen that each line on the one is parallel to a corresponding line on the other. Also, if a system of lines on the one meet at a point the corresponding lines on the other form a closed polygon. From these properties the diagrams (F) and (*f*) are called *reciprocal figures*.

No reference has yet been made to Fig. 177, but all that has been said with reference to Fig. 176 will also apply to Fig. 177, where the given forces are parallel to one another, except that the bars E and A are in compression, the remaining bars B, C, and D being in tension.

An examination of Figs. 176 and 177 will show that the simple rule to be remembered in drawing the link polygon is, that any side of that polygon has its extremities on the lines of action of two of the forces, and that that side is parallel to the line which joins the pole to the point of intersection of the lines which represent these two forces on the polygon of forces.

Referring to Figs. 176 and 177, it may be noted that the equilibrant of P and Q is represented in magnitude and direction by *ca*, and that the point of intersection of the sides A and C of the link polygon is a point in the line of action of this equilibrant. Also the equilibrant of P, Q, and R is represented in magnitude and direction by *da*, and the point of intersection of the sides A and D of the link polygon is a point in the line of action of this equilibrant.

Having shown that the link polygon together with the polygon of forces may be used to determine the equilibrant of any system of forces in a plane, it is obvious that the same construction will also determine the resultant of that system of forces, since the resultant acts along the same line and has the same magnitude as the equilibrant, but acts in the opposite direction or has the opposite sense.

Again referring to Figs. 176 and 177, it will be observed that the letters A, B, C, D, and E on the space diagrams are situated in the spaces between the forces P, Q, R, S, and T, and the letters P, Q, R, S, and T may be omitted and the forces referred to as AB, BC, CD, DE, and EA. Also the same forces in the force polygons are lettered *ab*, *bc*, *cd*, *de*, and *ea* respectively, which is Bow's notation. It will further be noticed that the sides A, B, C, D, and E of the link polygon are parallel to the polar lines *oa*, *ob*, *oc*, *od*, and *oe* respectively of the force polygon.

94. Examples.—The following two examples illustrate the application of the link polygon to the solution of problems on forces whose lines of action are either parallel or do not all meet at the same point.

(1) Using Bow's notation, AB, BC, and CD (Fig. 178) are three vertical forces acting on a horizontal beam. These forces are balanced by the vertical forces DE and EA whose magnitudes and senses are required.

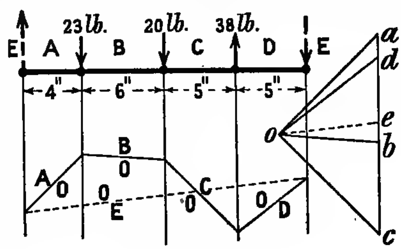


FIG. 178.

Since the forces are all parallel, the polygon of forces will be a straight line *abcdea*, the position of the point *e* being as yet unknown.

Choose a pole *o*, and join *oa*, *ob*, *oc*, and *od*. Draw OA, OB, OC, and OD parallel to *oa*, *ob*, *oc*, and *od* respectively as shown. These lines OA, OB, OC, and OD will form four sides of the link polygon of which OE will be the closing side. Draw *oe* parallel to OE to meet *ac* at *e*. This completes the solution. It will be found that

DE is 13·9 lb. and that it acts downwards. Also, EA is 18·9 lb. and acts upwards.

(2) LMN (Fig. 179) is a framed structure supported at M and N, MN being horizontal. There is a vertical load AB of 1100 lb. at L and a load BC of 1520 lb. at the middle point of LN and at right angles to LN. The supporting force CD at N is known to be vertical, but its magnitude is unknown. The supporting force DA at M is unknown both as regards magnitude and direction. It is required to complete the determination of the forces CD and DA.

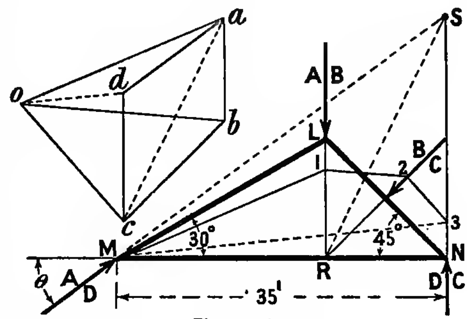


Fig. 179.

The triangle of forces *abc* determines the direction of the resultant of the given forces AB and BC and a line through R, the point of intersection of AB and BC, parallel to *ac* is the line of action of that resultant. Replacing AB and BC by their resultant there are now only three external forces acting on the frame and as they are not parallel they must meet at a point which is obviously the point S where the resultant of AB and BC meets CD. This determines the line of action of DA which passes through M and S. The polygon of forces *abcd* may now be completed. It will be found that CD = 1380 lb., DA = 1340 lb., and θ the inclination of DA to the horizontal is 36·5 degrees.

This example may be worked without using the point S, as follows. After drawing the sides *ab* and *bc* of the force polygon choose a pole *o*, and join *oa*, *ob*, and *oc*. Next proceed to draw the link polygon *starting at the point M* which is the only point in the line of action of DA which is known. Draw M1, across the space A, parallel to *oa*. Draw 1 2, across the space B, parallel to *ob*. Draw 2 3, across the space C, parallel to *oc*. Then M3 is the closing line of the link polygon. Draw *od* parallel to M3 to meet a vertical through *c* at *d* which is the remaining angular point of the force polygon. DA may now be drawn parallel to *da*.

95. The Centre of Parallel Forces.—If a system of parallel forces acts at fixed points, the resultant will act through another fixed point called *the centre of the system.* This centre is independent of the direction of the forces so long as the sense of each in relation to the sense of one of the forces is unaltered.

In Fig. 180, P, Q, R, and S are parallel forces acting at the fixed points A, B, C, and D respectively in a plane. By means of the force and link polygons the line of action LK of the resultant is determined. Let the direction of the forces be changed so that they act as shown by P′, Q′, R′, and S′. The line of action MK of the resultant is

determined as before. The point K, where LK and MK intersect, is the centre of the parallel forces P, Q, R, and S acting at the fixed points A, B, C, and D respectively. If the construction be repeated with the forces acting in any other direction, it will be found that the new resultant will act through the same point K.

In Fig. 180, the forces P, Q, R, and S have all the same sense, and therefore P′, Q′, R′, and S′ must have the same sense. But if the sense

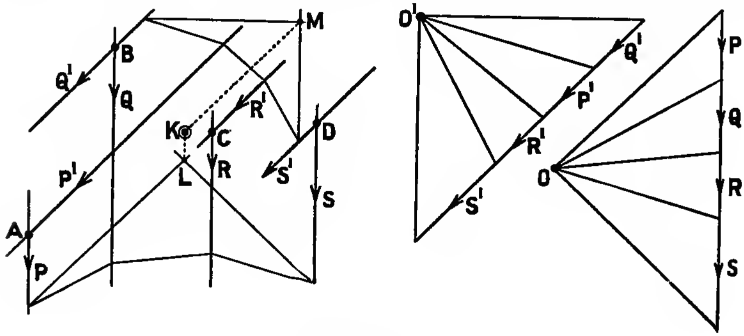


FIG. 180.

of Q, say, were opposite to that of P, then the sense of Q′ would be opposite to that of P′.

In applying the above method to the determination of the centre of a system of parallel forces, it is usually most convenient to take the two directions of the forces at right angles to one another.

96. Centres of Gravity or Centroids.—The particles of which any body is made up are attracted to the earth by forces which are proportional to the masses of these particles. For all practical purposes these forces may be considered to be parallel, and their resultant will pass through the centre of these parallel forces. In this case the centre of the parallel forces is called the *centre of gravity* or *centroid* of the body, and the determination of a centre of gravity resolves into finding the centre of a system of parallel forces.

The centre of gravity of a body may also be defined as that point from which if the body is suspended it will balance in any position.

When the term centre of gravity is applied to a line, the line is supposed to be an indefinitely thin wire; and when the centre of gravity of a surface is spoken of the surface is supposed to be an indefinitely thin sheet of material.

The following results, which are not difficult to prove, should be noted:—

The centroid of a straight line is at its middle point.

The centroid of a triangle is at the intersection of its medians.

The centroid of a parallelogram is at the intersection of its diagonals.

If a plane figure is symmetrical about a straight line, the centroid of the figure is in that straight line.

To find the centroid of a line made up of a number of straight lines. At the centres of the straight lines apply parallel forces whose magnitudes are proportional to the lengths of these lines. The centre of these parallel forces is the centroid required.

To find the centroid of a curved line. Divide the line into a number of parts, preferably of equal length. At the centres of these parts apply parallel forces whose magnitudes are proportional to the lengths of the parts. The centre of these parallel forces is approximately the centroid required. Theoretically the approximation is closer the more numerous the parts into which the curved line is divided, but practically when the parts are very numerous or very short the drawing of the link polygon becomes less accurate.

To find the centroid of any irregular figure. Divide the figure into parts whose centroids and areas are known or easily found. At the centroids of these partial areas apply parallel forces whose magnitudes are proportional to these areas. The centre of these parallel forces is the centroid required. If the given figure has an irregular curved boundary line such as is shown in Fig. 181, divide the figure into a number of parallel strips as shown by the full straight lines. Draw the centre lines of these strips, shown dotted. The centroids of these strips may be taken at the middle points of their centre lines, and the areas of the strips may be taken as proportional to the lengths of their centre lines.

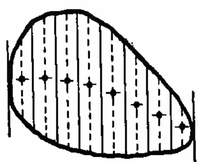


FIG. 181.

To find the centroid of a quadrilateral. Let ABCD (Fig. 182) be the quadrilateral. Draw the diagonals AC and BD, intersecting at O. Let OA be less than OC. Make CE equal to AO. Join BE and DE. G, the centroid of the triangle BDE is also the centroid of the quadrilateral ABCD.

To find the centroid of an arc of a circle. Let ABC (Fig. 183) be the arc of a circle of which O is the centre. Draw OB at right angles

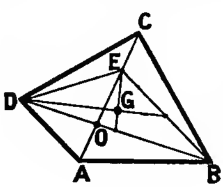


FIG. 182.

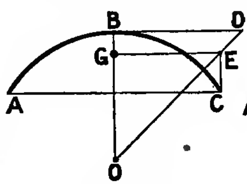


FIG. 183.

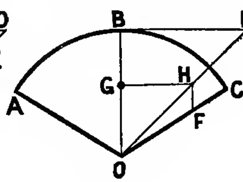


FIG. 184.

to the chord AC. Draw BD at right angles to OB and make BD equal to the arc BC. Join OD. Draw CE parallel to OB to meet OD at E. Draw EG parallel to AC to meet OB at G. G is the centroid of the arc ABC.

To find the centroid of a sector of a circle. Let OABC (Fig. 184) be the sector of a circle of which O is the centre. Draw OB at right angles to the chord AC. Draw BD at right angles to OB and make BD equal to the arc BC. Join OD. Find a point F in OC such that OF is two-thirds of OC. Draw FH parallel to OB to meet OD at H. Draw HG perpendicular to OB to meet OB at G. G is the centroid of the sector OABC.

To find the centroid of a figure considered as part of another figure. Frequently the addition of a simple figure to a given comparatively complicated one will make a simple figure. For example the figure ABCD (Fig. 185) only requires the addition of the triangle OCD to convert it into the sector of a circle whose centre is O. G_1 the centroid of the sector OAB is readily found and so is G_2 the centroid of the triangle OCD.

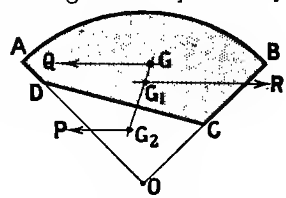


FIG. 185.

Let G be the centroid of the figure ABCD. Then if at G_2 and G parallel forces P and Q be applied, the magnitudes of P and Q being proportional to the areas of OCD and ABCD, the centre of these parallel forces will be at G_1 the centroid of the sector OAB.

Hence, if parallel forces P and R be applied at G_2 and G_1, *but in opposite directions*, the magnitudes of P and R being proportional to the areas of OCD and OAB, the centre of these parallel forces will be at G the centroid of the figure ABCD.

The area of a sector of a circle is equal to half the product of the arc and the radius.

97. Centre of Pressure and Centre of Stress.—If a plane figure be subjected to fluid pressure, the point in the plane of the figure at which the resultant of the pressure acts is called the *centre of pressure.* If the plane figure is a section of a bar or part of a structure which is subjected to stress, the point in the plane of the section at which the resultant of the stress acts is called the *centre of stress.*

In what follows "pressure" will be taken to include "stress."

If the pressure be uniform over the figure, then the centre of pressure coincides with the centroid of the figure.

A general construction for determining the centre of pressure of any plane figure when the pressure varies uniformly in one direction is illustrated by Fig. 186. ABCD is a plane figure supposed to be vertical, and AB and CD are horizontal. AA_1 is the altitude of the figure, and the pressure is supposed to vary uniformly from an amount represented by AP at the level AB to an amount represented by A_1Q at the level CD. AP and A_1Q are horizontal.

Join QP and produce it to meet A_1A at O. Draw any horizontal SRMN to cut the given figure. Draw the horizontal OF, and the verticals MM_1 and NN_1. Through K, the middle point of MN, draw the vertical KF. Join FM_1 and FN_1 cutting MN at *m* and *n*. If this construction be repeated at a sufficient number of levels, and all points

corresponding to m be joined, also all points corresponding to n, a figure abnCDma is obtained and the centroid of this figure will be the centre of pressure of the original figure.

The proof is as follows. Suppose that the line MN is the centre line of a *very narrow* horizontal strip of the original figure, and let the width of this strip be denoted by w. The magnitude of the resultant pressure on this strip is equal to MN $\times\, w \times$ RS, and it will act at K, the middle point of MN.

Since SRMN is parallel to QA_1DC,

$$M_1N_1 : mn :: OA_1 : OR,$$

and, $$MN : mn :: A_1Q : RS,$$

therefore, $$MN \times RS = mn \times A_1Q,$$

and $$MN \times w \times RS = mn \times w \times A_1Q,$$

that is, the resultant of the pressure on the strip of length mn when subjected to a pressure A_1Q will have the same magnitude as the resultant of the pressure on the strip of length MN when subjected to a pressure RS, and it will act at the same point K which is also the middle point of mn.

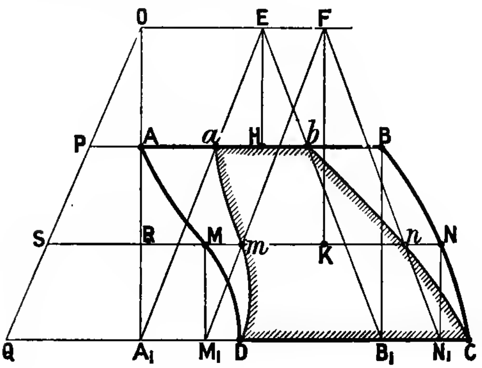


FIG 186.

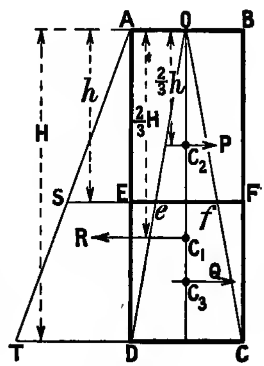


FIG. 187.

It follows that the resultant of the pressure on the figure abnCDma when subjected to a uniform pressure A_1Q will be the same as the resultant of the varying pressure on the original figure. But when the pressure on a plane figure is uniform, the centre of pressure is at its centroid. Therefore the centroid of the figure abnCDma is the centre of pressure of the original figure.

The figure abnCDma is called a *modulus figure* of the original figure.

A common and important practical case is that in which the given figure is a rectangle and the pressure on it varies uniformly in a direction parallel to a side of the rectangle.

In Fig. 187, ABCD is a rectangle on which the pressure varies uniformly from nothing at A to an amount DT at D. Applying the

construction just proved the student will have no difficulty in seeing that the modulus figure for the rectangle ABCD is the isosceles triangle OCD and that the centre of pressure for the rectangle ABCD is at C_1 on the vertical centre line of the rectangle and at a distance from AB equal to $\frac{2}{3}$H where H is the height of the rectangle. Also the magnitude of R the resultant of the pressure on ABCD is equal to the area of the rectangle multiplied by the half of DT, where DT is the intensity of the pressure or the pressure per unit area at the level CD.

If the rectangle ABCD (Fig. 187) be divided into two rectangles by the line EF, then, C_2, the centre of pressure of the rectangle ABFE is at a distance $\frac{2}{3}h$ from AB, where h is the height of the rectangle ABFE, and the magnitude of P the resultant of the pressure on ABFE is equal to the area of ABFE multiplied by the half of ES.

The centre of pressure of the rectangle CDEF will be at C_3 the centroid of the quadrilateral CDef and the magnitude of Q the resultant of the pressure on CDEF will be equal to the area of CDEF multiplied by half the sum of DT and ES. Q may however be determined both as regards magnitude and position by considering it as the equilibrant of the parallel forces P and R, the sense of R being opposite to that of P.

98. Masonry Dams.—A masonry dam is a wall for holding back the water at the end of a natural reservoir.

One form of dam section is shown in Fig. 188, AV being a vertical line.

The principal problems connected with dams are: (1) the determination of the *line of resistance* when the reservoir is empty; (2) the determination of the line of resistance when the reservoir is full, and (3) the determination of the stresses at various horizontal sections of the dam.

Reservoir empty. When the reservoir is empty the stresses in the dam are those due to its weight. Referring to Fig. 188. Consider a portion of the dam lying between two vertical cross sections one foot apart, then the weight of a part of this between two horizontal section planes will be equal to the area, in square feet, of the cross section between these planes multiplied by the weight of a cubic foot of the material of the dam.

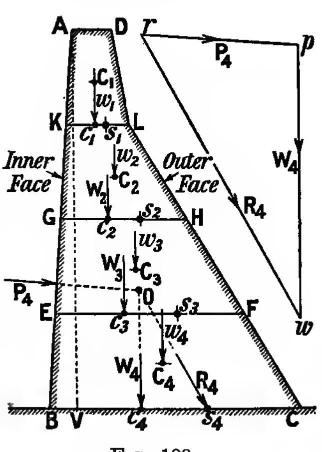


Fig. 188.

Let horizontal sections BC, EF, GH, and KL be taken. The resultant w_1 of the weight of the top portion ADLK acts vertically through C_1 the centroid of ADLK. w_1 cuts KL at c_1 which is the centre of pressure or centre of stress for the horizontal section KL.

The resultant w_2 of the weight of the portion KLHG acts vertically through C_2 the centroid of KLHG. The resultant of the load on GH is W_2 which is the resultant of w_1 and w_2. W_2 cuts GH at c_2 which is the centre of stress for the horizontal section GH.

Continuing in the same way, W_3, the resultant of w_1, w_2, and w_3, cuts EF at c_3 the centre of stress for EF, and W_4, the resultant of w_1, w_2, w_3, and w_4, cuts BC at c_4 the centre of stress for BC.

A fair curve drawn through the centres of stress of the various horizontal sections is called the *line of resistance* for the cross section of the dam.

Reservoir full. When the reservoir is full the gravity forces which have just been considered will still act but there will now be in addition the pressure of the water on the inner face.

The intensity of the pressure of the water per square foot at any depth h feet from the upper or free surface of the water is equal to h multiplied by the weight of a cubic foot of water or say $62{\cdot}3h$.

Still considering a portion of the dam one foot long. The resultant force on the horizontal section BC is the resultant of W_4 and P_4, where P_4 is the resultant of the pressure of the water on the face of the one foot length of dam from A to B. P_4 acts at two-thirds of AB from A and at right angles to AB. The magnitude of P_4 is the area of the face AB, one foot long, multiplied by half the intensity of the water pressure at B. Let P_4 and W_4 intersect at O. R_4 the resultant of P_4 and W_4 acts through O in a direction determined by the triangle of forces *rpw*. The line of action of R_4 cuts BC at s_4 which is the centre of stress for BC.

In a similar manner s_3, s_2, and s_1 the centres of stress for EF, GH, and KL may be determined. A fair curve through these centres of stress of the various horizontal sections determined as above is the line of resistance for the cross section of the dam when the reservoir is full.

If each of the horizontal lines BC, EF, etc. be divided into three equal parts then each of the middle parts is called a *middle third*. If the corresponding extremities of the various middle thirds be joined by fair curves[1] these curves will enclose the middle third of the cross section of the dam, and if a dam is properly designed the lines of resistance should fall within this middle third whether the reservoir is empty or full.

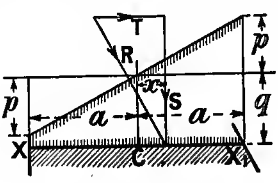


FIG. 189.

If R is the resultant force on any horizontal section XX_1 (Fig. 189) of the dam, then S the vertical component of R causes a normal stress on XX_1 and the horizontal component T causes a tangential or shear stress on XX_1.

The normal stress produced by S on XX_1 is not uniformly distributed but varies uniformly from $q - p$ at X which is furthest from S to $q + p$ at X_1.

[1] If the contour of the cross section of the dam is made up of straight lines these fair curves will become a series of straight lines.

Let C be the middle point of XX_1. Let $XX_1 = 2a$, and let the distance of S from C be x, then, still considering 1 foot length of the dam, $q = \dfrac{S}{2a}$ and $p = \dfrac{3qx}{a}$.

If $x = \dfrac{a}{3}$ then $p = q$ and there is no stress at X. If x is greater than $\dfrac{a}{3}$ then p is greater than q and there is a tension in the dam at X which should be avoided.

The maximum compressive stress $p + q$ should not exceed 7 tons per square inch.

The average weight of masonry dams is about 150 lb. per cubic foot.

Exercises VIIIa

1. The horizontal distance between the axles of a bicycle is 3 feet 6 inches, and its weight is 28 lb. Assuming that the weight on the saddle is vertically above a point 9 inches in front of the rear axle, and that a greater pressure than 200 lb. must not be brought on either wheel, find by a funicular polygon, and mark distinctly, the greatest weight the bicycle will bear. Scale of weight, 60 lb. to 1 inch. Scale of length, $\frac{1}{8}$th full size. [B.E.]

2. Using a funicular polygon, determine the resultant of the five parallel forces given in Fig. 190. The magnitudes of the forces are given in pounds. Use a force scale of 1 inch to 20 lb.

3. Determine the parallel forces, which, acting through the points L and M (Fig. 191), should balance the given forces. The magnitudes of the forces are given in pounds.

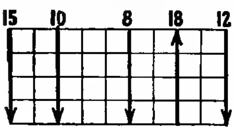


Fig. 190.

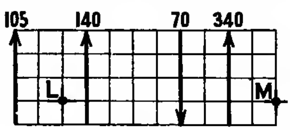


Fig. 191.

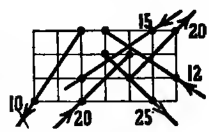


Fig. 192.

In reproducing the above diagrams take the small squares as of 0·5 inch side.

4. Six forces are given in Fig. 192, the magnitudes being in pounds. Find the resultant of these forces.

5. Find the resultant of the following three given forces, that is, find and measure r, θ, and a in the vector equation

$$R = {}_a r_\theta = {}_{1'}21_{14^\circ} + {}_{4'}33_{71^\circ} + {}_{8'}16_{115^\circ} \text{ pounds.}$$

Employ scales of $\frac{1}{2}$ inch to 1 foot, and 1 inch to 10 pounds. [B.E.]

6. Five coplanar forces P, Q, R, S, T are given as follows:—

Force.	Intercept.	Direction.	Magnitude.
P	10·5 inches	32°	65·1 lb.
Q	6·4 ,,	71°	28·5 ,,
R	2·8 ,,	153°	56·3 ,,
S	4·2 ,,	291°	15·2 ,,
T	−4·8 ,,	196°	50·1 ,,

(To explain this manner of defining a force see Art. 91, p. 91.)

Find the resultant of the given system of forces. Measure its intercept, direction, and magnitude.

Find also and measure the resultant of Q, R, and S.

Employ a linear scale of $\frac{1}{4}$ full size, and a force scale of $\frac{1}{2}$ inch to 10 lb. [B.E.]

7. Find the resultant of the forces given in Fig. 174, p. 91. The intercepts are in inches and the magnitudes of the forces are in pounds.

8. Determine r, x, and θ in the equation

$$R = {}_x r_\theta = {}_0 165_{0^\circ} + {}_{0\cdot7}355_{285^\circ} + {}_4 410_{33^\circ} + {}_{2\cdot7}323_{260^\circ} + {}_0 162_{140^\circ} + {}_{4\cdot3}205_{240^\circ}.$$

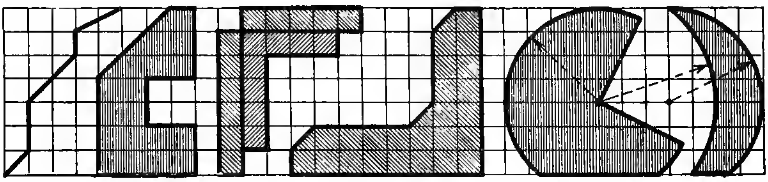
Force unit, 1 pound. Linear unit, 1 inch.

9. Given the equation of equilibrium of five coplanar forces

$${}_0 r_{90^\circ} + {}_{10} s_\theta = {}_1 200_{305^\circ} + {}_3 150_{270^\circ} + {}_8 500_{315^\circ}$$

find r, s, and θ. The unit of force is the pound and the unit of length is the foot. Employ a linear scale of $\frac{1}{2}$ inch to 1 foot, and a force scale of 1 inch to 100 pounds.

10. Find the centre of gravity of a piece of wire bent to the form shown at (*a*) Fig. 193.



(*b*) (*c*) (*d*) (*e*) (*f*)

FIG. 193.

In reproducing the above diagrams take the small squares as of 0·5 inch side.

11. Determine the centroids of the figures (*b*), (*c*), (*d*), (*e*) and (*f*) Fig. 193.

12. The triangle ABC (Fig. 194) is a diagram showing the intensity of earth pressure on a retaining wall, its area representing the total pressure P to the same scale that the area of the cross section of the wall represents the weight W of the wall. Determine the lines of action of P and W, the face AB of the wall being vertical. Find the centre of pressure on the base of the wall, this being the point where the resultant of P and W intersects the base.

Also find the centres of pressure on the sections D, E, and F. [B.E.]

13. A circle 4 inches in diameter is subjected to pressure which varies uniformly from 10 pounds per square inch at one end of a diameter to 5 pounds per square inch at the other end of that diameter. Find the centre of pressure and the magnitude of the resultant pressure.

14. A rail section is given in Fig. 195. Find the centroid of this section. Through the centroid draw a line parallel to the bottom line of the section; this is the "neutral axis" of the section. This section is subjected to stress which is normal to the section and varies uniformly from the top to the bottom, being zero at the neutral axis where it changes sign. Find the centres of stress of the portions of the section above and below the neutral axis. If the stress at the lower edge of the section is 4 tons per square inch what is the stress at the upper edge, and what are the magnitudes of the resultants of the stresses on the portions of the section above and below the neutral axis?

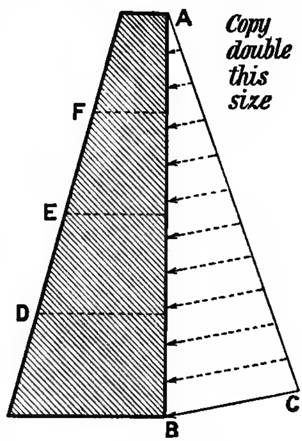


FIG. 194.

15. A section of a large dam is given in Fig. 196. The two curved faces are plotted from the vertical line shown. The weight of the masonry composing the dam is 125 lb. per cubic foot. The weight of the water may be taken as 62·5 lb. per cubic foot.

Draw the lines of resistance for this dam, (*a*) when the reservoir is empty, and (*b*) when the top surface of the water is 10 feet below the top of the dam.

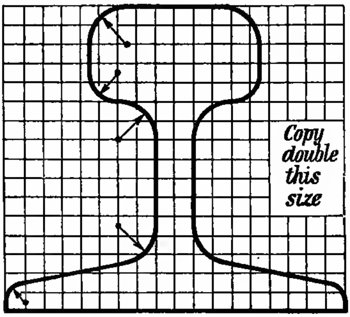


FIG. 195.

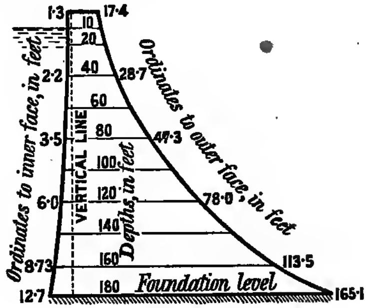


FIG. 196.

Draw also the "middle third" lines of the section of the dam, and the diagrams showing the distribution of the vertical stress on the lowest horizontal section when the reservoir is empty and when the reservoir is full.

Linear scale, say, 10 feet to 1 inch. [U.L. modified.]

99. Moment of a Force.—The moment of a force about a point or about an axis perpendicular to its line of action, is the measure of its turning power round that point or axis. The magnitude of the moment (generally called the moment) is the product of the magnitude of the force and the perpendicular distance of its line of action from the point or axis. For example, the moment of the force AB (Fig. 197) about the point M is equal to the magnitude of the force AB multiplied by MN, the perpendicular distance of M from the line AB. If the unit of force is the pound, and the unit of distance is the inch, then the unit of moment is the *inch-pound* or *pound-inch*. Other units of moment in common use are the *foot-pound* or *pound-foot*, the *foot-ton* or *ton-foot*, and the *inch-ton* or *ton-inch*.

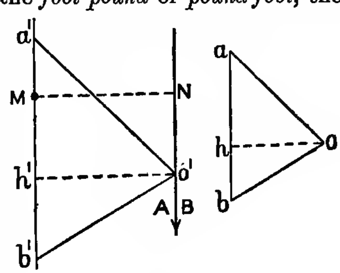


FIG. 197.

The construction shown in Fig. 197 is a very convenient one for determining graphically the moment of a force about a point. AB is the line of action of the force, and M is the point. The construction is as follows. Draw *ab* parallel to AB, and make the length of *ab* to represent the magnitude of the force. Through M draw *a*′M*b*′ parallel to AB. Choose a pole *o*. Join *oa* and *ob*. Take any point *o*′ in AB.

Draw $o'a'$ parallel to oa to meet $a'Mb'$ at a', and draw $o'b'$ parallel to ob to meet $a'Mb'$ at b'. Then $a'b'$ measured with a suitable scale will be the magnitude of the moment of the force AB about the point M.

Draw oh perpendicular to ab, and $o'h'$ perpendicular to $a'b'$. The triangles oab and $o'a'b'$ are obviously similar, and $ab : a'b' :: oh : o'h'$. Hence $ab \times o'h' = a'b' \times oh$. But ab is the magnitude of the force AB, and $o'h'$, which is equal to MN, is the perpendicular distance of M from AB. Therefore $ab \times o'h'$ is equal to the moment of AB about M, and therefore $a'b' \times oh$ is equal to the moment of AB about M.

If oh is made equal to the linear unit, then $a'b'$ measured with the force scale will give the moment required. For example, if oh is 1 inch and $a'b'$ measures 20 lb. on the force scale, then the required moment is 20 *inch-pounds*. It is not always convenient to make oh equal to the unit of distance, but it should be made a simple multiple or sub-multiple of it.

The following is the simple rule for determining the moment scale. Let oh be m times the linear unit, and let the force scale be n units of force per inch. Then the moment scale will be $m \times n$ units of moment per inch. For example, let the linear unit be *one foot*, and suppose that oh, *measured with the linear scale*, is 4 feet. Let the force scale be 100 lb. per inch, then the moment scale will be $100 \times 4 = 400$ *foot-pounds* per inch.

It may be pointed out that the figure $a'o'b'$ is the link polygon of the force AB with reference to the pole o.

The following problems are important.

(1) An unknown force P acting through the point A (Fig. 198)

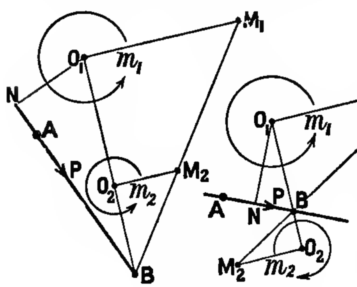


FIG. 198.

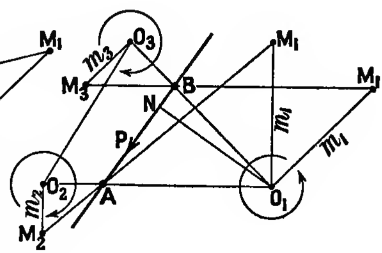


FIG. 199.

has a moment m_1 about the point O_1 and a moment m_2 about the point O_2: it is required to determine the force P.

Join O_1O_2. Draw O_1M_1 and O_2M_2 at right angles to O_1O_2. Make $O_1M_1 = m_1$ to any convenient scale and make $O_2M_2 = m_2$ to the same scale. O_1M_1 and O_2M_2 are on the same or on opposite sides of O_1O_2 according as m_1 and m_2 have the same or opposite signs. Join M_1M_2 and produce it if necessary to cut O_1O_2, or O_1O_2 produced at B. Then B is another point in the line of action of P. The sense of P is

readily determined by inspection. To find the magnitude of P, draw O_1N at right angles to AB. Then, magnitude of $P = \frac{m_1}{O_1N}$.

(2) An unknown force P has moments m_1, m_2, and m_3 about the points O_1, O_2, and O_3 (Fig. 199) respectively: it is required to determine the force P.

Draw O_1M_1 and O_2M_2 at right angles to O_1O_2. Make $O_1M_1 = m_1$ to any convenient scale, and make $O_2M_2 = m_2$ to the same scale. O_1M_1 and O_2M_2 are on the same or on opposite sides of O_1O_2 according as m_1 and m_2 have the same or opposite signs. Join M_1M_2 and produce it if necessary to cut O_1O_2, or O_1O_2 produced, at A. Then A is a point in the line of action of P. In like manner the point B is found as shown, and AB the line of action of P is determined. The sense of P is found by inspection, and the magnitude of P is equal to $\frac{m_1}{O_1N}$, where O_1N is the perpendicular from O_1 on AB.

100. Resultant Moment of a System of Forces.—The resultant moment of a system of forces about a point is equal to the algebraical sum of the moments of the separate forces about that point, and it is obvious that this sum must be equal to the moment of the resultant of the system about the same point. Hence the graphical determination of the resultant moment of a system of forces about a point resolves into constructing the resultant of the system, and the

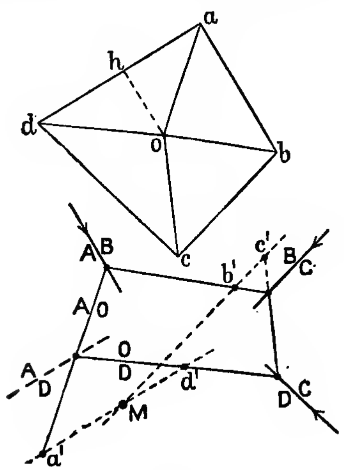


FIG. 200.

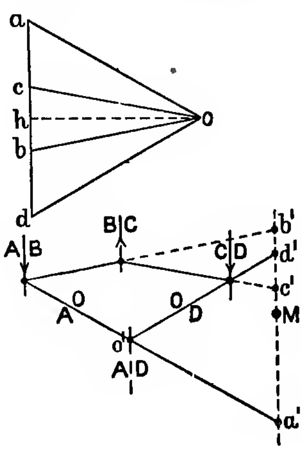


FIG. 201.

determination of the moment of this resultant about the given point by the construction of the preceding article. The two constructions may, however, be combined in one, as shown in Figs. 200 and 201. AB, BC, and CD are three given forces, and M is a given point. It

is required to determine the resultant moment of the given forces about the given point.

The polygon *abcda* is the force polygon; *ad*, the closing line, gives the magnitude and direction of the resultant of the three given forces. A pole *o* is taken at a perpendicular distance *oh* from *ad*, which is a simple multiple or sub-multiple of the linear unit. The link polygon of the forces with reference to the pole *o* is next drawn, and the intersection of the closing sides OA and OD determines a point on the line of action of the resultant force AD. A line through M parallel to *ad* intersects the closing sides OA and OD of the link polygon at a' and d'. The moment required is equal to $a'd' \times oh$. The triangle $o'a'd'$ is obviously similar to the triangle *oad*, and therefore, as shown in the preceding article, the moment of AD about M is equal to $a'd' \times oh$.

It may be observed that the moment of any one of the forces, say BC, is obtained by drawing through M a parallel to BC to intersect the sides of the link polygon which meet on BC at b' and c'; $b'c' \times oh$ is the moment of BC about M.

101. Principle of Moments.—When a number of forces acting on a rigid body are in equilibrium, then the moments of all the forces about any given axis being taken, the sum of the moments of those forces which tend to turn the body in one direction about the axis is equal to the sum of the moments of those forces which tend to turn the body in the opposite direction about the same axis.

102. Couples.—A *couple* consists of two equal parallel forces acting in opposite directions. The *arm* of a couple is the perpendicular distance between the lines of action of the two forces. The *moment* of a couple is the product of the magnitude of one of the forces and the arm of the couple. A couple tends to cause a body upon which it acts to rotate.

A couple cannot be balanced by any single force.

Two couples acting on a body will balance one another when (1) they are in the same plane or in parallel planes, (2) they have equal moments, and (3) their directions of rotation are opposite.

103. Bending Moment and Shearing Force Diagrams for Beams.—When a horizontal beam is acted on by vertical forces or loads, these forces tend to bend the beam, and the bending action at any transverse section is measured by the algebraical sum of the moments of the forces on one side of the section about a horizontal axis in that section. For example, the beams shown in Fig. 202 are acted on by forces P, Q, and R to the right of the transverse section XY, and the bending moment at XY is equal to

$$P \times l + Q \times m - R \times n.$$

The loads on a beam also tend to shear the beam transversely, and the shearing action at any transverse section is equal to the resultant of the transverse forces on one side of the section. For example, the shearing action at the section XY of the beams shown in Fig. 202 is equal to the resultant of the forces P, Q, and R which act to the right of the section, and this resultant is equal to $P + Q - R$.

The drawing of the bending moment diagram for a beam is simply the application of the construction explained in Art. 99. In Fig. 203 is shown a horizontal cantilever carrying vertical loads AB, BC, and CD. *abcd* is the line of loads, or polygon of forces. A pole *o* is chosen so that the pole distance *oh* is a simple multiple or sub-multiple of the linear unit. The link polygon $a'd'n'$ is then drawn. It is easy to show, as in Art. 100, that the bending moment at any section XY is equal to $a_1d_1 \times oh$, that is, the depth of the link polygon under the section multiplied by the pole distance. The depth of the link polygon is measured by the force scale, and the pole distance by the linear scale.

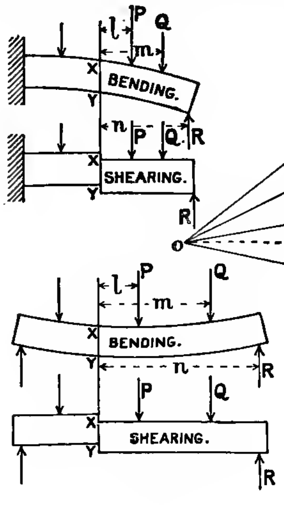


FIG. 202.

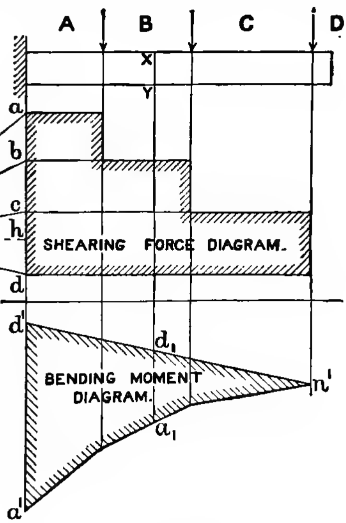


FIG. 203.

It follows that, since the pole distance is the same for all parts of the link polygon, the depth of the link polygon under any section of the beam is a measure of the bending moment on the beam at that section, the scale for measuring the bending moment being found as explained in Art. 99.

The shearing force diagram is constructed by drawing horizontals across the spaces A, B, C, and D at the levels *a*, *b*, c, and *d* respectively. The depth of this diagram under any section of the beam, measured with the force scale, gives the vertical shearing force on the beam at that section. For example, at the section XY the shearing force is the resultant of the forces to the right of XY, and is equal to $BC + CD = bc + cd = bd$.

Another example is illustrated by Fig. 204. Here a beam AB rests on supports at A and C and carries a distributed load, the intensity of which at any point is proportional to the height of the shaded diagram above AB at that point. The load diagram is divided

into a number of vertical strips of equal width as shown by the vertical dotted lines. If these strips are sufficiently numerous it will be sufficiently accurate to assume that the load which any one of these strips represents acts at the middle of its width and has a magnitude proportional to its mean height. Figures instead of letters have been used to denote the forces in Bow's notation.

The line of loads *rs* is drawn and a pole *o* chosen, and the link polygon (*a*) is determined, the closing line being *mn*. A line, shown dotted, through *o* parallel to *mn* to cut *rs* at the point 12 determines the reactions 11–12 and 12–1 at the supports.

The diagram (*a*) is the bending moment diagram. This has been redrawn at (*b*) on a horizontal base line. The heights in (*b*) and (*a*) are the same on the same vertical lines. It will be observed that the bending moment changes sign near to the line of action of the force 7–8.

The stepped diagram is the shear force diagram and is obtained by drawing the horizontal lines across the spaces between the forces and projected from the corresponding points on the line of loads. The shear force changes sign at a point between the supports, and again at the support C as shown by the part (*d*) of the shear force diagram, shown dotted. The part (*d*) has been transferred to (*d*′) so as to show the whole of the shear force diagram on the same zero base.

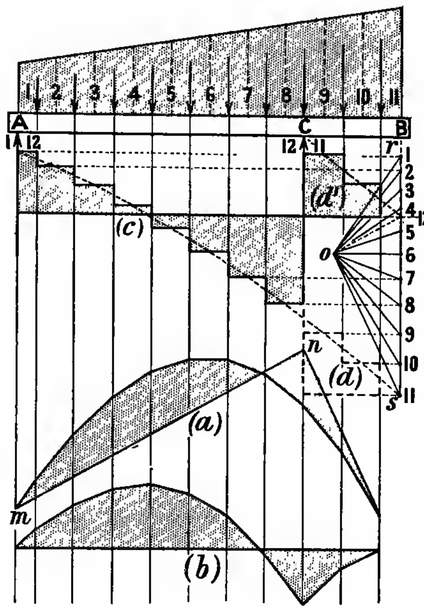


FIG. 204.

The shear forces as determined above are correct for the middle vertical lines of all the spaces between the forces except the end ones, and in the true shear force diagram the steps would be replaced by the dotted curve shown.

The link polygon is not quite exact because the forces will not act along the vertical middle lines of the strips but will act through the centroids of the strips which will be slightly to the right of the middle positions. The point 12 on the line of loads is therefore not quite exact but should be slightly higher.

If the forces be made to act through the centroids of the strips then the point 12 on the line of loads will be determined exactly. Also the link polygon then determines exactly the bending moments at

the middle vertical lines of all the spaces between the forces except the end ones, and the true bending moment diagram is obtained by drawing a fair curve through the points thus determined and continuing the curve to the verticals through A and B where the bending moments are zero.

104. Moment of Inertia.—The sum of the products of the mass of each elementary part of a body and the square of its distance from a given axis is called the *moment of inertia* of the body about that axis. Thus, if m_1, m_2, m_3, etc., be the masses of the parts of the body, and r_1, r_2, r_3, etc., be the distances of these parts respectively from the axis, then the moment of inertia $= I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 +$ etc. . . . $= \Sigma mr^2$.

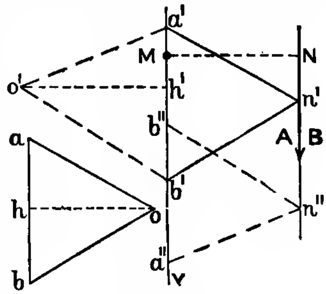


FIG. 205.

The *moment of inertia of an area* and the *moment of inertia of a line* are defined in a similar manner by substituting *area* or *length* for *mass*. But since areas and lines have no inertia, they have, strictly speaking, no moment of inertia.

The *moment of inertia of a force* about an axis perpendicular to the line of action of the force is the product of its magnitude and the square of the distance of its line of action from the axis.

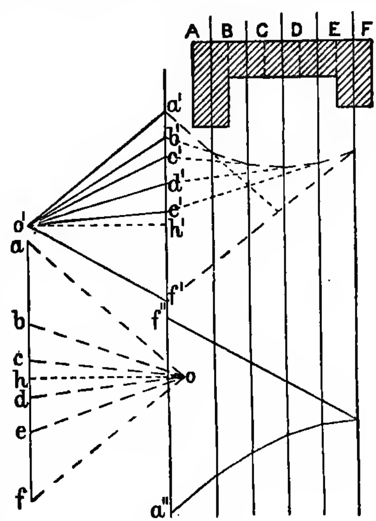


FIG. 206.

The graphic method of determining the moment of inertia of a plane area, or of a system of parallel forces, will be understood from the two examples worked out in Figs. 205 and 206.

Fig. 205 shows the application of the method to finding the moment of inertia of a force AB about a point M or about an axis through M and perpendicular to the plane of the paper. Through M draw MY parallel to AB. Draw MN perpendicular to AB. Applying the construction explained in Art. 59, $a'b' \times oh = AB \times MN$. Choose a pole o' at a distance $o'h'$ from $a'b'$, which is a simple multiple or submultiple of the linear unit. From a point n'' in AB draw $n''a''$ parallel to $o'a'$ and $n''b''$ parallel to $o'b'$. Since the triangle $a''b''n''$ is similar to the triangle

$a'b'o'$, it follows that $a''b'' \times o'h' = a'b' \times MN$, and therefore $a''b'' \times o'h' \times oh = a'b' \times oh \times MN$. But $a'b' \times oh = AB \times MN$. Therefore $a''b'' \times o'h' \times oh = AB \times MN^2 =$ moment of inertia of AB about M. $a'b'n'$ and $a''b''n''$ are link polygons, of which the first determines the moment AB × MN, and the second determines the moment of this moment, namely, (AB × MN) × MN. The lengths $a'b'$ and $a''b''$ must be measured with the force scale, and the lengths oh and $o'h'$ with the linear scale.

Fig. 206 shows the application of the method to the determination of the moment of inertia of the shaded figure about an axis $a'a''$ in the plane of the figure. The area is divided into parallel strips, and parallel forces AB, BC, CD, DE, and EF are supposed to act at the centres of gravity of these strips, the magnitudes of the forces being proportional to the areas of the strips. The sum of the moments of these forces about the given axis is equal to $a'f' \times oh$, and the sum of their moments of inertia is equal to $a''f'' \times o'h' \times oh$. The lengths $a'f'$ and $a''f''$ must be measured with the area scale and the lengths oh and $o'h'$ with the linear scale.

Exercises VIIIb

1. Three coplanar forces P, Q, and R (Fig. 207) act on a rigid body which is capable of rotation about an axis through O at right angles to the plane of the forces. P = 8 lb., Q = 10 lb., and R = 6 lb. Find the resultant moment of these forces about O. Determine the magnitude and sense of the force S which will prevent the rotation of the body.

2. AB (Fig. 208) is a lever whose fulcrum is at A. Forces P and Q act on the lever as shown. P = 10 lb., and Q = 15 lb. Rotation of the lever about its fulcrum is prevented by the force R. Find the magnitude of the force R and determine the reaction of the fulcrum on the lever.

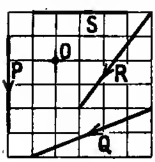


FIG. 207.

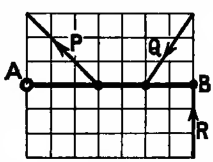


FIG. 208.

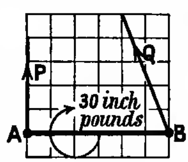


FIG. 209.

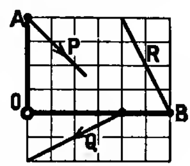


FIG. 210.

Reproduce the above diagrams three times this size and then consider that the reproduced diagrams are drawn to the scale of ¼ full size.

3. A bar AB (Fig. 209) is acted on by forces P and Q as shown. P = 20 lb., and Q = 25 lb. A couple whose moment is 30 inch-pounds also acts on the bar in the same plane as P and Q and tends to give clockwise rotation to the bar. Find the additional force acting on the bar which will produce equilibrium.

4. A bent lever AOB (Fig. 210) is capable of rotation about a pin at O. The lever is at rest under the action of the forces P, Q, and R, and the reaction S of the pin on the lever. P = 15 lb., and Q = 20 lb. Find the magnitude and sense of R and determine S.

5. Draw a square ABCD of 2 inches side, the lettering being in the clockwise direction. A force P acts through A in the plane of the square. P has a clockwise moment of 11 inch-pounds about B and an anti-clockwise moment of 19 inch-pounds about D. Determine the force P.

Another force Q also acts through A in the plane of the square. Q has a

clockwise moment of 24 inch-pounds about B and a clockwise moment of 8 inch-pounds about C. Determine the force Q.

6. ABC is a triangle. AB = 8 inches, BC = 9 inches, and CA = 10 inches. The lettering is clockwise. A force R acting in the plane of the triangle has a clockwise moment of 40 inch-pounds about A, an anti-clockwise moment of 30 inch-pounds about B, and a clockwise moment of 12 inch-pounds about C. Determine the force R, using a linear scale of $\frac{1}{4}$.

7. ABCD is a square of 2 inches side, lettered clockwise. A force P acts along the side AB and another force Q acts along the diagonal AC. The resultant moment of the forces P and Q about D is 10·4 inch-pounds, anti-clockwise. The resultant moment of P and Q about the middle point of BC is 25·5 inch-pounds, clockwise. Determine the forces P and Q.

8. The cantilever (Fig. 211) carries five loads each of 100 lb. as shown. Draw the bending moment and shear force diagrams.

9. The cantilever (Fig. 212) is loaded as shown, the loads being in tons. Draw the bending moment and shear force diagrams.

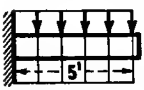


Fig. 211.

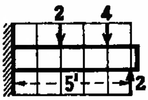


Fig. 212.

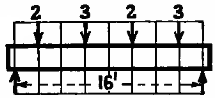


Fig. 213.

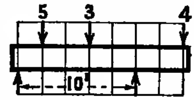


Fig. 214.

10. The beam (Fig. 213) rests on supports 16 feet apart and is loaded as shown, the loads being in tons. Draw the bending moment and shear force diagrams.

11. The beam (Fig. 214) rests on supports 10 feet apart and is loaded as shown, the loads being in tons. Draw the bending moment and shear force diagrams.

12. A beam is built firmly into a wall at one end and projects 24 feet from the face of that wall. The other end rests freely on a support. The beam carries a uniformly distributed load of 1 ton per foot run. It may be proved that the reaction at the support is three-eighths of the total load on the beam. Draw the bending moment and shear force diagrams.

13. A beam (Fig. 215) rests on supports 16 feet apart. The load at A is 2 tons and each of the other loads is 1 ton. Draw the bending moment and shear force diagrams.

14. A log of timber 24 feet long and of uniform cross section floats in still water. On the top of the log at points 6 feet from its ends are placed loads of 200 lb. each. Draw the bending moment and shear force diagrams.

[Note that the bending moments and shear forces on the log are those due to the given loads and a uniform upward water pressure whose total is 400 lb.]

15. The beam A (Fig. 216) rests freely on two cantilevers B and C as shown.

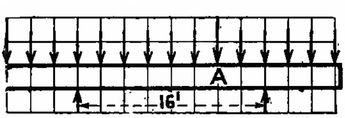


Fig. 215.

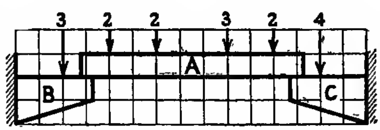


Fig. 216.

The given loads are in tons. Draw the diagrams of bending moment and shear force for this structure.

16. A beam rests on two supports 12 feet apart. This beam carries a distributed load which varies uniformly in intensity from nothing over one support to a maximum over the other support. The total load is 15 tons. Draw the bending moment and shear force diagrams.

17. Determine the moment of inertia of each of the figures given in Fig. 217 about a horizontal axis through its centroid.

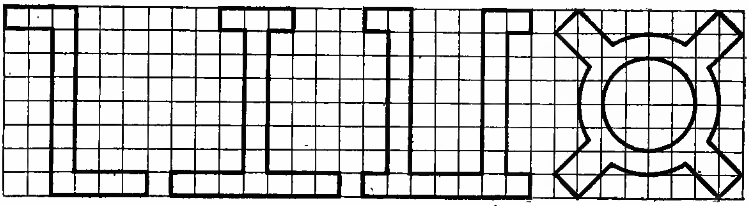


FIG. 217.

In reproducing the above figures take the small squares as of half inch side.

105. Stress Diagrams for Framed Structures.—It will be assumed that the framed structures considered are made up of bars which are connected by frictionless pin joints at their ends. It will also be assumed that the loads on the structure are concentrated at the joints. If a bar carries a load uniformly distributed over its length this load is divided into two equal parts, and one part is placed at each end of the bar. If a bar carries a load concentrated at an intermediate point, this load is divided into two parts, which are to one another as the distances of the load from the ends of the bar; these parts are then placed one at each end of the bar, the greater part being at that end of the bar which is nearest to the original load.

In studying the equilibrium of a structure, two kinds of forces have to be considered, (1) the external forces, which for the whole structure must balance one another, and (2) the internal forces. As a consequence of the two assumptions mentioned at the beginning of this article, the bars forming the structure are subjected either to direct compression or to direct tension under the action of the external forces. It follows, therefore, that the lines of action of the internal forces are the lines which represent the bars on the diagram of the structure (called the *frame diagram*). At any joint, therefore, the forces acting are the internal forces acting along the bars meeting at that joint, and the external forces, if there are any, acting at that joint.

If a sufficient number of the forces acting at any joint are known, the polygon of forces for that joint can be drawn and the unknown forces determined.

The general method of drawing the complete stress diagram for a framed structure will be understood by reference to the example worked out in Fig. 218. A simple roof truss is shown carrying a load AB at its apex. The other external forces are the reactions BC and CA at the supports. The internal forces are the forces acting along the bars AD, BD, and CD. The lines of action of all the forces are known, but AB is the only force whose magnitude is known as yet.

At each joint there are three forces acting, and the polygon of forces

for each joint is therefore a triangle. The triangle of forces for the joint **2** or for the joint **3** cannot yet be drawn, because the magnitudes of all the forces at these joints are as yet unknown, but the triangle of forces for the joint **1** may be drawn, and this is shown at (m). This triangle determines the magnitudes bd and da of the internal forces in the bars BD and DA respectively. The sense of these forces is also determined, and it will be observed that the internal forces in the bars BD and DA both act towards the joint **1**, therefore these bars are in compression. In drawing the triangle (m) the forces have been taken in the order in which they occur in going round the joint **1** in the watch-hand direction, beginning with the known force AB. Beginning with BA, and going round the joint in the opposite direc-

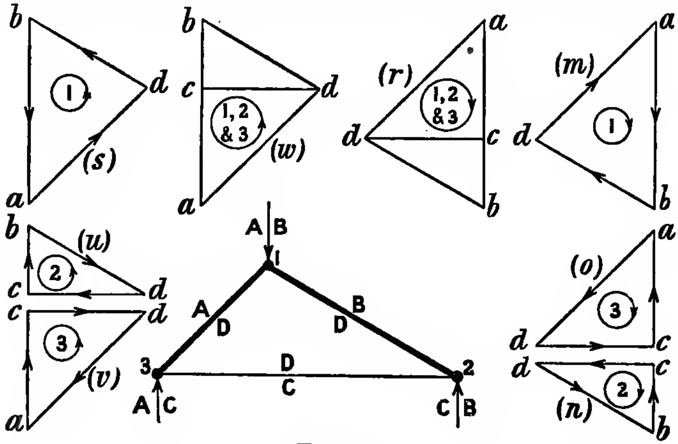


FIG. 218.

tion, the triangle (s), which is similar to (m) but differently situated, is obtained.

Passing next to the joint **2**, the three forces acting there are known in direction, and the magnitude of one of them, BD, has been determined by the drawing of the triangle (m) or the triangle (s). Beginning with DB, and taking the forces in the order in which they occur in going round the joint in the watch-hand direction, the triangle of forces (n) is drawn. If the forces be taken in the order in which they occur in going round the joint in the opposite direction, beginning with BD, the triangle (u) is obtained. Proceeding next to the joint **3**, the triangle (o) is obtained when the forces are taken in the watch-hand order, and the triangle (v) is obtained when the forces are taken in the opposite order.

The construction of the three triangles (m), (n), and (o), or the three triangles (s), (u), and (v), determines the magnitude and sense of each of the three internal forces, and also the magnitudes and sense of the external forces BC and CA.

It is obvious that the triangles (n) and (o) may be applied to the triangle (m) so as to form the figure (r), and this figure gives all the results which were found from the separate triangles (m), (n), and (o), and this figure (r) is the complete stress diagram for the given framed structure. The figure (r) may of course be drawn at once without drawing the triangles (m), (n), and (o). It should, however, be noticed that in order that the force polygons for the different joints may be combined into one diagram, these polygons must be drawn by taking the forces in the order in which they occur in going round each joint in the *same direction.* (r) is the form of the stress diagram when the forces are taken in the order in which they occur when going round each joint in the watch-hand direction, and (w) is the form of the diagram when the order is reversed.

It is important to observe that in going from the joint at one end of a bar to the joint at the other end the sense of the force in that bar is reversed, because if a bar is in tension it must exert a pull at each of the joints at its ends and if it is in compression it must exert a thrust at each of the joints at its ends.

The sense of the force in any bar may be determined from the polygon of forces for the joint at either end and from the sense of one of the forces represented by one side of that polygon because the polygon of forces represents forces which are in equilibrium, and if arrow heads be placed on the sides of the polygon to show the senses of the forces these arrow heads must follow one another round the polygon.

106. Examples of Stress Diagrams.—(1) The frame diagram for a "saw-tooth" roof truss is shown at (u), Fig. 219. The truss is loaded at the joints as shown, the loads being in pounds.

The first step is to draw the line of loads $ab \ldots g$. To determine the reactions GH and HA at the supports, which are obviously vertical, choose a pole o and draw the link polygon shown at (v). The closing line of this polygon is shown dotted. From the pole o draw oh parallel to the closing line of the link polygon to meet the line of loads at h, then gh and ha are the magnitudes of the reactions GH and HA respectively.

Commencing at the joint ABKHA at the left-hand support the polygon of forces *abkha* is completed. Proceeding to the joint BCLKB the polygon of forces *bclkb* is completed. Proceeding from joint to joint round the frame diagram the various polygons of forces are completed, the whole of the polygons of forces making up the complete force diagram for the truss as shown at (w).

The magnitudes of the forces in the various bars may now be scaled off from the diagram (w). The senses of the forces at any joint are determined by inspecting the polygon of forces for that joint and the sense of the force in a bar at a joint determines whether that bar is in tension or compression. Lines on the frame diagram which represent members in compression have been made bold.

(2) The lower part of Fig. 220 shows a roof truss under two systems of loads. One system is vertical and is due to the weight of

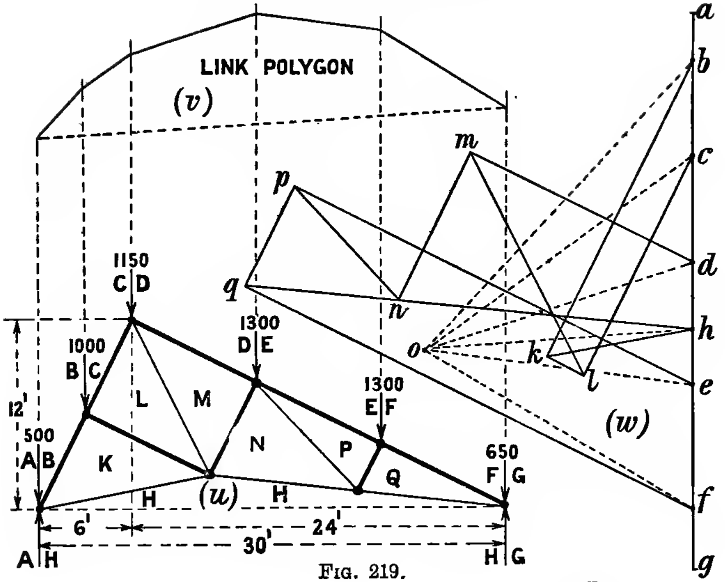


FIG. 219.

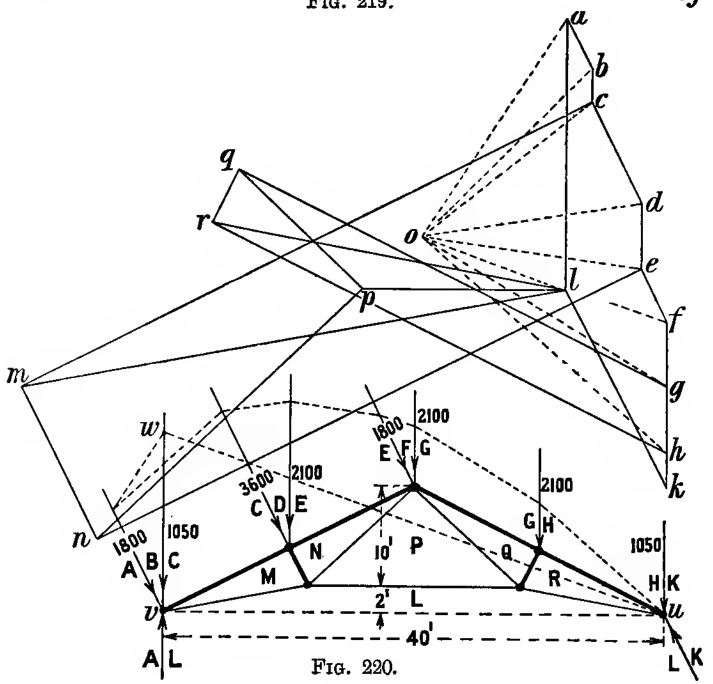


FIG. 220.

the roof covering. The other system is at right angles to the left-hand surface of the roof, and is due to wind pressure. All the loads are in pounds. The truss is supposed to be fixed at the right-hand support *u* but at the left-hand end *v* it is merely supported; hence the line of action of the reaction at *v* must be vertical. The direction of the reaction at *u* is unknown and the magnitudes of both reactions are unknown.

The part *abcdefghk* of the polygon of external forces acting on the truss is first drawn. The next step is to determine the reactions KL and LA at the supports.

Choose a pole *o* and draw the link polygon shown by dotted lines on the frame diagram, *starting at the point u.* By starting at the point *u* the space K, which is as yet not completely defined, is eliminated. The closing line of the link polygon is *wu*. A line through the pole *o* parallel to *wu* to meet the vertical through *a* determines the point *l* and therefore also *lk*. This completes the polygon of external forces.

Starting at the joint at one of the supports the complete stress diagram may now be drawn as in the preceding example.

107. The Method of Sections.—A little more than one half of a roof truss is shown in Fig. 221. Conceive that this truss is divided into two parts by a plane of section XX which cuts three bars FG, GH, and HA. Next suppose that the part of the structure to the right of XX is removed and that external forces are applied at the sections of the bars cut so as to balance the internal forces in these bars. These external forces will take the place of the forces exerted on the part of the structure to the left of XX through the parts of the cut bars which have been removed. The part of the structure to the left of XX will now be in equilibrium under the action of the given system of loads and the three external forces FG, GH, and HA. It will now be shown how these three applied external forces may be found. The determination of these three forces is evidently the determination of the forces in three of the members of the original structure.

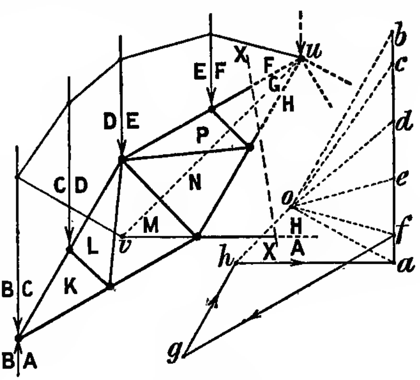


FIG. 221.

Draw the line of loads *abcdef.* To complete the polygon of external forces on the structure to the left of XX requires the locating of the points *g* and *h*. Choose a pole *o* and, *starting at the point u,* where the lines of action of two of the unknown external forces intersect, draw the link polygon of which *vu* is the closing line. A line through *o* parallel to *vu* to meet a line through *a* parallel to AH determines the point *h*. The intersection of *hg* parallel to HG and *fg*

parallel to FG determines the point g. The senses of the external forces FG, GH, and HA are found at once by going round the completed polygon of external forces in the direction fixed by the sense of any one of the known external forces. It will be seen that the bar FG is in compression while the bars GH and HA are in tension.

This method of sections is extremely useful, for it may be applied to determine directly the forces in three members of a structure without drawing the stress diagram for any other part of the structure.

In drawing the stress diagram for the truss, a part of which is shown in Fig. 221, the polygon of external forces would be first drawn, then the stress diagram would be built up polygon by polygon commencing at one of the supports where there are only two forces whose magnitudes have to be determined. In passing from joint to joint if a joint is arrived at where the magnitudes of *three* forces have to be determined the polygon of forces for that joint becomes indeterminate. This happens when the joint AMNHA or the joint DEPNMLD is reached. If however a section be taken as shown in Fig. 221 and the force AH determined as explained above the polygon of forces for the joint AMNHA may be completed and the stress diagram may then be proceeded with in the ordinary way.

108. Deflections of Braced Frames.—The bars which make up a braced frame are generally in tension or compression and the forces acting along the bars may be found as described in the preceding articles of this chapter. The bars have then to be proportioned to resist these forces. The pull or push on a bar will cause it to lengthen or shorten by an amount x and will produce a stress f in it. If E is the modulus of elasticity of the material in tension or compression and l is the length of the bar, then $E = \dfrac{\text{stress}}{\text{strain}} = \dfrac{f}{x/l} = \dfrac{fl}{x}$, and $x = \dfrac{fl}{E}$.

For steel E may be taken as 30,000,000 pounds per square inch and if f is, say, 15,000 pounds per square inch, then $x = \dfrac{15{,}000l}{30{,}000{,}000} = \dfrac{l}{2000}$, that is the bar will lengthen or shorten by an amount which is the 1–2000th part of its length. A bar 100 inches long would therefore lengthen or shorten 0·05 inch under these conditions.

It is outside the scope of this work to discuss further the design of structures or the determination of the strains produced in their members by given systems of loading, but it will now be shown how the deflections of braced frames may be determined when the alterations in the lengths of the members due to loading are given.

The lower part of Fig. 222 shows the frame diagram of a pent-roof truss loaded at the joints B, C, and D. It is given that the various bars of the truss alter in length by the amounts stated on the lines

which represent the bars. + denotes increase and − denotes decrease in length. These alterations in length are so small that the change in the form of the truss due to the loads cannot be clearly shown on an ordinary scale drawing of the truss. The displacements of the various joints may however be determined as follows.

The joints A and E are supposed to be fixed, that is these joints are not displaced when the truss is loaded. Choose an origin O and draw horizontal and vertical axes OX and OY. O is the origin and OX and OY are the axes of a displacement diagram which is to be drawn to a large scale. In Fig. 222 the displacement scale is five times full size, but on a drawing board the displacement scale would not be less than ten times full size.

Consider the joint B. Through O draw Ob_1 parallel to AB and make $Ob_1 = 0{\cdot}08$ inch on the displacement scale. Draw Ob_2 parallel to BE and make $Ob_2 = 0{\cdot}06$ inch on the displacement scale. Draw b_1b perpendicular to Ob_1 and b_2b perpendicular to Ob_2 and let these perpendiculars meet at b. If O is the original position of B then b is its new position when the bar AB has lengthened 0·08 inch and the bar BE has shortened 0·06 inch.

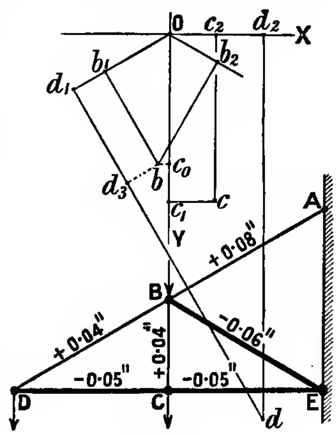


FIG. 222.

To be strictly accurate b_1b should be an arc of a circle whose centre is in b_1O produced at a distance from O equal to the length of AB measured on the displacement scale; but the radius of this arc is so large compared with the length of the arc that the arc is practically a straight line at right angles to b_1O. So also b_2b should be an arc of a circle whose centre is in Ob_2 produced at a distance from O equal to the length of BE measured on the displacement scale.

Consider next the joint C. C has a downward displacement whose vertical component is the vertical component of the displacement of B plus the displacement 0·04 inch due to the lengthening of the bar BC. The former is represented by Oc_0 and the latter by c_0c_1. The horizontal component of the displacement is 0·05 inch due to the shortening of the bar CE and this is represented by Oc_2. The horizontal c_1c to meet the vertical c_2c determines c the new position of C on the assumption that O was its original position.

Lastly consider the joint D. D has a displacement whose component in the direction BD is made up of the component of the displacement of B in that direction, that is Ob_1, and the amount 0·04 inch by which the bar BD lengthens and which is represented by b_1d_1. The total component of the displacement of D in the direction

BD is therefore Od_1. The horizontal component of the displacement of D is made up of the horizontal component of the displacement of C, that is Oc_2, and 0·05 inch the amount by which the bar CD shortens and which is represented by c_2d_2. The total horizontal component of the displacement of D is therefore Od_2. d_1d perpendicular to Od_1 meeting the vertical through d_2 determines d the new position of D assuming that its original position was O.

The angular deflection of any bar is measured by the ratio y/l where y is the relative displacement of the ends at right angles to the bar and l is the length of the bar. For example, consider the bar BD. The component of B's displacement at right angles to BD is bb_1 or d_3d_1 where bd_3 is parallel to Od_1. The component of D's displacement at right angles to BD is dd_1. Hence dd_3 is the relative displacement of D and B at right angles to BD, and the angular displacement of BD is $dd_3 \div$ BD. If BD is 14 feet and dd_3 is 0·292 inch, the angular displacement θ is $\frac{0 \cdot 292}{14 \times 12} = 0 \cdot 00174$. Whether this be taken as the circular measure, or the sine, or the tangent of the angle, the angle being very small, the angle is 6 minutes or 1-10th of a degree.

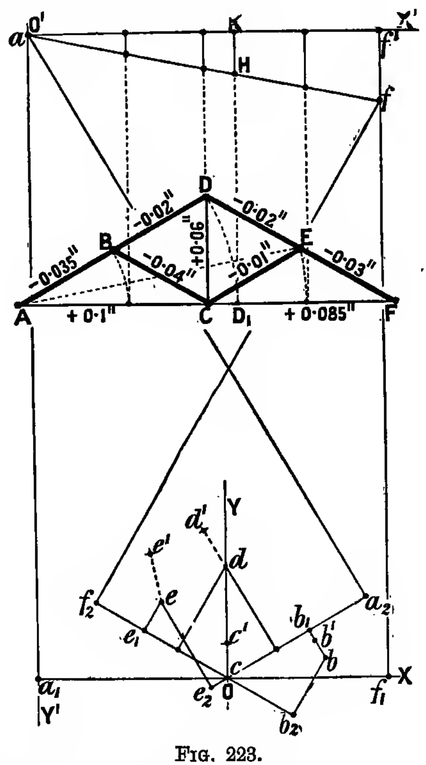


FIG. 223.

In the foregoing example one member AE (Fig. 222) was supposed to be fixed and to remain of the same length. An example will now be considered in which all the members are displaced. Fig. 223 shows a roof truss ADF in which the joint A is fixed and the joint F is free to move horizontally. The truss supports dead loads and in addition a wind load on the left-hand side. These loads cause alterations in the lengths of the bars which are stated on the lines which represent them.

Assume in the first place that, say, the bar CD remains vertical and that C is fixed in position. Choose an origin O and axes OX and OY and determine the displacements of the various joints, as in the previous example, the original position of each joint

being taken as at O. Then on the assumption that C is fixed and CD remains vertical, A suffers a horizontal displacement Oa_1 to the left and an upward vertical displacement a_1a. Also F suffers a horizontal displacement Of_1 to the right and an upward vertical displacement f_1f. But under the actual conditions A is fixed. Let, therefore, the whole truss be now lowered an amount aa_1, or, which is the same thing, let the axis OX be raised to O′X′ the level of a. Let, also, the whole truss be moved to the right a distance equal to Oa_1, or, which is the same thing, let the axis OY be moved to the left so as to pass through a. The axes are now O′X′ and O′Y′ and a which represents the position of the joint A is at the origin O′. It will be seen that the joint F has now a downward vertical displacement equal to $f'f$. To bring F to its original level, which is the level of A, the whole truss must now be rotated about A until F rises through a distance equal to ff'. This angular movement will cause the various joints to travel at right angles to the lines joining them to A distances proportional to the lengths of these lines.

Consider the joint D. Referred to the origin O′ and the axes O′X′ and O′Y′, d shows the displacement of D when A is fixed and F is at the distance $f'f$ below its proper level. It is now required to find the final displacement of D when the truss is turned about A to bring F to its proper level. With A as centre and AD as radius describe the arc DD_1 to cut the horizontal AF at D_1. Draw the vertical D_1HK to cut $O'f$ at H and $O'f'$ at K. Then HK is the amount that D will travel at right angles to DA when the truss is turned about A to raise F the distance ff'. Draw dd' perpendicular to AD and make dd' equal to HK, then d' shows the final displacement of D referred to the origin O′ and axes O′X′ and O′Y′. In like manner the points b', c', and e' are found.

109. The Three-Hinged Arch.—If the ends of a roof or bridge truss are secured to foundations by hinged joints, and there is another hinged joint at an intermediate point, say, at the middle of the truss, such a truss is known as a *three-hinged arch*, and it is said to be constructed on the *three-hinged system*. The determination of the stresses in the various bars of such a truss may be proceeded with as in an ordinary truss as soon as the reactions at the hinges are determined.

One method of finding the reactions at the hinges is as follows. Fig. 224 shows a truss hinged at A, B, and C. The resultant load on the part AB is the force P, and the resultant load on the part BC is the force Q. First neglect the load acting on the part BC. The part BC is then under the action of two forces only, namely, the reactions at B and C, and these forces must balance one another, and will therefore act in opposite directions along the straight line BC. The truss as a whole is now under the action of three forces, namely, the force P, the reaction T_1 at C, which acts along CB, and the reaction S_1 at A. Since these three forces are in equilibrium, and since the lines of action of two of them, T_1 and P, meet at m, therefore the line of action of the third one, S_1, must be Am. By means of the triangle of forces the magnitudes of S_1 and T_1 can be determined.

Next neglect the load on the part AC, and consider the load Q on the part BC. This load Q will cause reactions S_2 and T_2 at A and B respectively, and these reactions may be found in the same way as S_1 and T_1 were found.

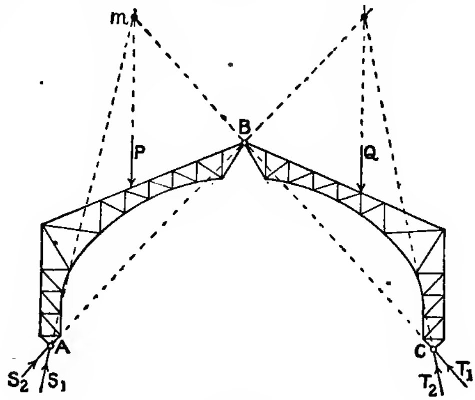


FIG. 224.

When both loads P and Q act, it is evident that the reaction at A will be the resultant of S_1 and S_2, and the reaction at B will be the resultant of T_1 and T_2.

The reaction of the part AB on the part CB at B will be the force which will balance the force Q and the reaction at C, and the reaction of the part CB on the part AB at B will be the force which will balance the force P and the reaction at A. These two reactions will, of course, be equal and opposite.

When the truss is symmetrical about a vertical centre line, and is symmetrically loaded, the reactions at B will be horizontal, and the line of action of the reaction at A will be the line joining A with the point of intersection of the line of action of the resultant load on the half truss AB with the horizontal line through B. The direction of the reaction at C is found in like manner.

The two parts of a three-hinged arch may be considered as two girders or two beams loaded obliquely. When these beams are solid or built up of plates and angles instead of being open brace work they are called *arched ribs.*

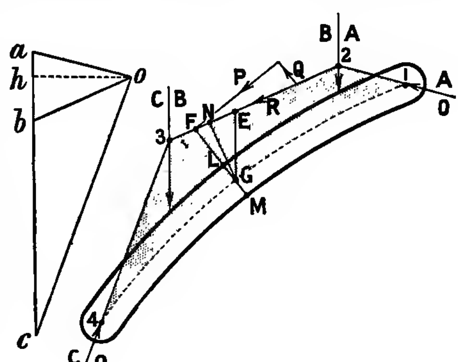


FIG. 225.

An arched rib of a three-hinged arch is shown in Fig. 225, the joints being at 1 and 4. This rib carries vertical loads AB and BC. The reactions CO and OA at the joints may be determined in the manner already explained, and *oabc*, the polygon of external forces, may then be drawn. If the link polygon 1 2 3 4 for the external forces be drawn so as to pass through the joints 1 and 4 this polygon

is called the *linear arch* for the rib. The sides of the linear arch are the lines of action of the thrusts which the rib has to support. The magnitudes of these thrusts are given by *oa*, *ob*, and *oc*. It will be observed that each of these thrusts has a horizontal component whose magnitude is given by *oh*, the perpendicular from *o* on *abc*.

Consider the normal section LM of the rib. Let G be the centroid of this section. The thrust R on LM has the magnitude *ob* and since R does not pass through G there is a bending moment on the rib at LM whose magnitude is R × GN, where GN is the perpendicular from G on **2 3**. Draw the vertical GE to meet **2 3** at E. Comparing the triangles GNE and *ohb* it will be seen that they are similar, and that GE : GN : : *ob* : *oh*, or GE × *oh* = *ob* × GN = R × GN. But R × GN is the bending moment at the section LM, therefore GE × *oh* is also the bending moment at LM. Hence the vertical intercept between the locus of G and the link polygon **1 2 3 4** at any point on that locus is a measure of the bending moment at that point, since *oh* is constant. The shaded figure is therefore the bending moment diagram for the rib.

Let **2 3** meet ML produced at F. Resolve R into components P and Q perpendicular and parallel respectively to LM. It is easily seen that P × GF = R × GN; therefore the bending moment on the rib at G is equal to P × GF.

In addition to the bending moment on the rib at the section LM there is a shear force equal to Q and a normal thrust whose resultant passes through G and is equal to P

Exercises VIIIc

In working the following exercises on forces in framed structures it is not sufficient to draw the force diagrams. The forces should be scaled off from the force diagrams and the results stated on the lines of the frame diagram. The lines of the frame diagram which represent bars which are in compression should be lined in considerably thicker than the others after the forces have been determined.

1. A crane frame is shown at (*a*) Fig. 226. The external forces are, the load

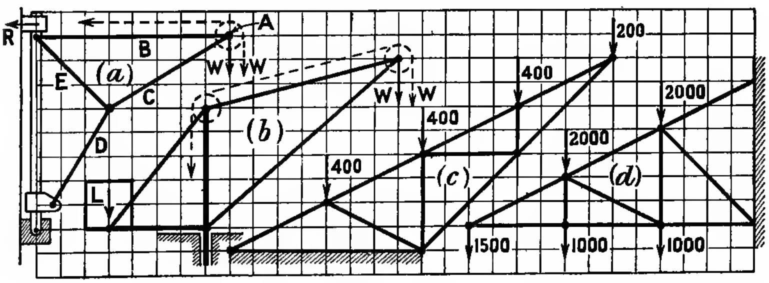


FIG. 226.

W of 2 tons, the horizontal force R and the reaction at the footstep at the lower end of the post. The lower end of the post is hemispherical and the reaction there may be assumed to pass through the centre of the sphere, the axis of the

lower sloping member also passes, when produced, through that point. Draw the frame diagram four times the given size and then determine the reaction at the footstep and the forces in the bars B, C, D, and E, (1), assuming that the load W is hung directly from the joint A, and, (2) assuming that the load W is suspended by a chain which passes over a pulley at A as shown by the dotted lines.

[Note that when a chain passes over a pulley the resultant force on the axle of the pulley is the resultant of two forces each equal to the tension in the chain and acting along the straight parts of the chain which proceed from the pulley.]

2. A crane frame is shown at (*b*) Fig. 226. The load W is 5 tons. Draw the frame diagram, say, three times the given size. Determine the balance weight L so that there shall be no bending action on the lower part of the crane post. Determine the forces in the various members of the frame, (1) assuming that the load W is applied directly at the point of the jib, (2) assuming that the load W is suspended by a chain which passes over two equal pulleys as shown by the dotted lines. [See note to preceding exercise.]

3. A projecting truss for the roof of an open shed is shown at (*c*) Fig. 226. The truss is loaded as shown, the loads being in pounds. Draw the frame diagram, say, three times the given size and then determine the forces in the various members.

4. A pent roof truss is shown at (*d*) Fig. 226. The truss is loaded as shown, the loads being in pounds. Draw the frame diagram three times the given size and then determine the forces in the various members.

5. The given roof truss (Fig. 227) is loaded as shown. Find and measure the

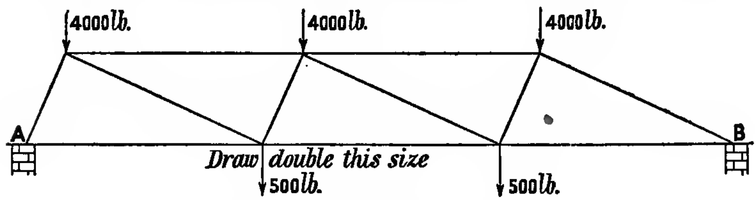


Fig. 227.

vertical supporting forces at A and B. Determine the thrust in each of the two upper horizontal members of the frame, writing the answers on the bars themselves. [B.E.]

6. A curb roof truss is shown at (*a*) Fig. 228. The truss is loaded as shown,

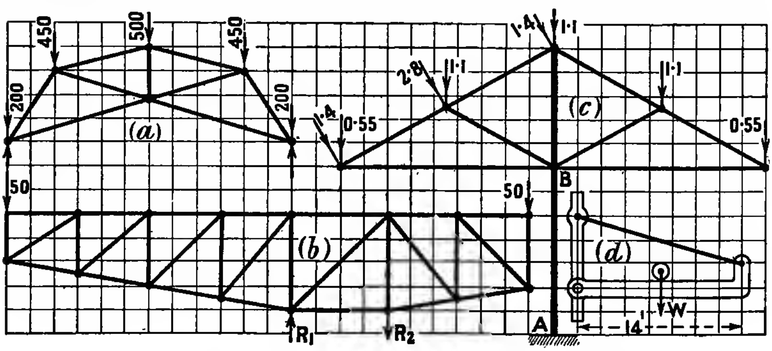


Fig. 228.

the loads being in pounds. Draw the frame diagram three times the given size and then determine the forces in the various members.

7. The diagram (*b*) Fig. 228 represents a girder supported and anchored down by vertical forces R_1 and R_2 respectively. The girder carries loads of 50 tons at each end as shown. Draw the frame diagram three times the given size and then determine the forces R_1 and R_2 and the forces in the various members.

8. At (*c*) Fig. 228 is shown the truss for the roof of an island platform of a railway station. The loads (in tons) due to the roof covering and wind pressure are given. Draw the frame diagram double the given size and determine the forces in the members of the frame.

The scale of the figure, when drawn double the given size, being $\frac{1}{8}$ inch to 1 foot, what are the bending moments on the supporting pillar at A and at B? [B.E.]

9. In the crane shown at (*d*) Fig. 228 the horizontal member has a pin joint at the left-hand end, and is supported at the other end by an inclined tie bar as shown. The load W of 1 ton is suspended from the middle of the horizontal member as shown. Draw the frame diagram to the scale of $\frac{1}{2}$ inch to 1 foot and then determine, (1) the magnitude and line of action of the resultant shear force on the pin at the left-hand end of the horizontal member, (2) the tension in the tie bar, (3) the thrust in the horizontal member, (4) the total bending moment at the middle of the horizontal member.

10. The diagram, Fig. 229, represents a swivel bridge, its pivot placed at AA.

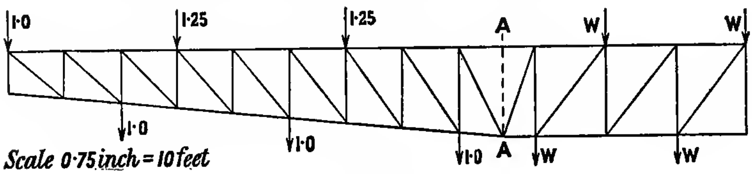


FIG. 229.

The loads, in tons, are placed as indicated. Draw the diagram double the given size. Determine the equal loads on the counterpoise to the right of AA placed as shown, then find the forces in the various members of the bridge. [B.E.]

11. You are given in Fig. 230 a bowstring truss under the action of wind

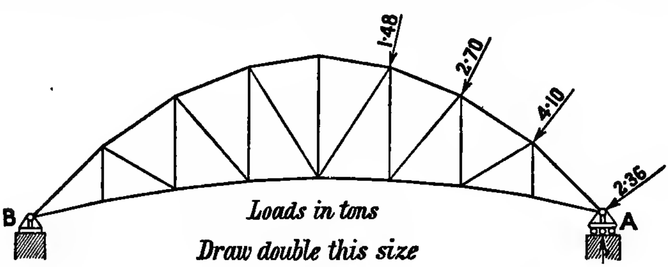


FIG. 230.

forces, and supported at the ends A and B. Determine the supporting forces at the ends, that at A acting in a vertical line. Also draw the force diagram or reciprocal figure for the truss. [B.E.]

12. Determine the reactions at the supports and draw the stress diagram for the truss of example (2) p. 115 with the same loads except that the wind forces act on the right-hand instead of on the left-hand side of the truss.

13. The given crane (Fig. 231) is set out to a scale of 1 cm. to 3 feet. A load W causes the bar AD to lengthen, and the three bars radiating from C to shorten, by the amounts written on the bars. Assuming AB to be rigid, find the vertical

component of the deflection of D. Find also the angular deflection of CD. [B.E.]

14. A crane has the form shown in Fig. 232. A load W causes the bars to alter in length by the amounts written thereon, the minus sign denoting shortening. The joint A being fixed and the joint B being free to move horizontally, find the deflection of D and measure its vertical and horizontal components.

What is the angular deflection of the bar CD, if its actual length is 100 inches? [B.E.]

15. Forces, not shown, acting on the given roof truss (Fig. 233), cause the

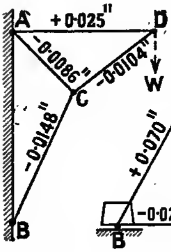


FIG. 231.

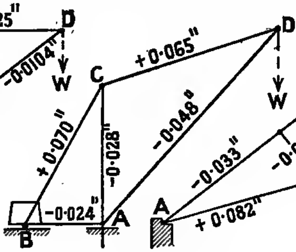


FIG. 232.

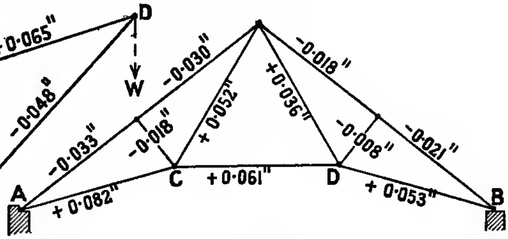


FIG. 233.

bars to alter in length by the amounts written thereon, the minus sign denoting shortening. If the end A is fixed, and B is free to move horizontally, find the displacement of B.

What is the angular displacement of the horizontal bar CD if its actual length is 110 inches? [B.E.]

16. The diagram (Fig. 234) shows a simple roof truss, resting on walls and loaded as shown, the span being 15 feet and the inclination of the rafters 39°.

(*a*) Find and measure the horizontal thrust on the walls.

(*b*) Draw diagrams showing the thrust, shearing force and bending moment throughout the length of one of the rafters. Measure the maximum values of these quantities. [B.E.]

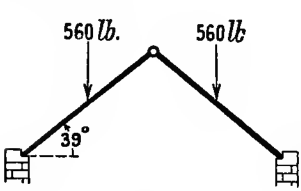


FIG. 234.

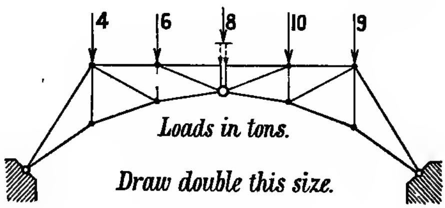


FIG. 235.

17. A braced arch, hinged at the crown and at each springing, is loaded as shown in Fig. 235. Determine the horizontal thrust on the abutments. Find the forces in all the members of the structure. What is the shearing force on the pin at the crown? [B.E.]

18. An arched rib in the form of a circular arc ACB, is hinged at each end A, B, and at the crown C. The span AB is 150 feet, and the rise 30 feet. Draw the curve of the arch to a scale of ½ inch to 10 feet.

A load of 10 tons passes over the arch. Confine your attention to one position only of this load, that for which its horizontal distance from A is 50 feet. Determine and measure the horizontal thrust of the arch in tons. Draw a diagram of bending moment on the rib, and measure the maximum bending moment in ton-feet. Determine also and measure the greatest thrust and the

greatest shear in the rib, and state the horizontal distances from A of the places where these occur. [B.E.]

19. The form of an arched rib is a circular arc of 100 feet span and 16 feet rise, the supports being at the same level. It is hinged at the ends and loaded with a weight of 12 tons at a horizontal distance of 30 feet from one end. The horizontal thrust of the arch is known to be 12·5 tons.

Draw a diagram of bending moment for the arch. Indicate the places where the shearing force, thrust, and bending moment on the rib have their maximum values, and give these values. [B.E.]

20. A flying buttress has the form and weight shown in Fig. 236. It is sub-

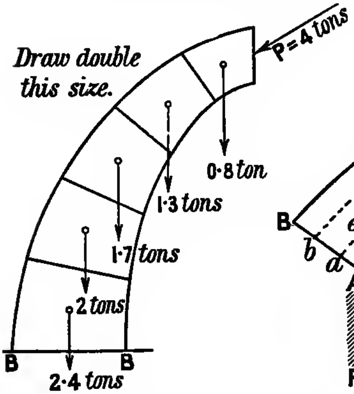


FIG. 236.

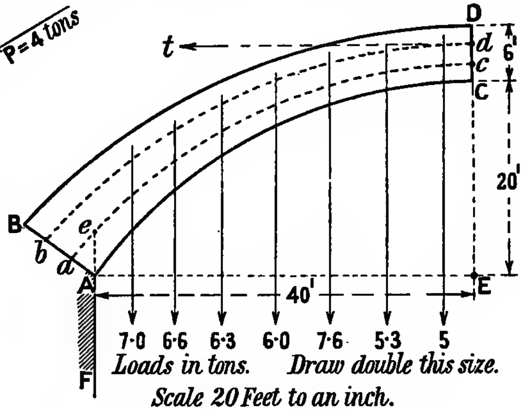


FIG. 237.

ject to a thrust P as indicated. Show the lines of the forces acting across the several joints. State the magnitude of the force on the bed joint BB. Adopt a force scale of ½ inch to 1 ton. [B.E.]

21. ABDC, Fig. 237, represents the section of a half-arch. The span of the half-arch is 40 feet, rise 20 feet, thickness of key-stone 6 feet, thickness of arch-ring at abutment, AB, 9 feet.

The loads on the half-arch are supposed concentrated as shown.

The dotted arcs *bd* and *ac* represent the limits of safety, and are at one-third of the thickness of the arch-ring from the extrados and intrados respectively. The horizontal thrust at the crown is assumed to be along the line *dt*, and the curve of thrust is further assumed to pass through the point *e* where the vertical through the abutment FA meets the dotted arc *ac*. Determine and draw the curve of thrust between the points *d* and *e*, by means of a funicular polygon, and give the value of the horizontal thrust at *d*.

Scale of lengths, 10 feet to an inch.

Scale of loads, 10 tons to an inch. [B.E.]

CHAPTER IX

PLANE CO-ORDINATE GEOMETRY

110. Co-ordinates of a Point.—The position of a point P in a plane (the plane of the paper) may be fixed by giving its distances x and y from two fixed intersecting straight lines OX and OY (Figs. 238 and 239), the distances x and y being measured parallel to OX and OY respectively as shown.

The fixed lines OX and OY are called the *axes*. The axes are of unlimited length and each extends both ways from O. The point O where the axes intersect is called the *origin*. The distance x is called the *abscissa* and the distance y the *ordinate* of the point P, and these two distances together are called the *co-ordinates* of the point P. A point P whose co-ordinates are x and y may be referred to as the point x,y.

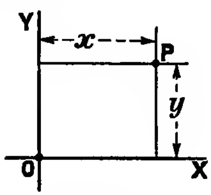


FIG. 238.

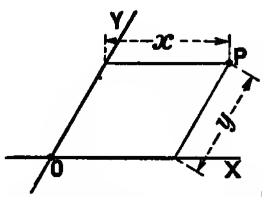


FIG. 239.

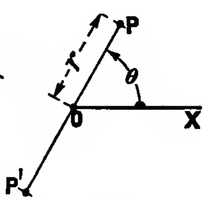


FIG. 240.

When the axes are at right angles to one another (Fig. 238) x and y are the *rectangular co-ordinates* of the point P, and when the angle between the axes is not a right angle (Fig. 239) x and y are the *oblique co-ordinates* of the point P.

The abscissa x is positive $(+)$ or negative $(-)$ according as the point P is to the right or left respectively of the axis OY, and the ordinate y is positive or negative according as the point P is above or below the axis OX.

The position of a point P in a plane (the plane of the paper) may be fixed in the manner illustrated by Fig. 240. O is a fixed point called the *pole*, and OX is a fixed straight line called the *initial line*, or *line of reference*. The point P is joined to O and the line OP whose length is r is called the *radius vector*. The angle θ which OP makes with OX, measured from OX in the anti-clockwise direction is called the *vectorial angle*. r and θ are called the *polar* co-ordinates of

the point P, and these co-ordinates fix the position of the point P. As defined above r and θ are both positive. If θ is measured in the clockwise direction from OX then it is negative. The angle θ whether it is positive or negative fixes the position of the positive radius vector OP. If the radius vector is negative then produce PO to P′ and make OP′ equal to OP, then OP being the positive radius vector, OP′ is the negative radius vector. A point P whose polar co-ordinates are r and θ may be referred to as the point r,θ.

Rectangular co-ordinates are the most common in practical problems and when co-ordinates are referred to, without qualification, rectangular co-ordinates will be understood.

111. The Straight Line.—AB (Figs. 241 and 242) is a straight line which intersects the axis OY at C. Let OC = c. Take any point P in AB. Draw the ordinate PM and through C draw CL parallel to OX to meet PM at L. Then $\frac{PL}{CL}$ is a measure of the slope of AB to OX, and since the slope of AB is independent of the position of the point P, $\frac{PL}{CL}$ = constant = m, say. But PL = PM − LM = PM − OC = $y - c$, and CL = x. Therefore $\frac{y-c}{x} = m$, or $y = mx + c$, and this is the equation to the straight line AB. The meaning of this equation is that, the constants m and c being known, the co-ordinates of every point in the straight line satisfy the equation. The distance OC = c is called the *intercept* of AB on the axis of y.

For the line shown in Fig. 241, $m = \frac{1}{2}$ and $c = 5$, hence the equation to AB is $y = \frac{1}{2}x + 5$.

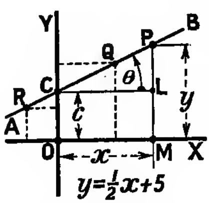


FIG. 241.

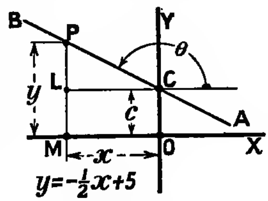


FIG. 242.

It will be found that for the point Q, $x = 6$, and $y = 8$. Inserting these values in the equation, $8 = \frac{1}{2} \times 6 + 5 = 8$, and the equation is satisfied. Again, for the point R, $x = -3$, and $y = 3\frac{1}{2}$. Inserting these values in the equation, $3\frac{1}{2} = -\frac{1}{2} \times 3 + 5 = -1\frac{1}{2} + 5 = 3\frac{1}{2}$, and the equation is again satisfied, and so it will be found for every point in AB.

The line AB (Fig. 241) is called the *graph* of the equation $y = \frac{1}{2}x + 5$. Conversely if any pair of values of x and y which satisfy the equation $y = \frac{1}{2}x + 5$ be taken as the co-ordinates of a point, and this point be plotted, all such points will lie in the straight line AB which is the graph of the equation.

An equation containing the variables x and y in the first power only is called an *equation of the first degree*. The general form of the equation of the first degree, with two variables, is $Ax + By + C = 0$, where A, B, and C are constants which are finite or zero and may be positive or negative. It may be proved that the graph of an equation

of the first degree is a straight line, and since a straight line is fixed by two points in it, the graph of such an equation may be drawn as soon as two values of x and the two corresponding values of y are known. For example, take the equation $-2x + 3y - 9 = 0$. Put $x = 0$, then $y = 3$. Put $x = 10$, then $y = 9\frac{2}{3}$. Plotting the points $x = 0$, $y = 3$, and $x = 10$, $y = 9\frac{2}{3}$, and joining them determines the graph of the equation $-2x + 3y - 9 = 0$.

If a straight line is parallel to OX and at a distance b from it, then for every point in the line $y = b$, and this is the equation to the line. The distance b is positive or negative according as the line is

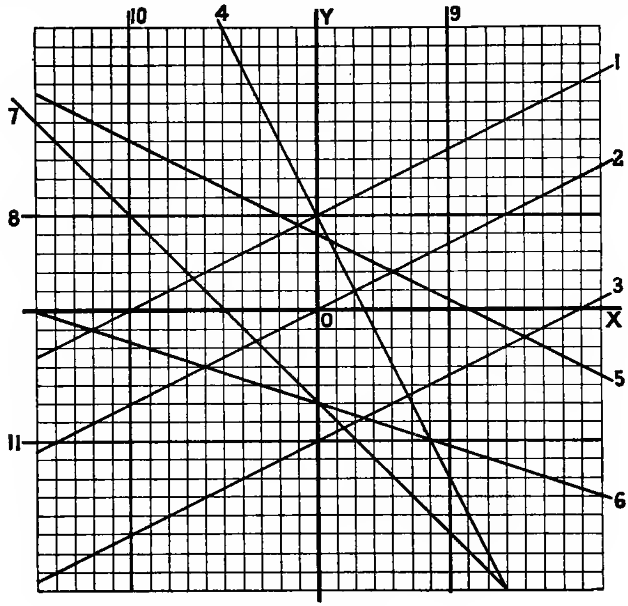


FIG. 243.

above or below OX. If a straight line is parallel to OY and at a distance a from it, then for every point in the line $x = a$, and this is the equation to the line. The distance a is positive or negative according as the line is to the right or left of OY.

For problems in co-ordinate geometry "squared paper" will be found very convenient. Fig. 243 represents a piece of squared paper on which are drawn eleven straight lines numbered 1 to 11.

The equations to the lines shown in Fig. 243 are given below and the student should carefully study these, comparing them with their graphs. The unit used is the length of side of the small squares.

1. $y = \frac{1}{2}x + 5$, or $2y = x + 10$.
2. $y = \frac{1}{2}x + 0$, or $2y = x$.
3. $y = \frac{1}{2}x - 7$, or $2y = x - 14$.
4. $y = -2x + 5$, or $2x + y = 5$.

5. $y = -\frac{1}{2}x + 4$, or $x + 2y = 8$.
6. $y = -\frac{1}{3}x - 5$, or $x + 3y + 15 = 0$.
7. $y = -x - 5$, or $x + y + 5 = 0$.
8. $y = 5$, or $y - 5 = 0$.
9. $x = 7$, or $x - 7 = 0$.
10. $x = -10$, or $x + 10 = 0$.
11. $y = -7$, or $y + 7 = 0$.

Comparing the lines 1, 2, and 3 it will be seen that they are parallel, and when their equations are written in the form $y = mx + c$, the coefficient of x is the same for each line. Line 4 is perpendicular to lines 1, 2, and 3, and the "m" for 4 is the reciprocal of the "m" for 1, 2, and 3, with the sign changed.

If the units for x and y are represented by the same length on the graph of the equation $y = mx + c$, then m is the tangent of θ (Figs. 241 and 242) the inclination of AB to the axis of x, θ being measured in the anti-clockwise direction. But in many practical problems in co-ordinate geometry the co-ordinates x and y represent different kinds of quantities and even when they represent the same kind of quantity the scales used for measuring them on the graphs may be different, and the slope of the line which is the graph of the equation $y = mx+c$, where m is a definite number, may be very different in different cases depending on the scales used for measuring x and y. These remarks will be better understood after studying the following two examples.

Example 1.—In a certain machine for raising weights the effort Q, in pounds, at the driving point required to raise a load W, in pounds, was found to be given by the equation $Q = 0{\cdot}2W + 4$. Since Q is much smaller than W a more satisfactory graph of the equation is obtained by using a larger scale for Q than for W. Fig. 244 shows a graph of the equation with the scale for Q five times that for W. Observe that to find m or $\frac{PL}{CL}$ from the graph, PL must be measured on the effort scale and CL on the load scale. On the effort scale PL = 80 and on the load scale CL = 400. Therefore $\frac{PL}{CL} = \frac{80}{400} = 0{\cdot}2$ as in the equation given.

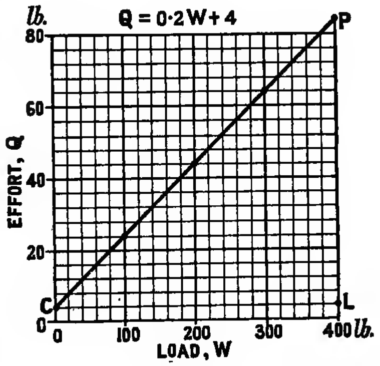


Fig. 244.

Example 2.—The horse-power, H, of a steam engine was varied by altering the initial steam pressure. The weight of steam, W, in pounds, used by the engine per hour, was measured at different powers and the results were as given in the table.

H	8	16	26	34	40
W	150	303	454	548	650

Plotting H horizontally

to a scale of 1 inch to 20 horse-power, and W vertically to a scale of 1 inch to 500 pounds, the results are shown by the dots in Fig. 245. It will be seen that these dots are approximately in a straight line, and that the straight line passing through the points $H = 0$, $W = 50$, and $H = 40$, $W = 650$, lies evenly amongst the dots. Hence the relation between W and H is given approximately by an equation of the form $W = mH + c$, where $c = 50$, and $m = \frac{PL}{CL} = \frac{600}{40} = 15.$ Observe that c and PL are measured on the W scale and that CL is measured on the H scale.

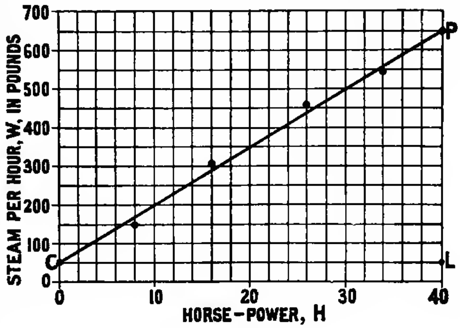


Fig. 245.

Judgment is required in choosing the scales for x and y in plotting. In general the scales should be such that, in the case of a straight line, the line should be inclined to the axis of x at an angle lying between 30° and 60°.

In co-ordinate geometry when the part of a line or a figure is being dealt with which is at considerable distances from the axes, a new pair of axes may be drawn nearer to the figure and parallel to the original axes which may now be off the paper. The new origin will however not be the zero point for x or y. For example if the new axis of y is 50 units to the right of the original axis of y the new origin will represent 50 on the axis of x.

112. The Circle.—Referring to Fig. 246, C is the centre of a circle, the co-ordinates of C being a and b. P is any point on the circumference of the circle, the co-ordinates of P being x and y. PM and CN are the perpendiculars from P and C on OX. CL is parallel to OX and meets PM at L. Let CP, the radius of the circle, be denoted by r.

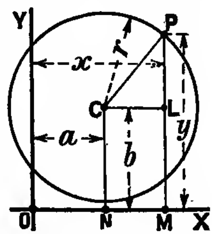


Fig. 246.

In the right-angled triangle CPL, $CL^2 + PL^2 = CP^2$.

But $\quad CL = NM = OM - ON = x - a,$

$\quad PL = PM - LM = y - b,$ and $CP = r$.

Therefore $\quad (x-a)^2 + (y-b)^2 = r^2$

or $\quad x^2 + y^2 - 2ax - 2ay + a^2 + b^2 - r^2 = 0.$

This is the equation to the circle. r must always be positive, but a and b may be either positive or negative.

Conversely the graph of an equation of the form

$$x^2 + y^2 + Ax + By + C = 0$$

is a circle whose centre has for co-ordinates, $a = \frac{A}{2}$ and $b = \frac{B}{2}$, and whose radius is $r = \frac{1}{2}\sqrt{A^2 + B^2 - 4C}$.

113. The Parabola.—OK (Fig. 247) is a fixed straight line, and F is a fixed point. P is a point which moves in the plane of F and OK, so that its distance from F is always equal to its distance from OK. The path of P is a *parabola*, whose axis is the line through F perpendicular to OK. The line OK is called the *directrix*, and the point F the *focus* of the parabola. The curve cuts the axis at A, the *vertex* of the parabola. FA is equal to AO.

Draw PK perpendicular to OK, and PN perpendicular to the axis. Draw the tangent to the parabola at A, and let it meet PK at K′. The tangent at A is obviously perpendicular to the axis of the parabola.

Let $\quad FA = a$, $PN = x$, and $PK' = y$.

Then $\quad PN^2 + FN^2 = PF^2 = ON^2$.

That is, $\quad x^2 + (y - a)^2 = (y + a)^2$.

Therefore $\quad x^2 = 4ay \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

which is the equation to the parabola referred to the axis of the parabola and the tangent at the vertex, the axis being the axis of y, and the tangent the axis of x

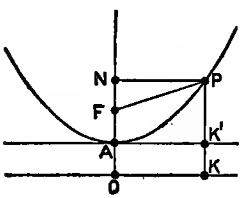


Fig. 247.

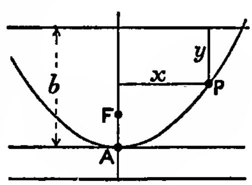


Fig. 248.

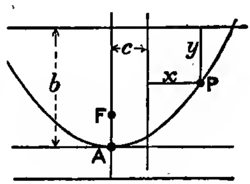


Fig. 249.

If the axis of x be moved parallel to itself until it is at a distance b from the vertex (Fig. 248), then y in (1) will become $y - b$, and the new equation will be

$$x^2 = 4a(y - b) = 4ay - 4ab \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the axis of y be moved parallel to itself until it is at a distance c from the axis of the parabola (Fig. 249), then x in (2) will become $x - c$, and the new equation will be

$$(x - c)^2 = 4a(y - b) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the axes of x and y be interchanged then x and y in the above equations will change places.

114. The Ellipse.—The equations to the ellipse are not of great importance in practical geometry.

If the origin O be taken at C (Fig. 250) the centre of the ellipse, and if the major and minor axes be taken as the axes of x and y respectively and if the semi-major axis be denoted by a and the

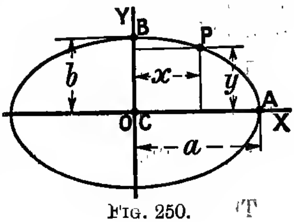


Fig. 250.

semi-minor axis by b, then the equation to the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, or $b^2x^2+a^2y^2=a^2b^2$.

If $a=b=r$, then $x^2+y^2=r^2$, which is the equation to the circle when the origin is at its centre.

115. The Hyperbola.—Let a denote the semi-transverse axis and b the semi-conjugate axis of an hyperbola. Let the origin O be taken at C (Fig. 251) the centre of the hyperbola, and let the transverse and conjugate axes be taken as the axes of x and y respectively.

If x and y are the co-ordinates of any point P on the hyperbola, then $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, or $b^2x^2-a^2y^2=a^2b^2$, and this is the equation to the hyperbola.

If x and y are the co-ordinates of any point Q on the conjugate hyperbola, then

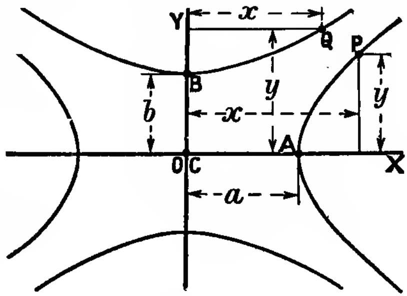


FIG. 251.

$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$, or $a^2y^2-b^2x^2=a^2b^2$, and this is the equation to the conjugate hyperbola.

If $b=a$, the hyperbola is equilateral or rectangular, then $x^2-y^2=a^2$ for the hyperbola and $y^2-x^2=a^2$ for the conjugate hyperbola. The rectangular hyperbola and its conjugate are identical but differently situated. In the rectangular hyperbola the asymptotes are at right angles to one another.

An important equation to the hyperbola is that obtained by taking the asymptotes as the axes of x and y as shown in Figs. 252 and 253, which show one branch only of the curve. OA the semi-transverse axis bisects the angle XOY and the curve is symmetrical about OA produced. In Fig. 252 the asymptotes are oblique while in Fig. 253 they are at right angles to one another. The hyperbola in Fig. 253 is therefore rectangular.

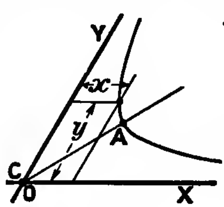


FIG. 252.

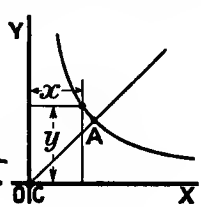


FIG. 253.

For Fig. 252, $xy=\frac{a^2+b^2}{4}=$ constant, where a and b are the semi-axes as before.

In Fig. 253, $b=a$ and $xy=\frac{a^2}{2}=$ constant.

The curve $xy=\frac{a^2}{2}=$ constant (Fig. 253) is an important one. In

the form PV = constant, the co-ordinates of this curve give the relation between the pressure and volume of a gas when it expands or is compressed isothermally, that is, at constant temperature.

A simple construction for finding any number of points on the curve $xy =$ constant when one point is given is described in Art. 50, p. 47.

116. The Curve $y = x^n$.—Whatever be the value of n in the equation $y = x^n$ if $x = 0$, then $y = 0$. Therefore the curve whose equation is $y = x^n$ passes through the origin.

Let $n = 1$, then $y = x$ which is the equation to a straight line.

Let $n = 2$, then $y = x^2$ which is the equation to a parabola. Whether x is positive or negative its square is always positive, therefore y is always positive and the curve lies entirely above the axis of x. Also since $x = \pm\sqrt{y}$ the two values of x corresponding to any given value of y are numerically equal, but one value is positive and the other negative. The curve is therefore symmetrical about the axis of y. Also if $x = +1$, $y = +1$, and if $x = -1$, $y = +1$.

Let $n = 3$, then $y = x^3$. If x is positive x^3 is positive, and if x is negative x^3 is negative. Hence the sign of y is the same as that of x, and the curve lies in the first and third quadrants. If $x = +1$, $y = +1$, and if $x = -1$, $y = -1$.

To plot the curve $y = x^n$ for any given value of n, first select values of x and calculate the corresponding values of y and tabulate as here shown for $n = 2$ and $n = 3$, then plot.

$x =$	0·0	±0·4	±0·8	±1·0	±1·2	±1·4
$n = 2, y =$	0·0	+0·16	+0·64	+1·0	+1·44	+1·96
$n = 3, y =$	0·0	±0·064	±0·512	±1·0	±1·728	±2·744

Fig. 254 shows the curves (1) $y = x$, (2) $y = x^2$, and (3) $y = x^3$. The curve $y = x^3$ is known as the *cubic parabola.*

In Fig. 254, the scale for y is the same as that for x and the true forms of the curves are shown. It will be seen that when n is greater than 1 y increases more rapidly than x, and for comparatively small values of x, y becomes very large when $n = 3$. To show a sufficient amount of the curve within a reasonable area when n is greater than 1 it is generally necessary to make the scale for y smaller than that for x. This will alter the shape of the curve without altering its character. This will be referred to again when considering the curve $y = bx^n$.

Considering further the influence of n on the form of the curve $y = x^n$, it is evident that so long as n is a positive integer, y will have the same sign as x when n is odd, and y will always be positive when n is even whether x is positive or negative. Calling the curves having n odd the *odd curves* and those having n even the *even curves* it follows that all the odd curves lie in the first and third quadrants while all the even curves lie in the first and second quadrants. Also,

all the odd curves pass through the points $(+1, +1)$ and $(-1, -1)$, that is through the points P and R, Fig. 254, and all the even curves pass through the points $(+1, +1)$ and $(-1, +1)$, that is through the points P and Q, Fig. 254.

It will be seen that for a given value of y, greater than ± 1 as n increases the curve moves nearer to the axis of y; but for a given value of y less than ± 1 the curve moves further from the axis of y as n increases.

Generally in practical problems only positive values of x and y have to be considered, and these relate to the curve in the first quadrant.

When n is positive but not a whole number, values of y corresponding to assumed values of x in the equation $y = x^n$ are found by using logarithms as follows. If $y = x^n$ then $\log y = n \log x$. For example, to plot the curve $y = x^{1 \cdot 3}$, $\log y = 1 \cdot 3 \log x$. Assume values for x, calculate and tabulate as follows.

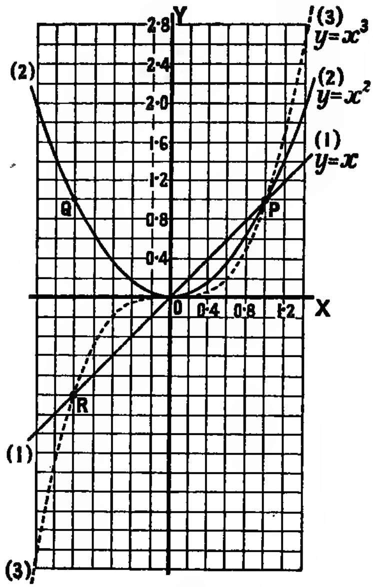


FIG. 254.

$x =$	1	2	3	4	5	6
$\log x =$	0	0·3010	0·4771	0·6021	0·6990	0·7782
$\log y = 1 \cdot 3 \log x =$	0	0·3913	0·6202	0·7827	0·9087	1·0117
$y =$	1	2·46	4·17	6·06	8·10	10·27

The curve is shown plotted in Fig. 255, and for comparison the curves $y = x$, and $y = x^2$ are also shown.

117. The Curve $y = bx^n$.—Let $y' = x^n$, then $y = by'$. Hence if the curve $y' = x^n$ be plotted, the curve $y = by' = bx^n$ will be obtained by enlarging (or reducing if b is a fraction) the ordinates of the curve $y' = x^n$ b times. This is evidently equivalent to drawing the curve $y' = x^n$ with another scale for the ordinates. The factor b may therefore be called a scale factor since it does not alter the character of the curve.

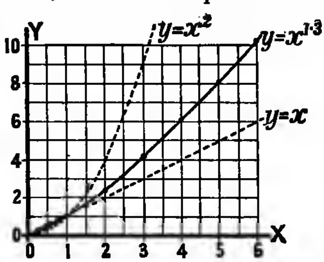


FIG. 255.

118. The Curve $y = a + bx^n$.—Let $y' = bx^n$, then $y = a + y'$. Hence if the curve $y' = bx^n$ be plotted, the curve $y = a + y' = a + bx^n$ will be obtained by raising (lowering if a is negative) the curve bodily through a height equal to a, or more simply, by lowering (raising if a is negative) the axis of x a distance equal to a.

An important problem of frequent occurrence is to find the equation to a given curve which may have been obtained by plotting a number of observations or the results of an experiment. Let AEF (Fig. 256) be such a curve referred to the axes OX and OY. An inspection of the curve suggests that it may be of the form $y = a + bx^n$. The value of a is found at once and is equal to OA which in this case is 80.

Take A as a new origin O′ and let y' denote the ordinate of any point on the curve, measured from O′X′; then $y' = y - a = y - 80 = bx^n$.

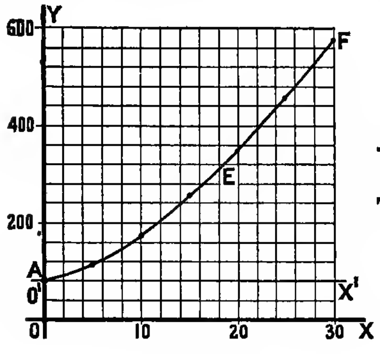


FIG. 256.

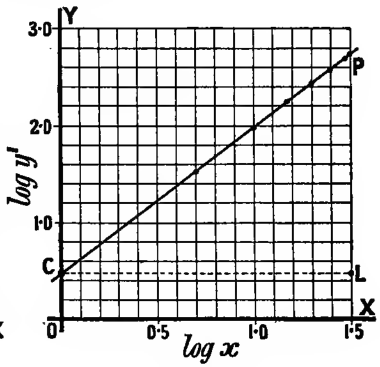


FIG. 257.

Values of x and y are given in the following table.

x	5	10	15	20	25	30
y	114	175	255	348	455	574
$y' = y - 80$	34	95	175	268	375	494
$\log x$	0·6990	1·0000	1·1761	1·3010	1·3979	1·4771
$\log y'$	1·5315	1·9777	2·2430	2·4281	2·5740	2·6937

If the curve is of the form $y' = bx^n$, then $\log y' = \log b + n \log x$.

Let $\log y' = y''$, $\log b = c$, and $\log x = x'$, then $y'' = c + nx'$, which is the equation to a straight line.

Values of y' corresponding to the given values of y are inserted in the table, also the values of $\log x$ and $\log y'$, taken from a table of logarithms.

Plotting the values of $\log x$ and $\log y'$, as shown in Fig. 257, it is

found that the points lie in a straight line for which $n = \frac{PL}{CL} = 1{\cdot}5$, PL being measured on the log y' scale and CL on the log x scale. $c = OC = 0{\cdot}477$. Hence log $b = 0{\cdot}477$, and the equation to the curve is, $y = 80 + 3x^{1{\cdot}5}$.

If the points (log x, log y') are not exactly in a straight line, a mean line may be drawn amongst them, from which n and log b may be taken to get an approximate equation to the given curve.

119. The Curve $yx^n = c$.—The most important application of the equation $yx^n = c$ is to the curve whose co-ordinates give the relation between the pressure and volume of a gas which expands or is compressed. If p is the pressure of a gas when its volume is v, then when the gas expands or is compressed the pressure and volume both vary and in general the relation between them is given by an equation of the form $pv^n = c$. In the case where $n = 1$, $pv = c$, and the curve is a rectangular hyperbola which was considered in Art. 115, p. 134.

In studying the curve $pv^n = c$ only positive values of p, v, and n need be considered. If $v = 0$, then $v^n = 0$, and in order that $p \times 0$ may have the finite value c, p must be infinite. Again, if $p = 0$, then v^n must be infinite, and v must also be infinite if n is finite. Hence the curve $pv^n = c$ approaches nearer and nearer to the axes but never actually meets them, or it meets them at infinite distances from the origin. The axes are therefore asymptotes to the curve.

As an example consider the curve $pv^{1{\cdot}3} = 292$. Taking logarithms of both sides of the equation,

$$\log p + 1{\cdot}3 \log v = \log 292.$$

Let $p = 60$, then

$$\log v = \frac{\log 292 - \log 60}{1{\cdot}3} = \frac{2{\cdot}4654 - 1{\cdot}7782}{1{\cdot}3} = 0{\cdot}5286,$$

and $v = 3{\cdot}38$.

Let $v = 4$, then

$\log p = \log 292 - 1{\cdot}3 \log 4 = 2{\cdot}4654 - 1{\cdot}3 \times 0{\cdot}6021 = 1{\cdot}6827$, and $p = 48{\cdot}2$.

In like manner for any given or assumed value of v or p corresponding values of p or v may be calculated.

Various corresponding values of v and p are given in the following table, and these are plotted in Fig. 258, giving the curve ABD.

v	3·38	4	5	7	10	15
p	60·0	48·2	36·0	23·3	14·6	8·6

An important practical problem is to determine whether a given curve is of the form $pv^n = c$, and if so to find the values of n and c. If $pv^n = c$ then $\log p + n \log v = \log c$, which is the equation to a straight line.

Let EFH (Fig. 258) be a given curve which appears to be of the form $pv^n = c$. Corresponding values of v and p for this curve are given in the following table, also the values of log v and log p.

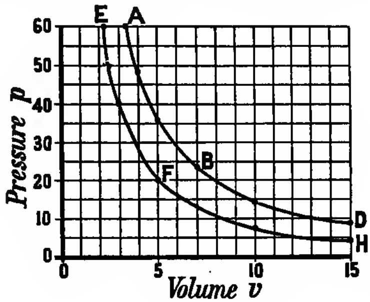


FIG. 258.

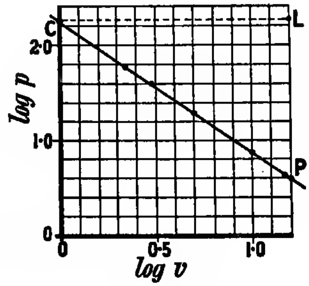


FIG. 259.

v	2·2	3	5	10	15
p	60	40	20	7·5	4·2
log v	0·3424	0·4771	0·6990	1·0000	1·1761
log p	1·7782	1·6021	1·3010	0·8751	0·6232

The values of log v and log p are plotted in Fig. 259, from which it is seen that the resulting points are in a straight line for which the equation is $\log p = -\frac{\text{PL}}{\text{CL}} \log v + \text{OC}$. Measuring PL on the log p scale and CL on the log v scale $\frac{\text{PL}}{\text{CL}} = 1·4$. Also OC, measured on the log p scale, is 2·267. Hence $\log p = -1·4 \log v + 2·267$, therefore $n = 1·4$, $\log c = 2·267$ and $c = 185$. The equation to the curve is therefore $pv^{1·4} = 185$.

In a problem such as the foregoing, if the points (log v, log p) are not exactly in a straight line, a mean line may be drawn amongst them, from which n and c may be found to get an approximate equation to the given curve.

120. The Curve $y = ae^{bx}$.—In the equation $y = ae^{bx}$, e is the base of the Napierian system of logarithms, its numerical value to four places of decimals being 2·7183. a and b are constants. The above equation is called an *exponential equation* because the variable x occurs as part of the index or exponent of the power of e. The curve which is the graph of $y = ae^{bx}$ is called an exponential curve.

Let $x = 0$, then $e^{bx} = e^0 = 1$, and $y = a$. The constant a is therefore the intercept of the curve on the axis of y.

Let $y = 0$, then $e^{bx} = 0$, and $x = -\infty$. The curve is therefore asymptotic to the axis of x.

Taking logarithms of both sides of the equation $y = ae^{bx}$,

$$\log y = \log a + bx \log e = \log a + 0{\cdot}4343bx.$$

From the equation $\log y = \log a + 0{\cdot}4343bx$, y may be calculated if x is given or assumed, or x may be calculated if y is given or assumed.

As an example take $a = 0{\cdot}8$, and $b = 2$, then $y = 0{\cdot}8e^{2x}$, and $\log y = \bar{1}{\cdot}9031 + 0{\cdot}8686x$. This curve is shown plotted in Fig. 260. The curve $y = 0{\cdot}8e^{-2x}$ is also shown in Fig. 260, and it will be seen that this second curve is the image of the first in a mirror represented by the axis of y.

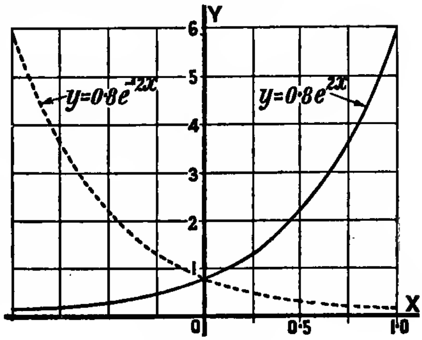


FIG. 260.

121. The Curve $y = a \log_e bx$.—The curve $y = a \log_e bx$ is called a *logarithmic curve*. Referring again to the exponential curve $y = ae^{bx}$, $\log_e y = \log_e a + bx \log_e e$, but $\log_e e = 1$, therefore $\log_e y = \log_e a + bx$, and $x = \dfrac{1}{b} \log_e \dfrac{y}{a}$, which is of the form $x = p \log_e qy$.

The logarithmic curve $y = a \log_e bx$ is therefore of the same form as the exponential curve $y = ae^{bx}$ with the variables x and y interchanged. The logarithmic curve will therefore be asymptotic to the axis of y while the exponential curve is asymptotic to the axis of x.

122. Spiral Curves.—If a straight line OP revolves in a plane about a fixed point O in a fixed straight line OX in the plane, and if at the same time a point P in OP moves along the line OP, the point P describes a curve called a *spiral*. The form of the spiral depends on the law connecting the displacement of the point P along OP with the angular displacement of OP.

The point O is the *pole* and the fixed line OX is the *initial line*. The position of OP at any instant is given by the angle θ which it makes with OX. θ is the *vectorial angle* and r the length of OP is the *radius vector*. The position of P at any instant is fixed by r and θ.

123. The Archimedean Spiral.—In the *Archimedean spiral* the displacement of P from O is proportional to θ and the equation to the curve is, $r = a\theta$ where a is a constant. It follows that for equal increments of θ there are equal increments of r. Stated in another way this means that if successive values of θ are in arithmetical progression successive values of r are also in arithmetical progression.

When $\theta = 0$, $r = 0$, and when $\theta = 2\pi$, $r = 2\pi a$. If θ be increased by 30° at a time from $\theta = 0$, then the constant increment of r is $\dfrac{2\pi a}{12}$.

The construction for plotting the curve when a is given is shown in Fig. 261. Radii are drawn at intervals of $\frac{\pi}{6}$ or 30°. OA is the radius vector when $\theta = 2\pi$, or 360°. Hence $OA = 2\pi a$. OA is divided into $\frac{360}{30} = 12$ equal parts. The remainder of the construction is clearly shown in the figure.

The tangent PQ at any point P makes an angle with the radius OP whose tangent is equal to $\frac{OP}{a}$. Hence if on a radius at right angles to OP, OL be made equal to a, PL will be the normal to the spiral at P, and the tangent PQ will be at right angles to PL.

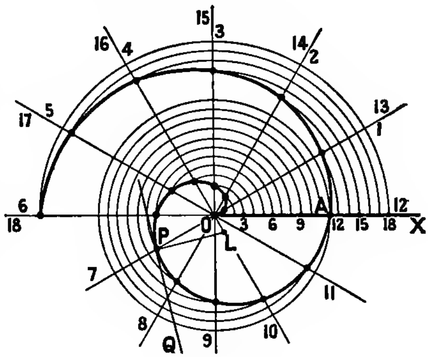


FIG. 261.

124. The Logarithmic or Equiangular Spiral.—If the successive values of θ are in arithmetical progression and the corresponding values of r are in geometrical progression the curve traced by P becomes a *logarithmic spiral.*

The equation to the logarithmic spiral is $r = a^{\theta}$. Hence $\log r = \theta \log a$, and this logarithmic equation may be used to calculate values of r for given or assumed values of θ when the constant a is known. If $\theta = 0$, then $r = 1$.

Fig. 262 shows one and a quarter convolutions of the spiral to a scale of $\frac{2}{5}$ when $a = 1{\cdot}2$, the unit being 1 inch.

$$\text{If } \theta = \frac{\pi}{6} \text{ or } 30°, \text{ then}$$

$$\begin{aligned} \log r &= \frac{\pi}{6} \log a \\ &= \frac{3{\cdot}1416}{6} \times 0{\cdot}0792 \\ &= 0{\cdot}04147, \\ \text{and } r &= 1{\cdot}10. \end{aligned}$$

If the increments of θ be taken equal to $\frac{\pi}{6}$ then the increments of $\log r$ are equal to $0{\cdot}04147$ and successive values of r are quickly calculated.

The logarithmic spiral has the property that the angle ϕ which the tangent at any point of the curve makes with the radius vector at that point is constant. On account of this property the logarithmic spiral is also known as the *equiangular spiral.*

The tangent of the angle ϕ is given by the formula

$\tan\ \phi = \dfrac{\log_{10} e}{\log_{10} a} = \dfrac{0{\cdot}4343}{\log a}$. For the curve shown in Fig. 262, $\tan \phi = \dfrac{0{\cdot}4343}{0{\cdot}0792} = 5{\cdot}48$.

To draw the tangent at the point L, on the radius OL produced make LK = 1 on any scale, and make KH at right angles to OL and equal to 5·48 on the same scale. Join LH, then LH is the tangent to the spiral at L.

If OP and OQ are any two radii of the logarithmic spiral, and if OR is a radius bisecting the angle POQ, then OR is a mean proportional between OP and OQ. This property may be used in constructing the curve geometrically when the pole and two points on the curve are known.

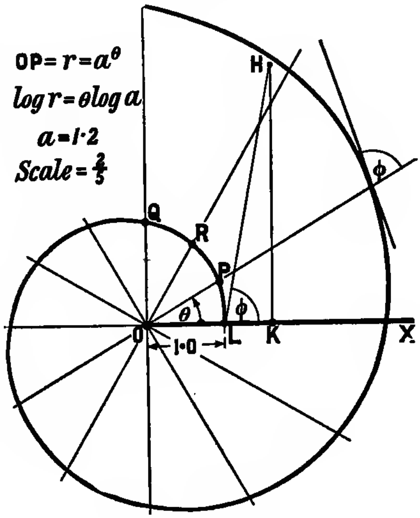


Fig. 262.

125. Graphic Solution of Equations.—Many equations which are difficult or impossible of solution by ordinary algebraic methods may be readily solved by graphic methods. On the other hand the graphic method is generally inferior to the algebraic method in the solution of the simpler forms of equations. But in any case a study of the graphic method will lead to a better understanding of the theory of equations.

Consider first the simple equation $ax + b = 0$, where a and b are constants. To solve this equation graphically, let $y = ax + b$. This is the equation to a straight line which may readily be drawn. This straight line cuts the axis of x where $y = 0$, and the distance of this point from the origin is the value of x which satisfies the original equation. In this case the graphic method is obviously inferior to the algebraic.

Next consider the pair of simultaneous equations $ax + by + c = 0$, and $a_1x + b_1y + c_1 = 0$, where a, b, c, a_1, b_1, and c_1 are constants, The graph of each of these equations is a straight line, and the lines may readily be drawn, and the co-ordinates of their point of intersection are the values of x and y which satisfy both equations.

Take the equation $1{\cdot}7x^2 - 0{\cdot}75x - 1{\cdot}5 = 0$. This is not a difficult equation to solve by the method of algebra but its graphic solution will now be considered in order to illustrate graphic methods.

First method. Let $y = 1{\cdot}7x^2 - 0{\cdot}75x - 1{\cdot}5$. Choose a number of simple values of x and calculate the corresponding values of y. Plot the points whose co-ordinates are these values of x and y. The

curve joining these points will cut the axis of x where $y = 0$. There will be two such points and their distances from the origin are the values of x which satisfy the original equation.

This curve is shown plotted in Fig. 263. The curve cuts the axis of x at A, giving $x = 1 \cdot 18$, and at B, giving $x = -0 \cdot 74$, and these are the roots of the equation. The curve should first be sketched to a small scale through a few calculated points in order to see the probable form and position of the curve. The parts of the curve in the neighbourhood of the axis of x should then be plotted carefully to a large scale in order to obtain more accurately the points A and B.

Second method. Rewrite the equation in the form $1 \cdot 7x^2 = 0 \cdot 75x + 1 \cdot 5$. Let $y' = 1 \cdot 7x^2$, and let $y'' = 0 \cdot 75\, x + 1 \cdot 5$. Draw the graph of $y'' = 0 \cdot 75x + 1 \cdot 5$ which is a straight line. Next draw the graph of $y' = 1 \cdot 7x^2$ which is a parabola. These two graphs are shown in

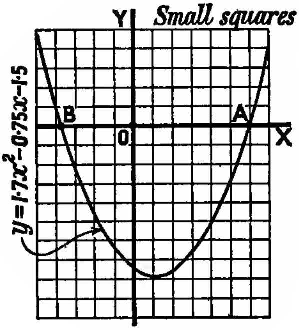


FIG. 263.

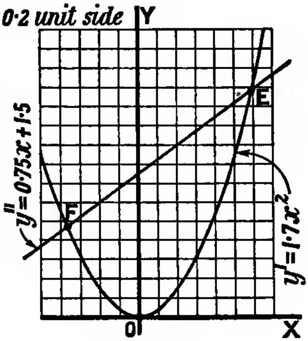


FIG. 264.

Fig. 264, and they intersect at the points E and F. The values of x for the points E and F are the required roots of the given equation. This is evident, for at these points $y' = y''$ and therefore $1 \cdot 7x^2 = 0 \cdot 7x + 1 \cdot 5$ when for x is substituted the abscissa of E or the abscissa of F.

The graph of $y' = 1 \cdot 7x^2$ should first be sketched through a few calculated points and the parts in the neighbourhood of the straight line $y'' = 0 \cdot 75x + 1 \cdot 5$ should then be drawn carefully to obtain as accurately as possible the points E and F.

In general the second method of solution just described will be found preferable to the first as it involves less calculation.

As a final example take the equation $x^3 - 0 \cdot 6x - 0 \cdot 1 = 0$. Rewrite the equation in the form $x^3 = 0 \cdot 6x + 0 \cdot 1$. Let $y' = x^3$, and $y'' = 0 \cdot 6x + 0 \cdot 1$. The graph of $y'' = 0 \cdot 6x + 0 \cdot 1$ is a straight line and the graph of $y' = x^3$ is a cubic parabola.

A rough sketch of the graphs to a small scale will show that their points of intersection are within the limits $x = +1$ and $x = -1$. The graphs should now be drawn to a large scale between the limits $x = +1$ and $x = -1$. They will appear as shown in Fig. 265.

Their points of intersection, A, B, and C, give the roots $x = 0{\cdot}85$, $x = -0{\cdot}18$, and $x = -0{\cdot}67$. In Fig. 265 the small squares are of 0·1 unit side.

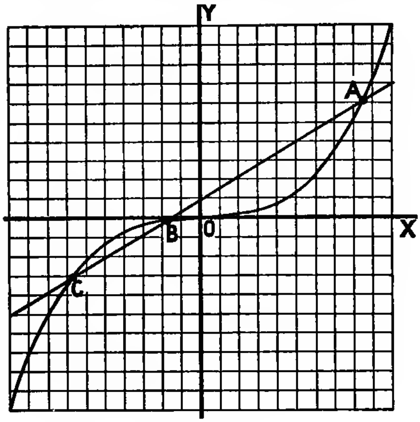


FIG. 265.

The given equation is of the form $x^3 + ax + b = 0$, and is a *cubic equation.* In the theory of equations it is shown that a cubic equation has three roots but one or more of these roots may be imaginary, that is, there may be three real roots, or two real roots, or one real root, or no real roots. This will be understood by reference to Fig. 265, for a straight line which is the graph of $y'' = -ax - b$ may meet the graph of $y' = x^3$ at three points, or two points only, or one point only, and the line may not meet the curve at all. When the straight line does not meet the curve there is no real value of x which will satisfy the original equation.

Similar remarks apply to the quadratic equation which may have two, but not more than two real roots.

Exercises IX

1. The axes being at right angles to one another and the unit of length being 1–10th of an inch, plot the points whose co-ordinates are given in the following table:—

Point	A	B	C	D	E	F	G	H
x	15	0	20	−17	−23	−18	0	16
y	20	25	0	16	0	− 9	−12	−8

2. Same as exercise 1 except that the co-ordinates are oblique, the angle XOY between the axes being 60°.

3. Plot the points whose polar co-ordinates are given in the following table, the unit of length being 1 inch.

Point	A	B	C	D	E	F	G	H
r	2·30	1·25	1·85	2·65	0·95	1·00	2·05	−1·70
θ	30°	90°	105°	180°	200°	−100°	320°	−120°

4. State in tabular form the polar co-ordinates of the points given in exercise 1, r and θ to be positive in each case.

5. The polar or vector co-ordinates of two points A and B are (3 inches, 30°) and (4·2 inches, 100°). That is, if O is the pole, and OX the line of reference, OA = 3 inches, XOA = 30°, OB = 4·2 inches, XOB = 100°.

Draw OX and plot the points A and B. Measure AB and the perpendicular from O on AB, and calculate the area of the triangle OAB. Verify your answer by calculating the value of $\frac{1}{2}$OA.OB sin AOB. [B.E.]

6. Given the equation $y = 0·6x + 4$ calculate the values of y corresponding to the following values of x, namely, 5, 10, 15, 20, 25, and 30. Taking the unit for x and y as 0·1 inch plot the six points of which the above values of x and the corresponding values of y are the co-ordinates. Show that all these points lie on the same straight line.

7. Draw the straight line (a) passing through the points (10, 15) and (−15, −15) and find its equation. Through the point (15, −3) draw a straight line (b) at right angles to (a) and find its equation. Find the co-ordinates of the point of intersection of (a) and (b).

8. Draw the graphs of the equations, $x - 2y = 9$, and $x + 3y = 24$ and determine the co-ordinates of their point of intersection.

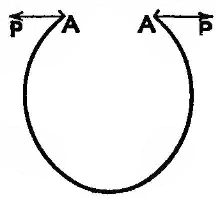


FIG. 266.

9. In arranging some elementary experiments in statics, cyclists' trouser clips are used as spring balances. In order to be able to measure pulls, a series of weights are hung on a clip, and the corresponding openings AA (Fig. 266) are measured. The results are as tabulated.

Pull = P ounces . .	0	4	8	12	16	20	24	28	32
Opening AA = x inches	0	0·51	1·15	1·83	2·53	3·24	3·89	4·44	4·96

Plot a curve showing the relation between P and x, the scale for P being $\frac{1}{8}$ inch to 1 ounce. Use this curve to graduate a decimal scale of ounces, which being applied to the spring at AA should measure any pull up to 32 ounces.

10. The co-ordinates of the centre of a circle being a and b, and the radius of the circle being r, draw the circles given in the following table. Dimensions are in inches.

	I	II	III	IV	V	VI	VII	VIII
a	1·5	0·0	0·0	−1·5	−0·5	1·2	−1·0	0·0
b	2·0	1·5	0·0	−2·0	1·0	−1·2	0·0	1·3
r	1·5	1·5	1·8	2·0	1·6	1·2	0·8	1·7

11. Draw the circles whose equations are—

I. $x^2 + y^2 = 4$. III. $x^2 + y^2 - 2x = 8$.
II. $x^2 + y^2 - 2y = 3$. IV. $x^2 + y^2 - 2x - y = 2·75$.

12. Draw the circle which passes through the three points (9, 8), (−16, 13), and (−5, −12). The unit is to be taken as 0·1 inch. Find the equation to this circle. Draw the straight line joining the points (10, 18), and (−25, 8). At the points where this line cuts the circumference of the circle draw the tangents and normals to the circle and find their equations.

13. Given the equation $x^2 = 2y$, take the following values of x, namely, 0, ±0·5, ±1·0, ±1·5, ±2·0, ±2·5, and ±3·0. Calculate the corresponding values of y, plot the points, and draw a fair curve through them. Name the curve. Unit = 1 inch.

14. Given the equation $y^2 = 2x - 1$, take the following values of y, namely, 0, ± 0.5, ± 1.0, ± 1.5, ± 2.0, ± 2.5, and ± 3.0. Calculate the corresponding values of x, plot the points, and draw a fair curve through them. Name the curve. Unit = 1 inch.

15. The equation to an ellipse is $0.49x^2 + y^2 = 1.96$. Take the following values of x, namely, 0, ± 0.4, ± 0.8, ± 1.2, ± 1.6, ± 1.8, and ± 2.0. Calculate the corresponding values of y plot the points and draw a fair curve through them. Unit = 1 inch.

16. Draw the rectangular hyperbola $xy = 200$ between the limits $x = 5$, and $x = 40$. Unit = 0·1 inch.

17. A cubic foot of gas at a pressure of 100 pounds per square inch expands isothermally until its volume is 5 cubic feet. Draw the expansion curve which shows the relation between the pressure and volume of the gas. Scales,—Pressure, 1 inch to 20 pounds per square inch. Volume, 1 inch to 1 cubic foot.

18. Plot the curves, $y = x^4$, and $y = x^5$, between the limits $x = \pm 2$. Scale for x, 1 inch to 1 unit. Scale for y, 1 inch to 10 units.

19. Plot the curves, $y = x^2$, $y = x^{2.3}$, and $y = x^3$, between the limits, $x = 0$ and $x = 3$. Scale for x, 1 inch to 1 unit. Scale for y, 1 inch to 5 units.

20. Plot the points whose co-ordinates are given in the following table, and find the equation to the fair curve joining them.

x	0·0	0·40	0·80	1·20	1·60	2·00	2·40	2·80
y	0·0	0·10	0·57	1·58	3·24	5·66	8·92	13·13

Take the scale for x as 1 inch to 1 unit and the scale for y as 1 inch to 5 units.

21. Plot the points whose co-ordinates are given in the following table, and find the equation to the fair curve drawn through them.

x	0	2	4	6	8	10
y	60	78	144	266	448	694

22. Plot the curve $y(x + 0.75)^{1.32} = 6$, between the limits $x = 0.75$ and $x = 4$, x and y being in inches.

23. The volumes (v cubic feet) and corresponding pressures (p pounds per square inch) of a pound weight of steam as it expands are given in the following table. Plot the expansion curve and test whether it approximately follows the law $pv^n = c$, and if it does, find the values of n and c.

v	3·0	3·7	4·7	6·1	7·9	10·6	14·5	20 2
p	153·3	118·3	90·0	67·2	49·3	35·5	25·0	17·2

24. Draw the first and second convolutions of an Archimedean spiral, having given that a certain radius vector cuts the first convolution at 0·7 inch from the pole and the second at 2 inches from the pole. Draw the tangent to the curve at the point which is 1·5 inches from the pole.

25. The equation to an Archimedean spiral is, $r = \frac{1}{4}\theta$ inches. Draw two convolutions of the spiral.

26. Draw a triangle POQ. OP = 1·1 inches, OQ = 1·4 inches, and the angle POQ = 45°. P and Q are points on the first convolution of an Archimedean spiral of which O is the pole. Find the initial line and draw the first convolution of the curve.

27. The equation to a logarithmic spiral is, $r = 1.3^{\theta}$ inches. Draw one convolution of the spiral and draw a tangent to the curve inclined at 60° to the initial line.

28. In a logarithmic spiral the common ratio of two radii containing an angle of 30° is 8 : 9. The longest radius vector in the first convolution is 3 inches. Draw one convolution of the spiral.

29. Plot the curve, $y = 6{\cdot}1'' e^{-0{\cdot}175x} \cos\left(x - \frac{\pi}{18}\right)$ from $x = 0$ to $x = 2\pi$.

You may obtain points on the curve by calculation, using mathematical tables; or by projection from the logarithmic spiral, $r = ae^{-0{\cdot}175\theta}$.

Observe that in this spiral the ratio of the lengths of two radii which differ in direction by $360° = e^{-0{\cdot}175 \times 2\pi} = 2{\cdot}718^{-1{\cdot}10} = 1 : 3$ very nearly. [B.E.]

30. The curve (Fig. 267) representing the curtail of a stone step is constructed of circular arcs, each of 90°. Set out the figure to the given dimensions, which are in centimetres. Scale $\frac{1}{4}$. [B.E.]

31. The given volute (Fig. 268) of one convolution is constructed of circular arcs each of 90°. Set out the curve, when the dimensions are as figured, in centimetres. [B.E.]

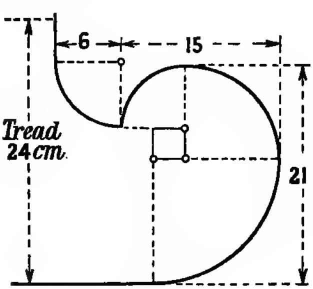


FIG. 267.

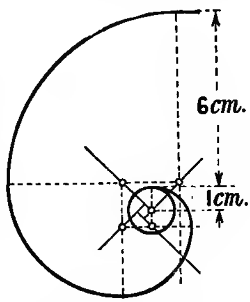


FIG. 268.

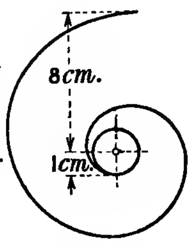


FIG. 269.

32. Fig. 268 shows a construction for describing a volute of one convolution. Set out a similar curve of one and a half convolutions, when the dimensions, in centimetres, are as in Fig. 269. [B.E.]

33. Solve, graphically, the equations:—

(1) $x^2 - 2{\cdot}5x + 1{\cdot}5 = 0$.
(2) $x^2 + 3x + 2 = 0$.
(3) $x^3 - 0{\cdot}5x + 0{\cdot}1 = 0$.
(4) $x^3 + 2x^2 - 3x - 3{\cdot}375 = 0$.

34. Solve, graphically, the simultaneous equations:—

(1) $\begin{cases} 2y - x - 1 = 0. \\ 4x + 3y + 1{\cdot}8 = 0. \end{cases}$
(2) $\begin{cases} x^3 + 1 = 9y. \\ x^2 + x = 6y. \end{cases}$

35. Solve, graphically, the equations:—

(1) $3^{2x} + 5x = 6$.
(2) $15x^2 - 2e^{2x} = 0$.

CHAPTER X

PERIODIC MOTION

126. Periodic Motion.—If any position of a moving body be selected and it is found that after a certain interval of time the body is again moving through that position in the same direction and with the same velocity, the body is said to have *periodic motion.* The interval of time between two successive appearances of the body in the same position moving in the same direction with the same velocity is called the *periodic time* or *period* of the motion. Periodic time is generally measured in seconds and will be denoted by T.

The simplest case of periodic motion is that of a body revolving about a fixed axis with uniform velocity. In this case the periodic time is the time taken to make one complete revolution. A swinging pendulum has periodic motion, but it has different velocities in different positions. In general all the parts of a machine have periodic motion although some parts may have very variable velocities and very complicated motions during a period.

The number of periods in the unit of time is called the *frequency* and is the reciprocal of the periodic time, or frequency $= N = 1/T$.

127. Simple Harmonic Motion.—P (Fig. 270) represents a point which is moving with a uniform velocity V along the circum-

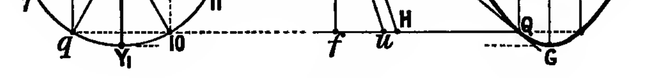


FIG. 270.

ference of the circle XYX_1Y_1 whose centre is C. M is another point which is moving backward and forward along the diameter YY_1 of the same circle in such a manner that PM is always at right angles to YY_1, in other words, M is the projection of P on YY_1. Under these circumstances the point M has *simple harmonic motion.* If XX_1 is a

diameter at right angles to YY_1 and if PN is perpendicular to XX_1 then the point N also has simple harmonic motion.

CP is the *representative crank* of the harmonic motion of the point M or the point N, and the circle XYX_1Y_1 is the *auxiliary circle* or *circle of reference.*

The *amplitude* of a harmonic motion of a point is the greatest displacement of the point from its mean position and is equal to the radius of the representative crank.

Denote the angle PCX by θ and let $CP = r$, $CN = x$, and $CM = y$. Then, $x = r \cos \theta$, and $y = r \sin \theta$.

If the angular velocity of CP be $\omega = V/r$, and if P moves from X to P in the time t, then $\theta = \omega t$. Hence, $x = r \cos \omega t$, and $y = r \sin \omega t$.

If the time t be plotted along the straight line OL and the corresponding displacement y be plotted at right angles to OL as shown in Fig. 270, the *harmonic curve* OEFGL is obtained. The length OL represents the periodic time and this has been divided into twelve equal parts. The circumference of the circle has been divided into the same number of equal parts to obtain the corresponding values of θ.

The distance y is to be taken as positive when measured above XX_1 and negative when measured below. Also, the distance x is to be taken as positive when measured to the right of YY_1 and negative when measured to the left.

Since θ is proportional to t, lengths along OL will also represent values of θ and the curve OEFGL is then called a *sine curve.* If θ is in degrees OL represents 360°, but if θ is in radians OL represents 2π.

The tangent to the harmonic or sine curve at any point Q is determined as follows. Draw Qq parallel to OL to meet the circle at q. Draw qn perpendicular to XX_1. Make $Qf = FO$. Draw fS at right angles to OL. Make $fS = \pi$ to any convenient scale and make $fH = 1$ on the same scale. Make $fu = Cn$. Draw uK parallel to HS to meet fS at K. KQ is the tangent required.

The mean height of the curve OEF is $\frac{2r}{\pi}$ and the mean depth of the curve FGL is also $\frac{2r}{\pi}$ or $0{\cdot}637r$. In other words, the mean ordinate for half a period is $0{\cdot}637r$. If s^2 is the mean of the squares of the ordinates for half a period, then $s^2 = \frac{1}{2}r^2$ and therefore $s = 0{\cdot}707r$. These results are of importance in connection with alternating electric currents.

Consider the equation $y = 2 \sin 1{\cdot}2t$. This is the equation to a simple harmonic curve for which $r = 2$ feet, and $\omega = 1{\cdot}2$ radians per second. The periodic time is $T = \frac{2\pi}{1{\cdot}2} = \frac{\pi}{0{\cdot}6} = 5{\cdot}236$ seconds. The curve may be drawn as shown in Fig. 270. OL will represent 5·236 seconds and will be measured on a time scale. If OL and the

circle be divided into twelve equal parts the successive values of t will be $\frac{\pi}{12 \times 0{\cdot}6}$, $\frac{2\pi}{12 \times 0{\cdot}6}$, $\frac{3\pi}{12 \times 0{\cdot}6}$, etc. This harmonic curve may also be constructed by calculating values of y from the equation $y = 2 \sin 1{\cdot}2t$. In this case values of t increasing by, say, 0·5 second may be taken. The values of t would be multiplied by 1·2 to obtain the values of θ which would be in radians. Taking the values of the sines of these angles from a table of sines and multiplying them by 2 would give the corresponding values of y. t would be plotted along OL on a time scale while y would be plotted at right angles to OL on a distance scale.

Consider the equation $y = 2{\cdot}5 \sin \theta$. This is the equation to a sine curve which may be constructed as in Fig. 270. The radius of the circle is 2·5 on any convenient scale and y will be measured on the same scale. If the circle be divided into twelve equal parts the successive values of θ will be 30°, 60°, 90°, 120°, etc. and these will be plotted along OL to any convenient scale. The corresponding values of y are projected from the circle as shown. This sine curve may also be constructed by calculating values of y from the equation $y = 2{\cdot}5 \sin \theta$, values of $\sin \theta$ being taken from a table of sines.

The exact position at any instant of the point M (Fig. 270) which has simple harmonic motion is defined by the angle θ which is called the *phase angle*. The term *phase* is generally used to designate the position of the point M at any instant by stating the fraction of its period of motion which it has performed reckoned from its middle position when moving in the positive direction. The first or zero phase is when M is passing through C towards Y. For this position $\theta = 0$. For a complete period $\theta = 2\pi$ or 360° and for any position the phase is the $\frac{\theta}{2\pi}$ period phase. For example, when θ is equal to π the phase is the half period phase.

A well known mechanism which gives simple harmonic motion is shown in Fig. 271. The crank pin of the crank works in the slot on the end of the piece which is made to reciprocate. The reciprocating piece has simple harmonic motion when the crank rotates with uniform velocity.

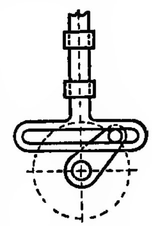


FIG. 271.

128. Lead and Lag.—Instead of using the phase angle θ to fix the position of a point having harmonic motion it is frequently more convenient to give the angular position of the representative crank with reference to a fixed radius of the auxiliary circle which is inclined at an angle α to the zero position of the representative crank. In Fig. 272 CX is the zero position of the representative crank and CA is a fixed radius inclined at an angle α to CX. CP is any position of the crank and ϕ is the angle which CP makes with CA. The angle α is positive when measured in the anti-clockwise direction and negative when measured in the clockwise direction.

Evidently $\theta = \phi + \alpha$ and $y = r \sin (\phi + \alpha)$.

If ω is the angular velocity of CP and t is the time taken by P to travel from A to P, then $\phi = \omega t$, and $y = r \sin (\omega t + \alpha)$.

When α is positive it is called the *advance* or *lead*, and when it is negative it is called the *lag*.

The curve which is the graph of the equation $y = r \sin (\phi + \alpha)$ is the same as the curve which is the graph of the equation $y = r \sin \phi$ except that it starts at a different point; this is clearly shown in Fig. 272. The dotted part of the curve corresponds to the motion

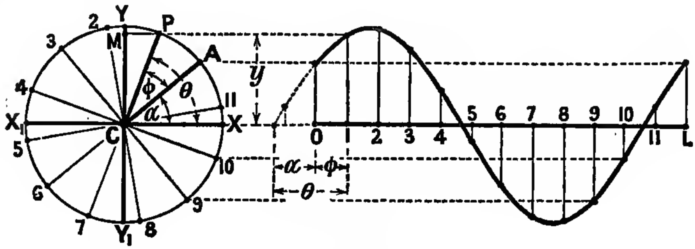


Fig. 272.

of the crank from the zero position CX to the position CA which is to be taken as the new starting position. Observe that the equal divisions on the circle which correspond to the equal divisions on OL start from the point A.

129. Composition of Collinear Harmonic Motions.—A simple way of giving a point a motion which is compounded of two or more simple harmonic motions is shown in Fig. 273. A and B are cranks which give simple harmonic motions to the rods C and D which carry pulleys E and F at their upper ends. G and H are suspended guide pulleys. A thin cord or fine wire is fixed at one end K and passes under or over the different pulleys as shown. The free end of the wire is loaded and guided in a vertical direction and carries a pen or pencil P which traces a curve on a sheet of paper stretched round a revolving drum L. The cranks and the drum are driven so that each has a uniform velocity.

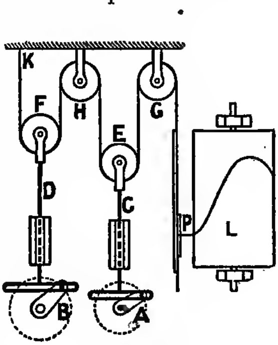


Fig. 273.

The vertical components of the motions of the crank pins are communicated to the pencil but the amplitudes of these motions are doubled by the action of the pulleys. If one of the cranks is stopped while the other rotates uniformly, the pencil will have simple harmonic motion of the same period as that of the rod driven by the moving crank but the amplitude of the motion of P will be twice the radius of

the moving crank. When both cranks rotate the pencil traces on the moving paper a curve whose ordinates are the sum of the ordinates of the harmonic curves due to the separate cranks. Any number of harmonic motions may be compounded in this way by introducing a crank for each.

The mechanism illustrated in Fig. 273 is the basis of Lord Kelvin's tide predicting machine.

If the crank shafts be at a considerable distance from the guide pulleys G and H compared with the radii of the cranks, the pulleys E and F may be mounted directly on the crank pins and the mechanism is considerably simplified but the motions communicated to the pencil by the separate cranks are not quite simple harmonic motions.

When two collinear simple harmonic motions *of the same period* and therefore of the same frequency are combined the resulting motion is also a simple harmonic motion.

Referring to Fig. 274, OApL is the harmonic or sine curve for the crank CP which starts from the zero position CX. The radius of the

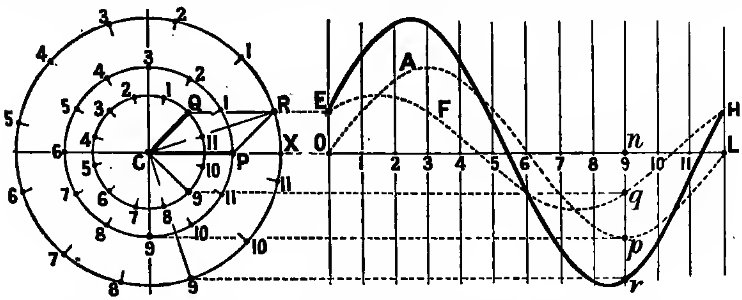


FIG. 274.

crank CP is 1·8 inches and the equation to the curve OApL is therefore $y_1 = 1{\cdot}8 \sin \theta$ in which θ is measured from the zero position of the crank. EFqH is the harmonic or sine curve for the crank CQ which has a lead of 45°. The radius of the crank CQ is 1·2 inches and the equation to the curve EFqH is therefore $y_2 = 1{\cdot}2 \sin (\theta + 45°)$ in which θ is measured from the radius which is 45° in advance of the zero position CX.

To combine the two curves the ordinates of the one must be added to the ordinates of the other. For example, $rn = pn + qn$. In the addition regard must of course be paid to the signs of the ordinates.

The equation to the resulting curve is $y = y_1 + y_2 = 1{\cdot}8 \sin \theta + 1{\cdot}2 \sin (\theta + 45°)$.

The resulting curve in this case may be drawn independently as follows. Complete the parallelogram PCQR. Then the diagonal CR is the representative crank whose harmonic curve, having an angle of advance RCP, is the curve which has just been obtained by combining the harmonic curves OApL and EFqH.

The general construction for determining the representative crank

which will give to a point a harmonic motion which is the resultant of the harmonic motions of the same period due to any number of given representative cranks is shown in Fig. 275. CP, CQ, CR, and CS are the relative positions and radii of the given cranks at a given instant. Draw Pq parallel and equal to CQ. Draw qr parallel and equal to CR. Draw rs parallel and equal to CS. Then CS, the closing line of the polygon CPqrs is the crank required. The point whose motion is being considered is supposed to be reciprocating in the line YCY$_1$. The proof of the above construction is as follows. The distance of the moving point from C at any instant is the sum of the projections of the given cranks on the line YCY$_1$ at the instant considered, regard being paid to the signs of these projections. But the sum of these projections is evidently equal to the sum of the projections of CP, Pq, qr, and rs on YCY$_1$, and this sum is also equal to the projection of Cs on YCY$_1$.

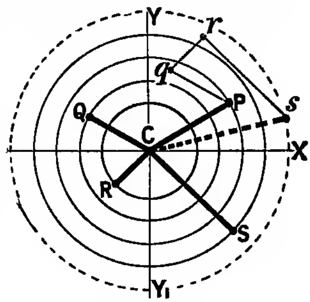


Fig. 275.

When the simple harmonic motions which have to be combined are not of the same period the resulting motion is not a simple harmonic motion. To construct the resulting displacement curve the separate simple harmonic curves are first drawn and then combined as already explained for motions of the same period (Fig. 274).

An example of the compounding of two simple harmonic motions of different periods is illustrated by Fig. 276, where the full curve is the graph of the equation

$$y = 2 \sin (\theta + 30°) + 1{\cdot}4 \sin (2\theta + 50°)$$

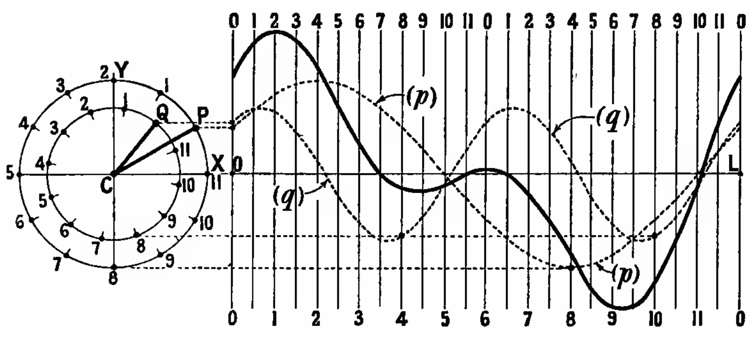


Fig. 276.

The curve (p) is the graph of the equation $y_1 = 2 \sin (\theta + 30°)$ and is the harmonic curve for the crank CP which starts with an advance of 30° from the zero position CX.

The curve (q) is the graph of the equation $y_2 = 1{\cdot}4 \sin (2\theta + 50°)$ and is the harmonic curve for the crank CQ which starts with an advance of 50° from the zero position CX.

It will be observed that the frequency of the second motion is double that of the first, that is, the crank CQ rotates twice as fast as the crank CP.

130. Composition of Harmonic Motions at Right Angles to one another.—The diagram (a) Fig. 277 shows two slotted bars at right angles to one another and driven by two cranks so that each of the slotted bars has simple harmonic motion. If a pencil P fitting both slots be passed through them where they overlap, then the pencil will have a motion which is compounded of two simple harmonic motions at right angles to one another. The remainder of Fig. 277 shows how the curve traced by the pencil P may be drawn.

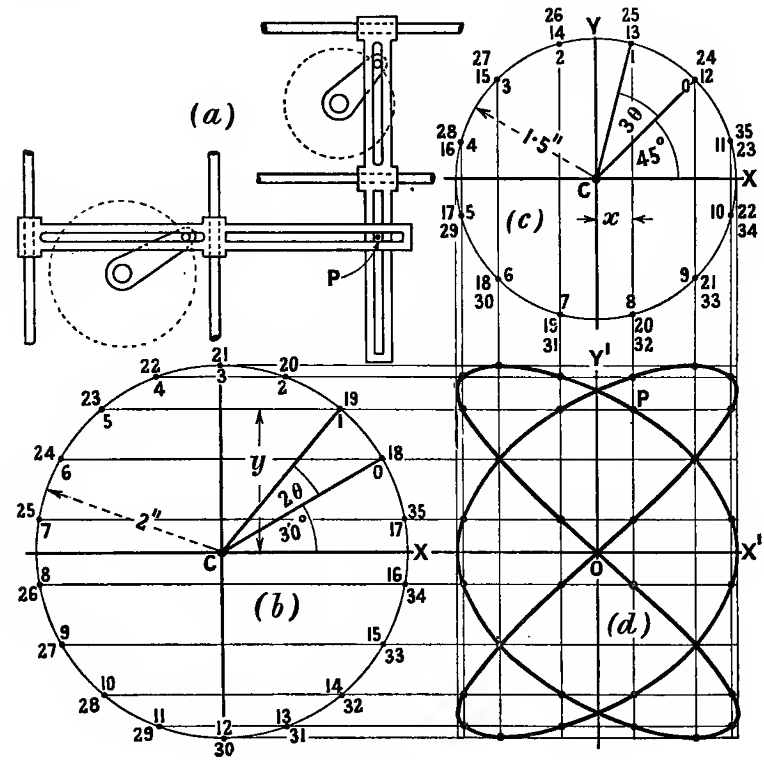


FIG. 277.

In the example worked out in Fig. 277 the crank at (b) has a radius of 2 inches while that at (c) has a radius of 1·5 inches. The

first crank makes two revolutions while the second makes three revolutions. The frequencies of the two harmonic motions are therefore in the ratio of 2 : 3. When the first crank makes 30° with the horizontal position CX the second makes 45° with CX. Calling these the initial positions, if the first crank turns through an angle 2θ from its initial position the second crank will turn through an angle 3θ from its initial position, in the same time.

The position of the point P is determined, with reference to the axes OX′ and OY′, (*d*) Fig. 277, by the two equations

$$x = 1{\cdot}5 \cos (3\theta + 45^\circ) \text{ and } y = 2 \sin (2\theta + 30^\circ)$$

The circle (*b*) has been divided into 18 equal parts, starting from the initial position of its crank, while the circle (*c*) has been divided into 12 equal parts, starting from the initial position of its crank. Observe that the numbers 18 and 12 are in the inverse ratio of the frequencies.

The construction for determining points in the path of P is clearly shown and need not be further described.

A great variety of curves may be obtained by taking different frequencies and different angles of lead. A few of the simpler cases may be mentioned here.

(i) *Equal frequencies and equal cranks.*
- (1) One crank 0° or 180° in advance of the other. The result is a circle.
- (2) One crank 90° in advance of the other. The result is a straight line.
- (3) For any other angle between the cranks the result is an ellipse.

(ii) *Equal frequencies and unequal cranks.*
- (1) One crank 90° in advance of the other. The result is a straight line.
- (2) For any other angle between the cranks the result is an ellipse.

It is easy to show that when the path of P is a straight line its motion in that straight line is a simple harmonic motion.

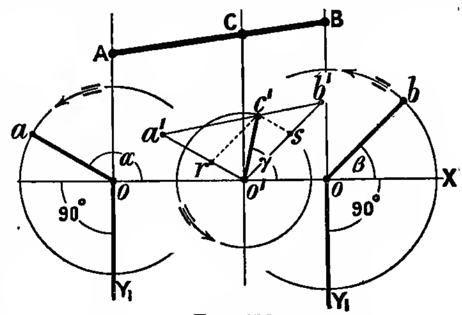


FIG. 278.

131. Composition of Parallel Harmonic Motions. — AB (Fig. 278) is a vibrating link. The vertical component of the motion of A is a simple harmonic motion of which *oa* is the representative crank. The vertical component of the motion of B is a simple harmonic motion of which *ob* is the representative crank. The motions

of A and B have the same frequency. The advance angles of the cranks *oa* and *ob* are α and β respectively.

C is a point in AB or in AB produced either way. The vertical component of the motion of C is obviously a simple harmonic motion and it is required to find the representative crank for this motion.

From a point o' draw $o'a'$ parallel and equal to *oa*. Draw $o'b'$ parallel and equal to *ob*. Join $a'b'$ and divide it at c' so that $a'c' : c'b' :: AC : CB$. Then $o'c'$ is the crank required and γ is its advance angle. The proof is as follows. Draw $c'r$ parallel to $b'o'$ to meet $o'a'$ at r. Draw $c's$ parallel to $a'o'$ to meet $o'b'$ at s. Let the vertical motion of B be destroyed, then the vertical motion of C will be to that of A as CB : AB or as $c'b' : a'b'$ or as $o'r : o'a'$. $o'r$ is therefore the representative crank for the vertical harmonic motion of C when B has no vertical motion. In like manner when the vertical motion of A is destroyed and B is driven, the representative crank for the vertical motion of C will be $o's$. Hence, when A and B are both driven the representative crank for the vertical motion of C will be $o'c'$ which is the diagonal of the parallelogram $o'rc's$ as was proved in Art. 129.

Denoting the lengths of the cranks *oa*, *ob*, and *oc* by a, b, and c respectively the equations for the vertical displacements of A, B, and C from their mean positions are

$$y_1 = a \sin(\theta + \alpha),\ y_2 = b \sin(\theta + \beta), \text{ and } y_3 = c \sin(\theta + \gamma)$$

where θ is measured from the initial positions *oa*, *ob*, and *oc*.

The above problem occurs in connection with reciprocating steam engine valve gears. *oa* and *ob* are eccentrics which drive the link AB. The valve is driven from a point C in AB. $o'c'$ is the *equivalent eccentric*, that is, $o'c'$ is an eccentric which would give the same motion to the valve, driving it directly, as the two eccentrics and the link AB give it.

In a vertical engine if the crank is in the vertical position oY_1 when *oa* and *ob* are in the positions shown, then $180 - \alpha$ is the *angle of advance of the eccentric oa*, and β is the angle of advance of the eccentric *ob*. In the actual engine the eccentrics *oa* and *ob* are on the same shaft.

132. Velocity and Acceleration in Harmonic Motion.—CP (Fig. 279) is the representative crank for the simple harmonic motions of the points M and N which reciprocate along the vertical and horizontal diameters respectively of the auxiliary circle.

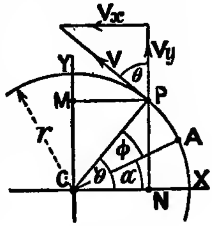


FIG. 279.

Let CP $= r$ and let the position of CP be defined by the angle θ which it makes with CX. Let ω be the angular velocity of CP and let V be the linear velocity of P. Then $V = \omega r$. Let the velocities of M and N when in the positions shown be V_y and V_x respectively. These velocities are the vertical and horizontal components of V respectively. Hence $V = V \cos\theta = \omega r \cos\theta = \omega r \cos \omega t$, and

$V_x = V \sin \theta = \omega r \sin \theta = \omega r \sin \omega t$, where t is the time taken by P to travel from X to P.

If a scale for velocity be chosen such that CP = V, then PM = V_y, and PN = V_x. The auxiliary circle therefore becomes a velocity diagram on a displacement base for the simple harmonic motion of the point M or the point N.

If the angular position of CP be measured from a radius CA which makes an angle α with CX and if the angle ACP = ϕ

then $V_y = V \cos (\phi + \alpha) = \omega r \cos (\phi + \alpha) = \omega r \cos (\omega t + \alpha)$

and $V_x = V \sin (\phi + \alpha) = \omega r \sin (\phi + \alpha) = \omega r \cos (\omega t + \alpha)$

where t is now the time taken by P to travel from A to P.

The velocities V_y and V_x may be plotted on a time t or angle θ or angle ϕ base in exactly the same way as displacements were plotted in Arts. 127 and 128.

The radial acceleration of the point P is $f = \frac{V^2}{r} = \omega^2 r$. Let the acceleration of the points M and N be f_y and f_x respectively. These accelerations are the vertical and horizontal components of f respectively. Hence (Fig. 280)

$$f_y = f \sin \theta = \frac{V^2}{r} \sin \theta = \omega^2 r \sin \theta = \omega^2 r \sin \omega t,$$

and
$$f_x = f \cos \theta = \frac{V^2}{r} \cos \theta = \omega^2 r \cos \theta = \omega^2 r \cos \omega t,$$

where t is the time taken by P to travel from X to P.

Since $\sin \theta = \frac{y}{r}$, and $\cos \theta = \frac{x}{r}$ the above equation may be written,

$$f_y = f\frac{y}{r} = \frac{V^2}{r^2} y = \omega^2 y,$$

and
$$f_x = f\frac{x}{r} = \frac{V^2}{r^2} x = \omega^2 x.$$

If a scale for acceleration be chosen such that CP = f, then $f_y = y$ and $f_x = x$.

The acceleration of the point M is shown plotted on the vertical diameter YY_1 as a base.

If the angular position of CP be measured from a radius CA making an angle α with CX and if the angle ACP = ϕ then for θ in the above equations substitute $\phi + \alpha$.

The accelerations f_y and f_x may be plotted on a time t or angle θ or angle ϕ base in exactly the same way as displacements were plotted in Arts. 127 and 128.

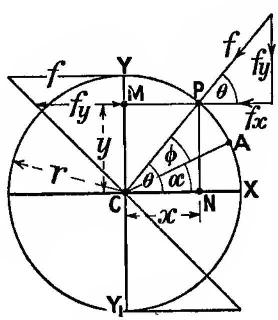


FIG. 280.

When a point has a motion which is compounded of two or more simple harmonic motions the resultant velocity and resultant acceleration at any

instant are determined from the component velocities and component accelerations exactly as for displacements.

133. Harmonic Analysis.—It has been shown that the displacement of a point from its mean position, when it has simple harmonic motion in a straight line, is given by the equation $y = r \sin(\theta + \alpha)$ or by $x = r \cos(\theta + \alpha)$.

Again, if a point has periodic motion in a straight line, and if its motion is compounded of a number of simple harmonic motions in which the separate displacements from the mean position are given by the equations—

$x_1 = r_1 \cos(\theta + \alpha_1)$, $x_2 = r_2 \cos(2\theta + \alpha_2)$, $x_3 = r_3 \cos(3\theta + \alpha_3)$, and so on,

then the resultant displacement is given by the equation—

$$x = r_1 \cos(\theta + \alpha_1) + r_2 \cos(2\theta + \alpha_2) + r_3 \cos(3\theta + \alpha_3) + \ldots$$

and it has been shown how such a resultant displacement may be found graphically and plotted to obtain a curve of resultant displacement on a time t or angle θ base.

If instead of measuring the resultant displacement from the mean position it is measured from a point at a distance c from it, then,

$$x = c + r_1 \cos(\theta + \alpha_1) + r_2 \cos(2\theta + \alpha_2) + r_3 \cos(3\theta + \alpha_3) + \ldots$$

The right hand side of this equation is known as a *Fourier series.*

Any periodic curve of displacement being given it may be resolved into a number of simple harmonic components. The number of these components is greater the greater the complexity of the given periodic curve.

The process of breaking up a given periodic curve into a number of simple harmonic curves, or the process of finding the equation to the curve in the form

$$x = c + r_1 \cos(\theta + \alpha_1) + r_2 \cos(2\theta + \alpha_2) + r_3 \cos(3\theta + \alpha_3) + \ldots$$

is called *harmonic analysis.*

The process of harmonic analysis is one of great importance in connection with the study of the periodic motions of machines and of alternating electric currents.

A graphic method of harmonic analysis which is comparatively simple and easily applied will now be described. This method is due to Mr. Joseph Harrison and was described by him in *Engineering* of Feb. 16th, 1906.

The procedure will be illustrated by reference to a definite example. The full curve EFGH (Fig. 281) has been plotted on the base line OL. The base represents one complete revolution of a shaft upon which there is an eccentric which drives a slide valve through certain intermediate link-work. The ordinates of the curve represent the displacements of the valve from a certain fixed position. In order that the student may transfer this curve accurately and of full size to his drawing paper, and work out the example for himself, the ordinates have been dimensioned. Twelve ordinates at equal intervals will be used and these are numbered 0 to 11.

In most engineering problems in harmonic analysis it will be found that not more than three of the harmonic terms are required and in many cases two terms are sufficient. In the example to be worked three terms in addition to the constant will be found.

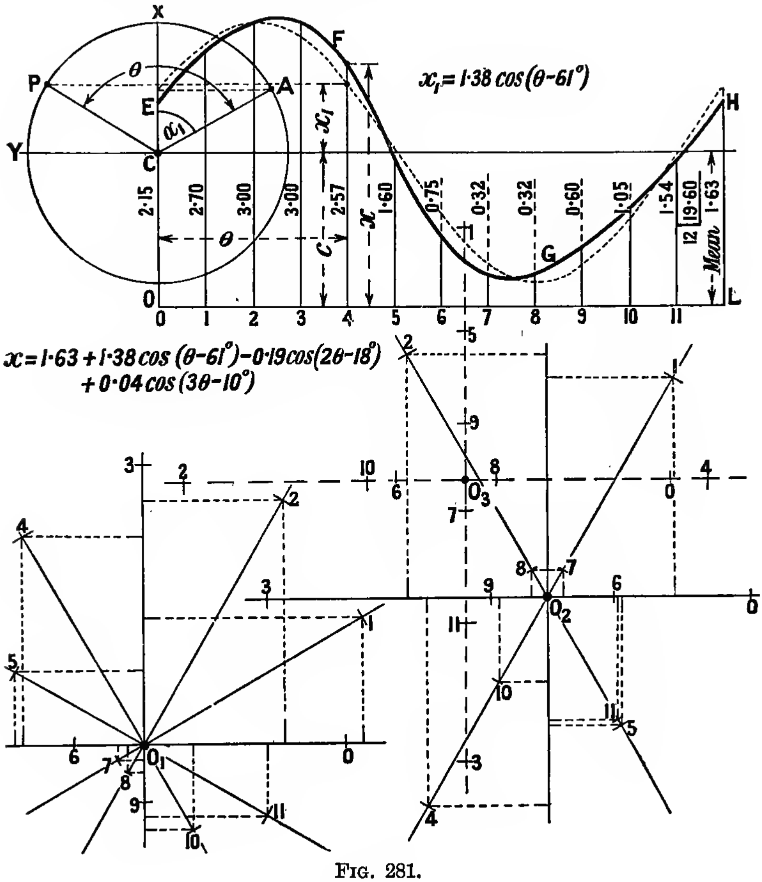


FIG. 281.

The equation is

$$x = c + r_1 \cos(\theta + \alpha_1) + r_2 \cos(2\theta + \alpha_2) + r_3 \cos(3\theta + \alpha_3)$$

and it is required to find c, r_1, r_2, and r_3, also α_1, α_2, and α_3.

Expanding the right hand side, the equation may be written

$$\begin{aligned} x = c &+ r_1 \cos \alpha_1 \cos \theta - r_1 \sin \alpha_1 \sin \theta \\ &+ r_2 \cos \alpha_2 \cos 2\theta - r_2 \sin \alpha_2 \sin 2\theta \\ &+ r_3 \cos \alpha_3 \cos 3\theta - r_3 \sin \alpha_3 \sin 3\theta. \end{aligned}$$

Let $r_1 \cos \alpha_1 = a_1$, $r_2 \cos \alpha_2 = a_2$, and $r_3 \cos \alpha_3 = a_3$.

Also, let $-r_1 \sin \alpha_1 = b_1$, $-r_2 \sin \alpha_2 = b_2$, and $-r_3 \sin \alpha_3 = b_3$.

Then
$$x = c + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta$$
$$+ b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta.$$

Since $r_1 \cos \alpha_1 = a_1$, and $-r_1 \sin \alpha_1 = b_1$ it follows by division that $\tan \alpha_1 = -\frac{b_1}{a_1}$. Also $r_1^2 = a_1^2 + b_1^2$ and $r_1 = \pm \sqrt{a_1^2 + b_1^2}$. There are similar relations for the other corresponding constants.

The constant c is the mean of all the given ordinates and in this case it is 1·63.

From an origin O_1 beginning with the zero direction draw 12 radial lines at equal angular intervals. Along these radial lines, beginning with the zero line, mark off in succession from O_1 the ordinates 0 to 11. Find the horizontal and vertical projections of these radiating vectors as shown. Find the sum of the horizontal projections having regard to their signs. This sum is 4·00 and is equal to $6a_1$, the coefficient 6 of a_1 being *half the number of ordinates used.* Therefore $a_1 = 0·67$. Find the sum of the vertical projections, having regard to their signs. This sum is 7·26 and is equal to $6b_1$. Therefore $b_1 = 1·21$.

$$\tan \alpha_1 = -\frac{b_1}{a_1} = -1·806 \text{ and } \alpha_1 = -61° \text{ to the nearest degree}$$

$r_1 = \pm\sqrt{a_1^2 + b_1^2} = \pm 1·38$. Since $r_1 \cos \alpha_1 = a_1$ and $\cos \alpha_1$ and a_1 are here both positive it follows that $r_1 = +1·38$.

The first harmonic term or the fundamental term is therefore $+1·38 \cos (\theta - 61°)$.

Next, from an origin O_2, beginning with the zero direction, draw, at equal angular intervals, half the number of radial lines that were drawn from O_1. On these radial lines, beginning with the zero line, mark off in succession from O_2 the ordinates 0 to 11, going twice round. Find the sums of the horizontal and vertical projections of these radiating vectors. In this case these sums are $-1·09$ and $-0·36$ respectively.

Now, $6a_2 = -1·09$, therefore $a_2 = -0·18$.
Also, $6b_2 = -0·36$, therefore $b_2 = -0·06$.

$$\tan \alpha_2 = -\frac{b_2}{a_2} = -0·333 \text{ and } \alpha_2 = -18° \text{ to the nearest degree.}$$

$r_2 = \pm\sqrt{a_2^2 + b_2^2} = \pm 0·19$. Since $r_2 \cos \alpha_2 = a_2$, and since $\cos \alpha_2$ is positive and a_2 is negative r_2 must be negative. Hence $r_2 = -0·19$.

The second harmonic term or the octave term is therefore $-0·19 \cos (2\theta - 18°)$.

Lastly, from an origin O_3, beginning with the zero direction, draw, at equal angular intervals, one third of the number of radial lines that were drawn from O_1. On these radial lines, which in this case are at right angle intervals, mark off in succession from O_3 the ordinates 0 to 11, beginning with the zero line, and going three times round as shown. The sums of the horizontal and vertical projections of these radiating vectors in this case are $+0·24$ and $+0·04$ respectively.

Now, $6a_3 = 0{\cdot}24$, therefore $a_3 = 0{\cdot}04$.
Also, $6b_3 = 0{\cdot}04$, therefore $b_3 = 0{\cdot}007$.

$$\tan \alpha_3 = -\frac{b_3}{a_3} = -0{\cdot}175 \text{ and } \alpha_3 = -10^\circ \text{ to the nearest degree.}$$

$r_3 = \pm\sqrt{a_3^2 + b_3^2} = \pm 0{\cdot}04$. Since $r_3 \cos \alpha_3 = a_3$ and since $\cos \alpha_3$ and a_3 are both positive, $r_3 = +0{\cdot}04$.

The third harmonic term or the term of the third order is therefore $+0{\cdot}04 \cos (3\theta - 10^\circ)$.

The complete equation is now

$$x = 1{\cdot}63 + 1{\cdot}38 \cos (\theta - 61^\circ) - 0{\cdot}19 \cos (2\theta - 18^\circ) + 0{\cdot}04 \cos (3\theta - 10^\circ).$$

The crank circle for the fundamental term $x_1 = 1{\cdot}38 \cos (\theta - 61^\circ)$ is shown and the curve for this has been projected as explained in Art. 128, and is shown dotted.

If the student will work out this example, step by step as described, and also the last four exercises at the end of this chapter, he will find that the method is simpler than it looks from the foregoing somewhat lengthy description.

After finding the Fourier equation it is a useful exercise to construct the component simple harmonic curves and then construct the resultant curve to see how near it approaches to coincidence with the original curve.

If the angular velocity ω in the equation $\theta = \omega t$ is known, curves of velocity and acceleration for the component harmonic motions may be constructed, and from these the resultant velocity and acceleration curves may be determined. Or the equations for the component velocities and accelerations may be found, and then by addition the equations for the resultant velocity and acceleration may be obtained and then used to calculate the velocity or acceleration for any position of the moving body.

Exercises X

1. A point has simple harmonic motion of amplitude 2 inches and periodic time 2 seconds. Construct the curve which shows the relation between displacement from mean position and time. Scales.—Displacement, full size; time 3 inches to one second. Draw the tangents to the curve where the time is 1 second and 1·3 seconds.

2. Taking the equation to a simple harmonic curve, $y = r \sin \omega t$, construct the curve for the case where $r = 0{\cdot}75$ foot and the frequency is 2 periods per second. Scales.—Displacement, 3 inches to 1 foot; time 10 inches to 1 second. What is the value of ω in radians per second?

3. Construct the sine curve $y = 1{\cdot}75 \sin \theta$, from $\theta = 0^\circ$ to $\theta = 360^\circ$ taking intervals of 30°, y being in inches. Scales.—For y, full size; for θ, 1·5 inches to 90°. Find the mean of the mid positive ordinates.

4. Given the equation, $y = 2 \sin \theta$, fill up the following table, by calculation, taking $\sin \theta$ from a table of sines.

θ	5°	15°	25°	35°	45°	55°	65°	75°	85°	Mean.
Sin θ										—
y										
y^2										

Plot θ and y and θ and y^2 from $\theta = 0°$ to $\theta = 90°$. Scales.—y in inches, full size; θ, 5 inches to 90°.

5. Given the equations, $y_1 = 1{\cdot}7 \sin \theta$, and $y_2 = 2 \sin (\theta + 30°)$, plot the values of $y_1 + y_2$ and θ, and $y_1 y_2$ and θ for values of θ from 0° to 360° at 30° intervals.

6. A point M has a simple harmonic motion, in which the displacement x from the mid position C is given in inches by $x = 2 \sin (1{\cdot}5t + 0{\cdot}4)$, t being time in seconds and the angle being in radians.

Draw a horizontal line (along the tee square) for the path of M, on which mark the centre C and the limits of the swing. Let positive displacements be those to the right of C. Draw the *representative crank*, its rotation being clockwise. Find the positions of M when $t = 0$ and when $t = 2$ seconds and measure CM in both cases. [B.E.]

7. A weight hangs by a spring, and has an up and down simple harmonic motion of period $T = 2$ seconds and amplitude $a = 2{\cdot}25$ inches, the advance α being $\frac{\pi}{2}$ radians (at the instant when time begins to be reckoned). The displacement y from mean position at any time t is thus given by the equation

$$y = a \sin \left(\frac{2\pi}{T} t + \alpha\right) = 2{\cdot}25 \sin \left(\pi t + \frac{\pi}{2}\right)$$

Draw a curve showing the relation between y and t at any time from $t = 0$ to $t = T = 2$ seconds.

Adopt as the horizontal scale for time 3 inches to 1 second, and take the vertical scale for y full size. Read off the displacement y when $t = 0{\cdot}25$ second. [B.E.]

8. A motion in a straight line, which is compounded of two simple harmonic motions of the same period, is itself a simple harmonic motion of that period.

If x is displacement of a point at time t, this theorem is represented by the equation:—

$$x = a_1 \sin (qt + e_1) + a_2 \sin (qt + e_2) = A \sin (qt + E).$$

Determine graphically and measure the amplitude or radius A, and the advance E of the resultant motion, having given the corresponding elements of the component motions, viz.:—

$$a_1 = 2 \text{ inches}, a_2 = 3 \text{ inches}; \; e_1 = 0{\cdot}25 \text{ radian}, e_2 = 1{\cdot}1 \text{ radians}.$$

Find and measure the displacement x when $t = 0$, and also when $t = 3$ seconds, the angular velocity q being $\frac{1}{2}$ radian per second. [B.E.]

9. Three simple harmonic motions in a straight line are represented by the equations:—

$$y_1 = 1{\cdot}5 \sin 0{\cdot}4t, \quad y_2 = 1{\cdot}2 \sin (0{\cdot}8t + 1), \quad \text{and } y_3 = 0{\cdot}9 \sin (1{\cdot}6t + 0{\cdot}5),$$

where y_1, y_2, and y_3 are displacements in inches from the mean position, the angles being measured in radians and the time in seconds.

These three motions are combined. Plot the separate displacements and the resulting displacements on a time base for a complete cycle.

10. The motion of a point in a straight line is compounded of two simple harmonic motions of nearly equal periods, represented by the following equation:—

$$x = 2{\cdot}1 \sin \left(9t + \frac{\pi}{4}\right) + \sin 8t,$$

where x is displacement in inches from mean position and t is time.

Let the complete period of vibration be divided into nine equal intervals. Taking only the first, fourth, and seventh of these intervals, in each case draw a curve in which abscissæ shall represent time, and ordinates the corresponding displacements of the point.

Let the time of one of the intervals be represented on the paper by a length of 8 inches. In determining successive ordinates the method of projection from the resultant crank may be used with advantage. [B.E.]

11. In the equation

$$y = 2{\cdot}6 \sin(\theta + 31^\circ) + 0{\cdot}33 \sin(2\theta + 112^\circ),$$

which represents a simple vibration with a small superposed octave, the displacement y for any value of θ is given approximately by the construction defined on the diagram, Fig. 282.

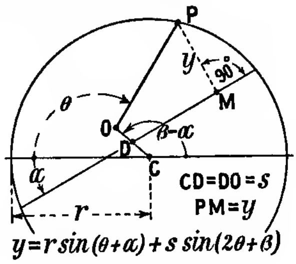


FIG. 282.

By means of this construction, or otherwise, determine y for values of θ of $0^\circ, 30^\circ, 60^\circ \ldots 360^\circ$. Plot a curve with y as ordinate on a θ base. Scale for θ.—1 inch to 60°. From your figure measure y when $\theta = 192^\circ$, and compare this with the true value of y as calculated from the equation, using the table of sines. [B.E.]

12. A point P in a plane has a compound harmonic motion, whose components parallel to two perpendicular axes OX, OY are given by the equations

$$x = a\cos(\omega t + \alpha) = 2{\cdot}5\cos\left(\theta + \frac{\pi}{3}\right) \text{ inches.}$$

$$y = b\sin(2\omega t + \beta) = 1{\cdot}5\sin\left(2\theta + \frac{\pi}{2}\right) \text{ inches.}$$

Plot the complete locus of P. [B.E.]

13. Referring to Fig. 277, p. 154, draw, full size, the locus of the point P when the frequencies of the component simple harmonic motions are as 3 : 4 instead of 2 : 3.

14. A, B, C are three collinear points in a vibrating link. AB : BC : AC = 5 : 1 : 6. The component motion of A in a certain direction is simple harmonic, with half travel 4 inches, advance -90°; that is, displacement

$$x_1 = 4\sin\left(\omega t - \frac{\pi}{2}\right) = 4\sin(\theta - 90^\circ).$$

The component motion of B in the same direction is given by

$$x_2 = b\sin\omega t = b\sin\theta.$$

Let the component motion of C be defined by the equation

$$x = a\sin(\theta + \alpha).$$

Find the half travel a and the advance α for the following values of b, and tabulate the results, as indicated :—

Values of b in inches.	0 0	0·5	1·0	1·5
Half travel a				
Advance α				

15. Taking the data of exercise 7 plot the velocities and accelerations of the weight on a time base. Scales.—Time, 3 inches to one second; velocity, maximum velocity in feet per second = 2·25 inches; acceleration, maximum acceleration in feet per second per second = 2·25 inches. Determine the lengths which represent a velocity of 1 foot per second and an acceleration of 1 foot per second per second.

16. Taking the data of exercise 9 plot the velocities and accelerations of the component simple harmonic motions also the resultant velocities and accelerations, all on the same time base, for a complete cycle. Construct the velocity and acceleration scales.

17. The slide valve of a steam engine is actuated by a Joy gear. Fig. 283

FIG. 283.

gives eight positions, 0, 1, 2, . . . 7 of the valve, corresponding to the eight crank positions θ of 0°, 45°, 90°, . . . 315°.

Measuring from the point A, the displacement x of the valve for any crank position θ is given approximately by the Fourier equation—

$$x = c + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + a_4 \cos 4\theta \\ + b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta.$$

Determine the eight constants in this equation.

If the speed of the crank shaft is 10 radians per second, what is the velocity of the valve when $\theta = 0$? [B.E.]

18. One cycle of a periodic curve A is given in Fig. 284. Express y as a

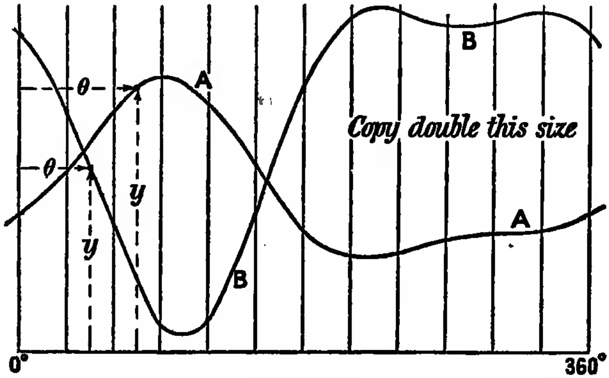


FIG. 284.

function of θ, assuming that all the Fourier vectors of the fourth and higher orders are negligible.

19. The curve B (Fig. 284) shows the displacement y inches of the block in the link of a Stephenson valve gear, for any crank position θ degrees, during one revolution of the crank shaft. Express y approximately in terms of θ by the first three terms of the Fourier series—

$$y = r_1 \sin (\theta + a_1) + r_2 \sin (2\theta + a_2) + r_3 \sin (3\theta + a_3) \\ + a \text{ constant}$$

Give values of r_1, r_2, r_3, and a_1, a_2, a_3.

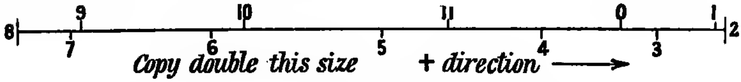


FIG. 285.

20. A point P oscillates in a straight line, the motion being repeated indefinitely. The period of oscillation being divided into twelve equal intervals,

beginning when $t = 0$, you are given, in Fig. 285, the twelve corresponding positions of P, numbered 0 to 11.

Suppose the displacement from mean position at any time t is given very approximately by the first three terms of the Fourier series—

$$x = a \sin(\omega t + \alpha) + b \sin(2\omega t + \beta) + c \sin(3\omega t + \gamma).$$

Find the elements of the motion, that is, find the three half travels a, b, c and the three angles of advance, α, β, γ for this case.

CHAPTER XI

PROJECTION

134. Descriptive Geometry.—*Practical Solid Geometry*, or *Descriptive Geometry* is that branch of geometry which treats (1) of the representation, on a plane surface, of points, lines, and figures in space, in such a way that the relative positions of the points, lines, and figures, and also the exact forms of the lines and figures are determined, and (2) of the graphic solution of problems connected with points, lines, and figures in space. The problems of descriptive geometry are best solved by means of the method of projections.

135. Projection.—When an object is seen by the eye of a spectator, rays of light come from all the visible parts of the object and converge towards a point within the eye. Now suppose that a flat sheet of glass is placed between the object and the eye of the spectator, and that each ray of light, in passing through the glass from the object to the eye, leaves a mark on the glass of the same colour and tint as the part of the object from which the ray came. In this way a picture would be produced on the surface of the glass, and if the object be removed while the picture and the eye remain stationary, the picture would convey to the mind of the spectator the same knowledge of the object as was conveyed by the presence of the object itself. Again, if instead of the rays of light from all the visible points of the object leaving an impression on the glass, only those which came from the edges of the object were to do so, an outline would be produced on the surface of the glass which, although it would not convey to the mind of the spectator the same impression as the presence of the object itself might still give a good idea of its form.

The foregoing remarks are illustrated by Fig. 286, where AB represents an object viewed by an eye at E; CD is a plane interposed between E and AB; the thin dotted lines represent a few of the rays of light passing from the edges of the object to the eye, and A′B′ is the outline obtained from the intersections of the rays of light with the plane CD. The figure A′B′ is called a *projection* of the object AB on the plane CD.

The plane upon which a projection is drawn is called a *plane of projection.*

The rays of light or imaginary lines passing from the different points of the object to the corresponding points of the projection are called *projectors.*

When the projectors converge to a point the projection is called a *radial*, *conical*, or *perspective* projection.

When the point to which the projectors converge is at an infinite distance from the object the projectors become parallel, and the projection is called a *parallel* projection.

If besides being parallel the projectors are also perpendicular to the plane of projection the projection becomes a *perpendicular*, an *orthogonal*, or an *orthographic* projection.

For the purposes of descriptive geometry orthographic projections are the most convenient and most commonly used, and when the term projection is used without any qualification orthographic projection is generally understood. In what follows projection will mean orthographic projection.

The *projection of a point* upon a plane is the foot of the perpendicular let fall from the point on to the plane.

The *projection of a line* upon a plane is the line which contains the projections of all the points of the original line.

The *projecting surface* of a line is the surface which contains the

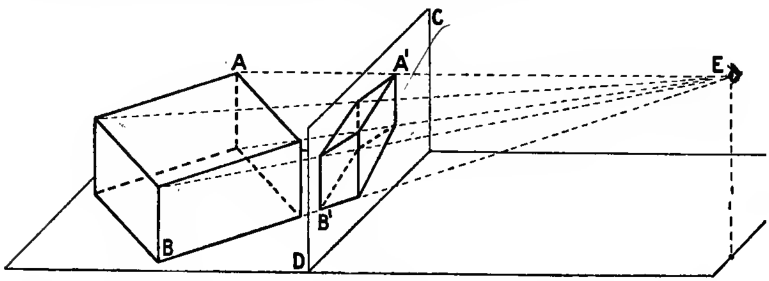


FIG. 286.

projectors of all the points of that line. When the projecting surface of a line is a plane it is called the *projecting plane* of the line. The projecting surface of a straight line is always a plane, but a line is not necessarily straight because its projecting surface is a plane. These definitions of projecting surface and projecting plane of a line and the statements which follow them apply to all kinds of projection.

One projection alone of a figure is not sufficient for determining its exact form. For example if a triangle *abc* drawn on a sheet of paper be taken as the projection on the paper of a triangle ABC somewhere above it, it is clear that the exact form of the triangle ABC will depend on the relative distances of its angular points from the paper, but the projection *abc* gives no information about these distances. If, however, another projection *a′b′c′* of the triangle ABC be obtained on a sheet of paper at right angles to the former one, then, as will be shown later, the true form of the triangle ABC may be obtained from these two projections.

The representation of an object by means of two projections, one on each of two planes at right angles to one another, and how these

projections are drawn on a flat sheet of paper will be understood by reference to Figs. 287 and 288.

Fig. 287 is to be taken as a pictorial projection of a model. Two planes of projection are shown one being vertical and the other horizontal. These planes, called *co-ordinate planes,* divide the space surrounding them into four dihedral angles or quadrants which are named, *first, second, third,* and *fourth dihedral angles* or *first, second, third,* and *fourth quadrants.* If an observer be facing the vertical plane of projection, then the first quadrant is above the horizontal plane of projection and in front of the vertical plane of projection. The second quadrant is behind the first and the others follow in order

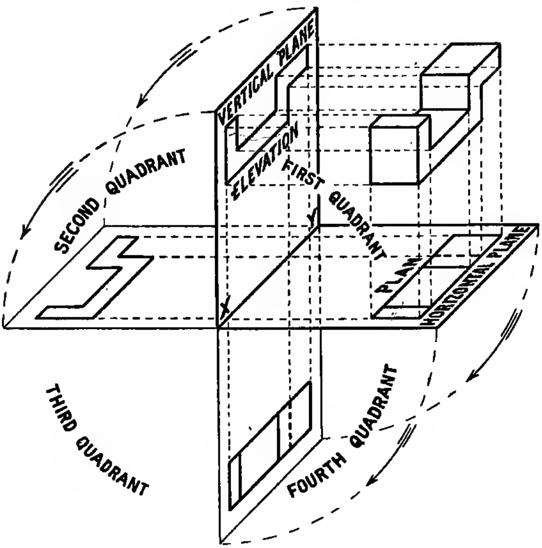


Fig. 287.

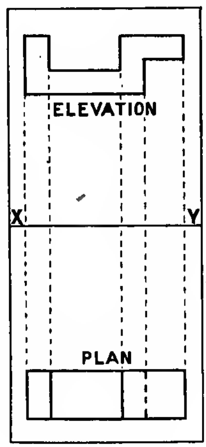


Fig. 288.

as shown. The line of intersection of the planes of projection is called the *ground line* and is lettered XY.

An object is shown in the first quadrant and projections of it on the horizontal and vertical planes of projection are also shown. The projection on the horizontal plane is called a *plan* and the projection on the vertical plane is called an *elevation.*

Now imagine the vertical plane to turn about XY as an axis, carrying with it the elevation, until it is in a horizontal position. The horizontal and vertical planes of projection will now coincide and the plan and elevation of the object will be on one flat surface and their exact forms may be drawn as shown in Fig. 288.

Instead of imagining the vertical plane to turn about XY until it is horizontal, the horizontal plane may be imagined to turn about XY until it is vertical as shown in Fig. 289.

In British and European countries the general practice in making working drawings is to conceive the object to be placed in the first quadrant as shown in Fig. 287 and the working plan and elevation are then in the positions shown in Fig. 288, the plan being below and the elevation above XY. In the United States of America the practice of conceiving the object to be placed in the third quadrant as shown in Fig. 289 is now very general and the working plan and elevation are then in the positions shown in Fig. 290, the plan being above and the elevation below XY.

Whether the object be placed in the first quadrant or in the third quadrant it is supposed to be viewed from above in obtaining the plan, consequently when the object is in the first quadrant it lies between

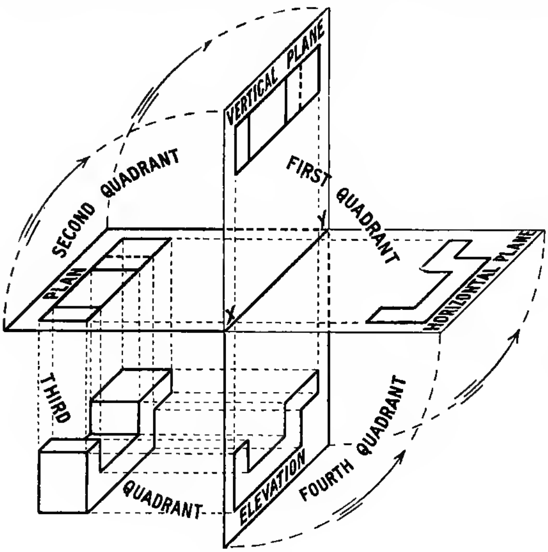


Fig. 289.

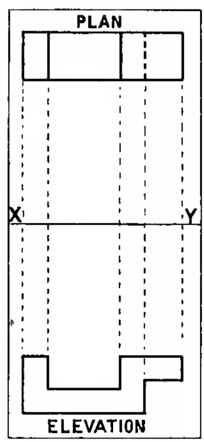


Fig. 290.

the observer and the plane of projection and the projectors from the visible parts have to go through the object to the plan, but when the object is in the third quadrant the plane of projection lies between the observer and the object, hence the projectors from the visible parts to the plan are not obstructed by the object. In like manner for the elevation, the projectors from the visible parts of the object go through the latter when it is in the first quadrant but are clear of it when in the third quadrant; hence the advantage claimed for placing the object in the third quadrant. The practice of placing the object in the first quadrant is however so well established and the advantage claimed for placing it in the third quadrant is so small, being probably more imaginary than real, that it is doubtful whether a change from the older practice should be encouraged. In any case drawings on

either system can be made with equal facility if the principles are understood.

In working problems in descriptive geometry by the method of projections on co-ordinate planes, the given points and lines may be in any one quadrant, but the lines for the solution may extend into any or all of the other quadrants.

136. Notation in Projection.—For the purpose of reference and for clearness, points, lines, and figures may be lettered. In general a point in space is denoted by a capital letter, its plan by a small italic letter, and its elevation by a small italic letter with a dash over it. Thus P denotes a point in space, p its plan and p' its elevation. A line AB in space would bave its plan lettered ab and its elevation $a'b'$. A point P in space may be referred to as the point P or as the point pp'. In like manner a line AB in space may be referred to as the line AB or as the line ab, $a'b'$.

The horizontal and vertical planes of projection may be referred to by using the abbreviations H.P. and V.P. respectively.

CHAPTER XII

PROJECTIONS OF POINTS AND LINES

137. Rules relating to the Projections of a Point.—Fig. 291 is a pictorial projection of a model showing the horizontal and vertical planes of projection in their natural positions together with four points A, B, C, and D in space, A being in the first quadrant or first dihedral angle, B in the second, C in the third, and D in the fourth. The plans and elevations of these points as obtained by dropping perpendiculars from them on to the horizontal and vertical planes of projection respectively are also shown. The positions of the

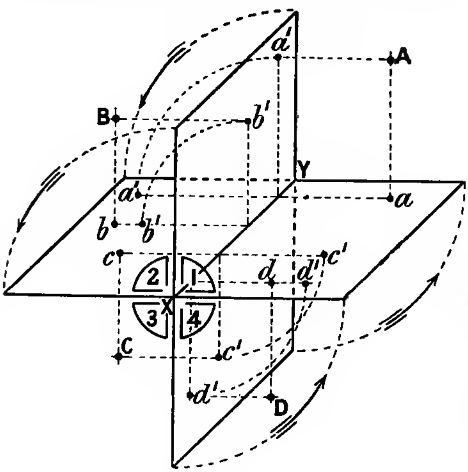


FIG. 291.

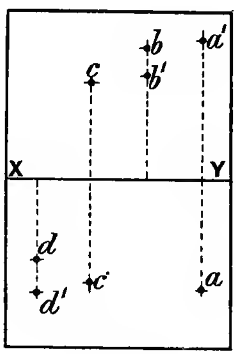


FIG. 292.

elevations of the points when the vertical plane of projection is turned about XY until it coincides with the horizontal plane are indicated, and in Fig. 292 the various plans and elevations are shown as they appear when the planes of projection are made to coincide and then placed flat on this paper.

A careful study of Figs. 291 and 292 should convince the student of the truth of the following rules relating to the projections of a point.

(1) The plan of a point is *below* or *above* XY according as the point is *in front* or *behind* the vertical plane of projection.

(2) The elevation of a point is *above* or *below* XY according as the point is *above* or *below* the horizontal plane of projection.

(3) The distance of the plan of a point from XY is equal to the distance of the point from the vertical plane of projection.

(4) The distance of the elevation of a point from XY is equal to the distance of the point from the horizontal plane of projection.

(5) The plan and elevation of a point are in a straight line perpendicular to XY.

The term *projector* has already been defined and is the line joining a point and a projection of it, but the line joining the plan and elevation of a point is also called a projector. A plan and elevation are also said to be projected, the one from the other.

138. True Length, Inclinations, and Traces of a Line.—

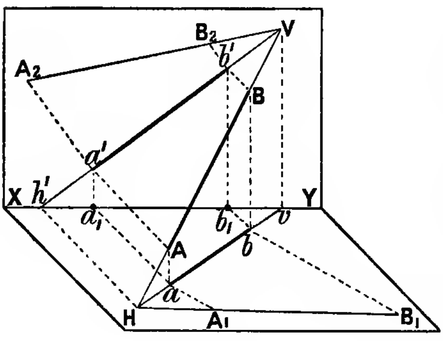


Fig. 293.

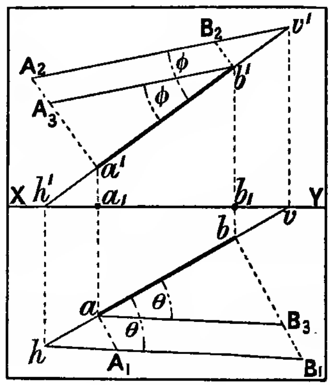


Fig. 294.

The projection of a line on a plane will be shorter than the line itself except when the line is parallel to the plane; in the latter case the line and its projection have the same length.

A line, its projection on one of the co-ordinate planes, and the projectors from its ends to that plane, form a quadrilateral concerning which everything required for constructing it is known if the plan and elevation of the line are given. One method of finding the true length of a line is therefore to construct this quadrilateral.

Let the plan ab and elevation $a'b'$ of a line AB be given as in Fig. 294. Referring to the pictorial projection in Fig. 293, it will be seen that the line AB, its plan ab, and the projectors Aa and Bb form a quadrilateral, the base ab of which is given. Also Aa is equal to $a'a_1$, Bb is equal to $b'b_1$, and the angles Aab and Bba are right angles. Hence to find the true length of AB, draw (Fig. 294) aA_1 at right angles to ab and equal to a_1a'. Next draw bB_1 at right angles to ab and equal to b_1b'. A_1B_1 is the true length of AB.

Note that if the extremities of the line AB are on opposite sides of the horizontal plane, the perpendiculars aA_1 and bB_1 must be drawn on opposite sides of the plan ab.

The inclination of a line to a plane is the angle between the line and its projection on that plane.

Referring to Figs. 293 and 294, it is evident that the inclination of AB to the horizontal plane is the angle between ab and A_1B_1, so that the construction just given for finding the true length of AB also serves for finding its inclination to the horizontal plane.

The inclination of AB to the vertical plane of projection is found by constructing the quadrilateral $a'b'B_2A_2$, in which $a'A_2$ and $b'B_2$ are at right angles to $a'b'$ and equal to a_1a, and b_1b respectively. A_2B_2 is the true length of AB and the angle between $a'b'$ and A_2B_2 is the inclination of AB to the vertical plane of projection.

If aB_3 be drawn parallel to A_1B_1 (Fig. 294) to meet bB_1 at B_3, then aB_3 will also be the true length of AB, and the angle baB_3 will be the inclination of AB to the horizontal plane. Also the length of bB_3 is equal to the difference between the distances of B and A from the horizontal plane. Hence the true length of AB and its inclination to the horizontal plane may be found by constructing the triangle abB_3. In like manner the true length and the inclination of AB to the vertical plane may be found by constructing the triangle $a'b'A_3$, in which $a'A_3$ is equal to the difference between the distances of A and B from the vertical plane.

The inclination of a line to the horizontal plane is usually denoted by the Greek letter θ (theta) and its inclination to the vertical plane by the Greek letter ϕ (phi). Notice that θ is the letter O with a *horizontal* line through it, while ϕ is the same letter with a *vertical* line through it.

The true length of a line AB and its inclinations to the planes of projection may also be found as follows. Referring to Fig. 295, through b draw a_1bb_1 parallel to XY. With centre b and radius ba describe the arc aa_1, cutting a_1bb_1 at a_1. Draw a_1a_1' perpendicular to XY to meet a line through a' parallel to XY at a_1'. $a_1'b'$ will be the true length of AB and the angle $b'a_1'a'$ will be its inclination θ to the horizontal plane. An inspection of the figure will be sufficient to make clear the corresponding construction for finding the inclination ϕ of the line to the vertical plane.

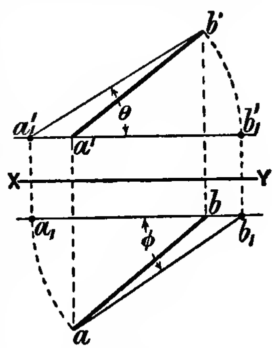


Fig. 295.

Comparing the constructions shown in Figs. 294 and 295, it will be seen that in both a quadrilateral is drawn having a base equal to one of the projections of the line, and in Fig. 294 this base is made to coincide with that projection while in Fig. 295 the base is made to coincide with XY.

When the inclination of a line is mentioned without reference to

any particular plane, inclination to the horizontal plane is generally understood.

The trace of a line on a surface is a point where the line intersects the surface. When the *traces of a straight line* are mentioned without reference to any particular surface, the points in which the line or the line produced cuts the planes of projection are understood.

The horizontal trace of a line is the point where the line or the line produced cuts the horizontal plane of projection, and *the vertical trace of a line* is the point where the line or the line produced cuts the vertical plane of projection.

After making the construction for finding the true length of a line, shown in Fig. 294, if B_1A_1 be produced to meet ba at h then h will be the horizontal trace of the line AB. In like manner if A_2B_2 be produced to meet $a'b'$ at v' then v' will be the vertical trace of AB. The correctness of these constructions is obvious from an inspection of Fig. 293.

If it is only the traces of a line which are required, then it is only necessary to produce the elevation $a'b'$ (Fig. 294) to meet XY at h' and then draw $h'h$ perpendicular to XY to meet the plan ab produced at h in order to determine the horizontal trace. In like manner, by producing the plan ab to meet XY at v, and drawing vv' perpendicular to XY to meet the elevation $a'b'$ produced at v', the vertical trace is determined. This construction fails however when the projections of the line are perpendicular to XY but the construction previously given will apply in this case also, provided that the plans and elevations of two points in the line are definitely marked and lettered.

When the projections of a straight line are perpendicular to XY the line itself is perpendicular to XY although it may not meet XY.

When a line is parallel to one of the planes of projection it has no trace on that plane.

When a line is perpendicular to one of the planes of projection it has no trace on the other plane of projection, and its projection on the plane to which it is perpendicular is a point.

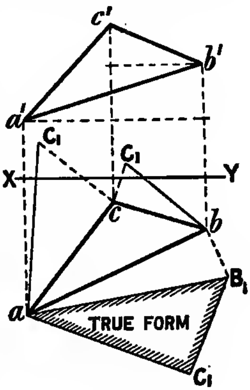


FIG. 296.

139. True Form of a Plane Figure.—The projection of a plane figure on a plane will not have the same form or dimensions as the figure itself excepting when the plane of the figure is parallel to the plane of projection, in which case the figure and its projection will be exactly alike. To determine the true form of any plane figure, whose projections are given, it is necessary to know the true distances of a sufficient number of points in it from one another; now these distances may be found by one of the constructions given in the preceding article.

An example is shown in Fig. 296, where *abc* and *a′b′c′* are the given projections of a triangle. The true lengths of the sides are found by the constructions shown, and the triangle which is the true form of the triangle whose projections are given is then drawn.

If the figure given by its projections has more than three sides it should first be divided into triangles; the true forms of these triangles are then found and assembled to give the true form of the whole figure.

In many cases the true form of a plane figure whose projections are given is best determined by the method of rabatment described in Art. 189, p. 221.

The traces of the sides of a plane figure whose projections are given may be determined by one of the constructions of the preceding article, and it will be found that all the horizontal traces will lie in one straight line, and all the vertical traces will lie in another straight line, and these two straight lines will intersect on XY, excepting when they are parallel, in which case they will be parallel to XY. This matter will be referred to again in Chapter XVI.

140. To mark off a given Length on a given Line.—Let *ab*, *a′b′* (Fig. 297) be the projections of the line, it is required to find the projections of a point C in this line so that AC shall be a given length.

Determine A_1B_1 the true length of AB. Make A_1C_1 equal to the given length. Through C_1 draw C_1c perpendicular to *ab* to meet *ab* at *c*. Through *c* draw *cc′* perpendicular to XY to meet *a′b′* at *c′*. *c* and *c′* are the projections required.

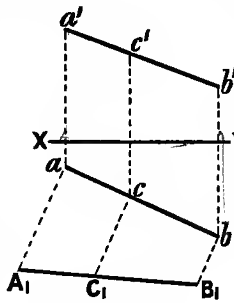


Fig. 297.

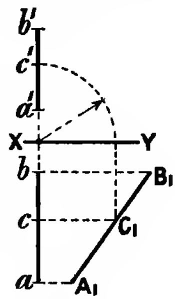


Fig. 298.

If the given projections of the line are perpendicular to XY (Fig. 298), determine *c* as before then make the distance of *c′* from XY equal to C_1c.

141. Given the True Length of a Line and the Distances of its Extremities from the Planes of Projection, to draw its Projections.—First determine the projections *aa′* (Fig. 299) of the end A of the line. On a_1a make a_1c equal to the distance of the end B of the line from the vertical plane of projection, and on a_1a' make a_1c' equal to the distance of B from the horizontal plane. Through *c* and *c′* draw lines parallel to XY. These parallels to XY will contain the plan and elevation respectively of B. With *a′* as centre and the given true length of the line as radius describe an arc to cut the parallel to XY through *c′* at

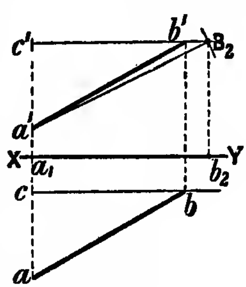


Fig. 299.

B_2. From B_2 draw the perpendicular B_2b_2 to XY. Then since $a'B_2$ is the true length of the line and a_1a' and b_2B_2 are the distances of its extremities from the horizontal plane it follows (see Fig. 293, p. 172) that a_1b_2 is the length of the plan of the line. Hence, if with centre a and radius equal to a_1b_2 an arc be described to cut the parallel to XY through c at b then ab is the required plan of the line. A perpendicular to XY from b to meet the parallel to XY through c' determines b', and $a'b'$ is the required elevation of the line.

142. Given the Projection of a Line on one of the Planes of Projection, its Inclination to that Plane, and the Distance of one end from it, to determine its other Projection.—Let ab (Fig. 300) be the given projection, θ the inclination to the horizontal plane and let the distance of the end A of the line from the horizontal plane be given.

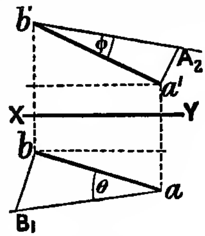


FIG. 300.

The distance of a' from XY is equal to the given distance of A from the horizontal plane. At a make the angle baB_1 equal to θ and draw bB_1 at right angles to ab to meet aB_1 at B_1. The distance of b' from XY is equal to bB_1 plus the distance of a' from XY.

If the elevation $a'b'$ is given and the inclination ϕ of the line to the vertical plane of projection and also the distance of A from the vertical plane of projection, the construction is similar to that just given and is shown in Fig. 300.

143. Given the Inclination of a Line to one of the Planes of Projection and the Angle which its Projection on that Plane makes with the Ground Line, to draw its Projections.—Let the line be inclined at an angle θ to the horizontal plane, and let its plan make an angle α with XY. From a point C in XY (Fig. 301) draw Cb' inclined at an angle θ to XY. Draw $b'b$ at right angles to XY. Then bC is the length of the plan of the line whose true length is Cb' and whose inclination to the horizontal plane is θ. With b as centre and bC as radius describe an arc, and from b draw ba to meet this arc at a and make an angle α with XY. ab is the required plan of the line. A perpendicular from a to XY determines a' and $a'b'$ is the required elevation of the line.

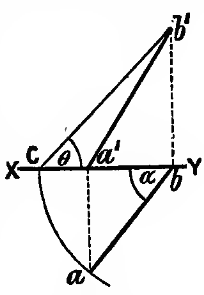


FIG. 301.

144. Given the Inclinations of a Line to the Planes of Projection, to determine its Projections.—Let the line be inclined at an angle θ to the horizontal plane and at an angle ϕ to the vertical plane of projection. From a point C in XY (Fig. 302) draw Cb' inclined at an angle θ to XY. Draw $b'b$ at right angles to XY. Then bC is the length of the plan of the line whose true length is Cb' and whose inclination to the horizontal plane is θ.

From b' draw $b'D$ making the angle $Cb'D$ equal to ϕ. Draw CD

perpendicular to $b'D$. Then $b'D$ is the length of the elevation of the line whose true length is Cb' and whose inclination to the vertical plane of projection is ϕ.

With centre b' and radius $b'D$ describe an arc to cut XY at a'. With centre b and radius bC describe the arc Ca to meet the perpendicular to XY from a' at a. ab is the plan and $a'b'$ is the elevation required.

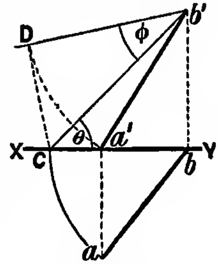


FIG. 302.

Note. The sum of the angles θ and ϕ may vary between 0° and 90°. When $\theta + \phi = 0°$ the projections of the line are parallel to XY, and when $\theta + \phi = 90°$ the projections of the line are perpendicular to XY.

145. Projections of Parallel Lines.—The projections of parallel lines on to the same plane are parallel. Hence if it is required to draw the projections of a line which shall pass through a point whose projections pp' are given and which shall be parallel to another line whose projections ab, $a'b'$ are given, draw through the plan p a line pq parallel to ab and through p' draw $p'q'$ parallel to $a'b'$. Then pq and $p'q'$ are the projections required.

The projections on the same plane of equal parallel lines are equal.

146. Conditions that Two Lines whose Projections are given may intersect.—If two lines intersect, their point of intersection is a point in each of the lines, therefore the plan of that point must be on the plan of each of the lines, and therefore the plans of the lines must intersect, and the point of intersection of the plans is the plan of the point of intersection of the lines. In like manner the elevations of the lines must intersect at a point which is the elevation of the point of intersection of the lines. But the plan and elevation of a point are in the same straight line at right angles to the ground line.

Hence the conditions that two lines intersect is that their plans and elevations respectively intersect and that the points of intersection are in the same straight line at right angles to the ground line.

There are exceptions to this rule. When the lines are perpendicular to the ground line and lie in the same vertical plane or when one of the lines only is perpendicular to the ground line the lines may or may not intersect. In such cases an auxiliary projection will show whether the lines intersect or not.

147. Angle between Two Intersecting Lines.—Let AC and BC be two intersecting lines whose projections are given; it is required to find the true angle between these lines. Take any point D in AC and any point E in BC. Determine by Art. 139, p. 174, the true form of the triangle DCE. The angle C of this triangle is the angle required.

The construction is simplified in many cases by taking for the points D and E the horizontal or vertical traces of the lines. In Fig. 303, D and E are the horizontal traces of the lines AC and BC respectively. dC_1e is the true angle between AC and BC.

Fig. 303 also shows how to determine the projections of a line which bisects the angle ACB. Draw C_1r bisecting the angle dC_1e and intersecting de at r. Find r' the elevation of R. Join cr and $c'r'$. These are the projections of the line which bisects the angle between AC and BC. This also suggests the construction for finding the projections of a line which shall intersect the lines AC and BC and make a given angle with one of them.

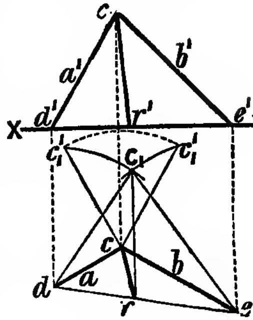


FIG. 303.

148. Angle between Two Non-intersecting Lines.—Let AB and CD be two non-intersecting lines whose projections are given; it is required to find the true angle between these lines. Through any point P in one of the lines, say AB, draw a line PQ parallel to the other line. The angle between these two intersecting lines PQ and AB is the angle required.

Exercises XII

1. Show the projections of the following points, using the same ground line for all the projections, and making the projectors 0·5 inch apart.

A, 1·2 inches in front of the V.P. and 1·8 inches above the H.P.
B, 1·9 inches in front of the V.P. and 1·7 inches below the H.P.
C, 1·4 inches in front of the V.P. and in the H.P.
D, 2 inches behind the V.P. and 1·6 inches above the H.P.
E, 2 inches behind the V.P. and 1·6 inches below the H.P.

2. Show the projections of the following points as in the preceding exercise.

A, 1·3 inches behind the V.P. and in the H.P.
B, in the V.P. and in the H.P.
C, 1·4 inches in front of the V.P. and 1·6 inches below the H.P.
D, in the V.P. and 1·2 inches below the H.P.
E, 1·5 inches behind the V.P. and 1·5 inches above the H.P.

3. State the exact positions of the points whose projections are given in Fig. 304 with reference to the planes of projection.

4. $a'b'$, the elevation of a straight line, is 2 inches long and it is inclined at 30° to XY. The end A is in the horizontal plane and 2 inches from the vertical plane. The end B is in the first vertical plane and the whole line is in the first dihedral angle. Draw the plan and elevation of the line.

5. A triangle ABC is in the first dihedral angle. A is in the V.P. and 2 inches above the H.P. B and C are in the H.P. ab makes 45° with XY and abc is an equilateral triangle of 2 inches side. Draw the plan and elevation of the triangle ABC.

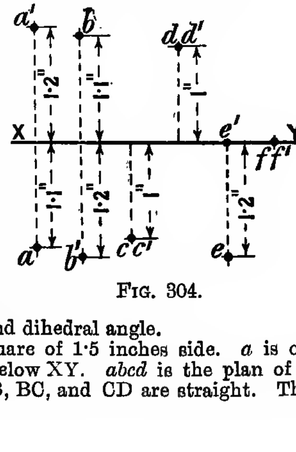


FIG. 304.

6. The same as the preceding exercise except that while the side AB remains in the first dihedral angle the side AC is placed in the second dihedral angle.

7. ab, bc, and cd form three sides of a square of 1·5 inches side. a is on XY, ab is inclined at 30° to XY, and $abcd$ is below XY. $abcd$ is the plan of a piece of thin wire ABCD of which the parts AB, BC, and CD are straight. The

heights of the points A, B, C, and D above the H.P. are 0·5, 0·9, 1·2, and 1·1 inches respectively. Draw the plan and elevation of the wire.

8. $a'b'$, 2·5 inches long, is the elevation of a straight line which is parallel to the V.P. The end A is in the H.P. and 1 inch in front of the V.P. The end B is 1·5 inches above the H.P. Draw the plan and elevation of AB.

9. Draw the projections of the following lines, and then find their traces where possible.

AB, 2 inches long, parallel to XY, 1 inch above the H.P. and 1·3 inches in front of the V.P.

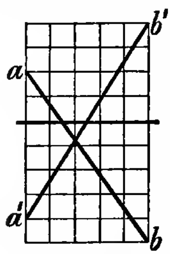
CD, 2·2 inches long, parallel to the H.P., inclined at 30° to the V.P., the end C to be in the V.P. and 1·2 inches above the H.P.

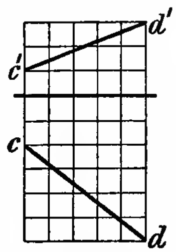
EF, 1·5 inches long, perpendicular to the H.P., the end E to be 0·5 inch above the H.P. and 1 inch in front of the V.P.

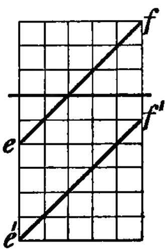
GH, 2 inches long, perpendicular to the V.P., 1 inch above the H.P., the end G to be 0·5 inch in front of the V.P.

10. The plan of a line is 2 inches long and it makes 35° with XY, the elevation makes 45° with XY, and the line intersects XY. Draw the plan and elevation of the line and then find its true length and its inclinations to the planes of projection.

11. Find the true length, the inclinations to the planes of projection, and the horizontal and vertical traces of each of the lines whose projections are given in







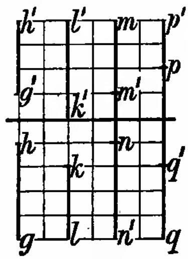
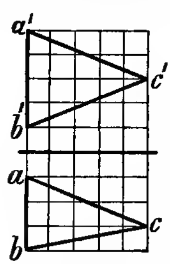


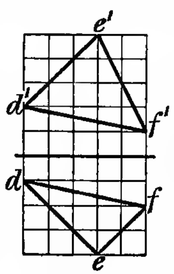
FIG. 305. FIG. 306. FIG. 307. FIG. 308.

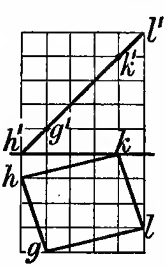
In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

Figs. 305–308. Show also in each case the projections of a point in the line whose true distance from the lower end of the line is 1 inch.

12. Determine the true form of each of the plane figures whose projections are given in Figs. 309–312. Find also for each figure the horizontal and vertical







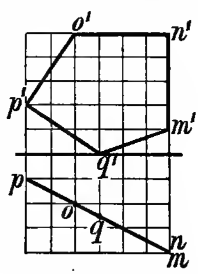


FIG. 309. FIG. 310. FIG. 311. FIG. 312.

In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

traces of each side, where possible, and show that the horizontal traces are in one straight line and the vertical traces in another.

13. A straight line 2·5 inches long has one end 0·3 inch in front of the vertical plane and 1·5 inches above the horizontal plane while the other end is 1·75 inches in front of the vertical plane and 0·6 inch above the horizontal plane. Draw the plan and elevation of the line.

14. A straight line inclined at 50° to the horizontal plane has one end 2 inches above the horizontal plane and 0·5 inch in front of the vertical plane of projection. The other end of the line is on the horizontal plane and 2 inches in front of the vertical plane of projection. Draw the projections of the line.

15. A straight line is 2·75 inches long. One end is in the horizontal plane and the other end is in the vertical plane of projection. The line is inclined at 30° to the horizontal plane and its plan makes an angle of 45° with XY. Draw the projections of the line.

16. *a'b'* (Fig. 313) is the elevation of a straight line. B is in the vertical plane of projection. The line is inclined at 35° to the horizontal plane. Draw the plan of the line.

17. *cd* (Fig. 314) is the plan of a straight line whose true length is 3 inches. The end C of the line is 0·5 inch below the horizontal plane and the end D is above the horizontal plane. Draw the elevation of the line.

18. *abc* (Fig. 315) is the plan of a triangle. The point A is on the horizontal plane. The point B is above the horizontal plane and is higher than the point C. The true length of AB is 2·5 inches and the inclination of BC to the horizontal plane is 40°. Draw the elevation of the triangle and find the true length of AC.

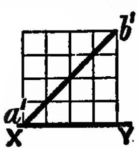


FIG. 313.

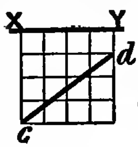


FIG. 314.

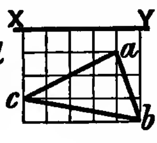


FIG. 315.

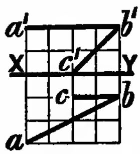


FIG. 316.

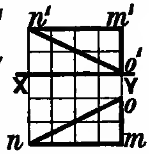


FIG. 317.

In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

19. The middle point of a straight line 3 inches long is 1 inch above the horizontal plane and 1·25 inches in front of the vertical plane of projection. The line is inclined at 30° to the horizontal plane and 40° to the vertical plane of projection. Draw the projections of the line.

20. Find the real angle between the lines whose projections are given in Fig. 316. Show also the projections of the line bisecting the angle ABC.

21. Determine the angle MON of the triangle whose projections are given in Fig. 317, and draw the projections of the line which passes through the point M and intersects the line ON at right angles.

22. ABC is an equilateral triangle of 2·5 inches side. A is in the horizontal plane and B is in the vertical plane of projection. AB is inclined at 45° to the horizontal plane and 30° to the vertical plane of projection. BC is inclined at 35° to the horizontal plane. Draw the plan and elevation of the triangle.

CHAPTER XIII

PROJECTIONS OF SIMPLE SOLIDS IN SIMPLE POSITIONS

149. Projections of Solids.—It has already been indicated (Chap. XI.) that one object of Descriptive Geometry is to convey to the mind an impression of the exact form and size of objects, which have length, breadth, and thickness, by means of representations of them on a surface, which has length and breadth only.

Now, a solid may be conceived to be made up of an immense number of small particles or material points, the relative positions of which may be represented by means of their projections on two planes, as explained in the two preceding chapters; but as in looking at a solid it is generally only the points on its external surface which are seen, all impressions of the form and size of objects are derived from the form and extent of their surfaces. It is therefore unnecessary in representing an object on paper to give the projections of points in its interior, that is, it is only necessary to represent the surface of the object.

Again, the form and extent of a surface are generally known when the forms, lengths, and relative positions of a sufficient number of lines on that surface are known. But it has been seen that by the method of projections the forms, lengths, and relative positions of lines may be represented on a single flat surface. Hence, a solid may be represented on paper by the projections of a sufficient number of lines on its surface.

When the solid has plane faces, the projections of the boundary lines of the faces are all that is necessary in order to represent them.

150. Definitions of Solids.—There are certain solids, of definite and simple form, which are of frequent occurrence and which will now be defined.

A *polyhedron* is a solid bounded entirely by planes. The *edges* of a polyhedron are the lines of intersection of its bounding planes. The *sides* or *faces* of a polyhedron are the plane figures formed by its edges.

A polyhedron is said to be *regular* when its faces are equal and regular polygons. Generally when a polyhedron is referred to a *regular polyhedron* is understood.

There are only five regular polyhedra, namely, the *tetrahedron*, the *cube*, the *octahedron*, the *dodecahedron*, and the *icosahedron*.

The *tetrahedron* (Fig. 318) has four faces, all equilateral triangles.

The *cube* (Fig. 319) has six faces, all squares.

The *octahedron* (Fig. 320) has eight faces, all equilateral triangles.

The *dodecahedron* (Fig. 321) has twelve faces, all pentagons.

The *icosahedron* (Fig. 322) has twenty faces, all equilateral triangles.

TETRAHEDRON.

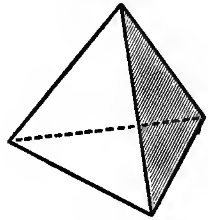


FIG. 318.

CUBE.

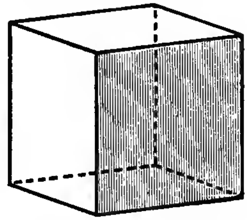


FIG. 319.

A *prism* is a polyhedron having two of its faces, called its *ends* or *bases*, parallel, and the remaining faces are parallelograms. Those faces of a prism which are parallelograms are generally called the *sides* of the prism.

OCTAHEDRON.

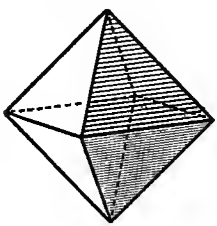


FIG. 320.

DODECAHEDRON.

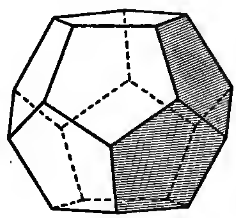


FIG. 321.

ICOSAHEDRON.

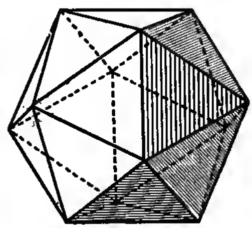


FIG. 322.

A *parallelepiped* is a prism whose bases are parallelograms.

A *pyramid* is a polyhedron having a polygon for its base, and for its sides it has triangles which have a common vertex and the sides of the polygon for their bases. The common vertex of the triangles is called the *vertex* or *apex* of the pyramid.

The *axis* of a prism is the straight line joining the centres of its ends; and the axis of a pyramid is the straight line joining its vertex to the centre of its base.

A *right prism* has its axis at right angles to its ends; and an *oblique prism* has its axis inclined to its ends.

A *right pyramid* has its axis at right angles to its base; and an *oblique pyramid* has its axis inclined to its base.

The *altitude* of a prism is the perpendicular distance between its ends; and the altitude of a pyramid is the perpendicular distance of its vertex from its base.

Prisms and pyramids are named from the form of their bases—as, triangular, square, pentagonal, hexagonal, etc. Examples of prisms and pyramids are shown in Figs. 323–326.

A *cylinder* resembles a prism. If the sides of the bases of a prism be continually diminished in length and increased in number, the ultimate form of the boundary lines of the bases will be curved lines,

and the ultimate form of the prism will be a cylinder. A *right circular cylinder* has its axis at right angles to its ends, which are equal circles.

RIGHT SQUARE PRISM.

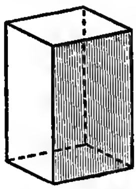


FIG. 323.

OBLIQUE PENTAGONAL PRISM.

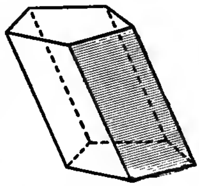


FIG. 324.

RIGHT HEXAGONAL PYRAMID

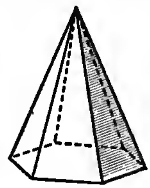


FIG. 325.

OBLIQUE HEPTAGONAL PYRAMID.

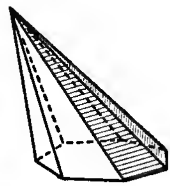


FIG. 326.

A right circular cylinder may also be defined as a solid described by the revolution of a rectangle about one of its sides, which remains stationary. The fixed line about which the rectangle revolves is the *axis*, and the circles described by the opposite revolving sides are the *bases* or *ends* of the cylinder.

The diameter of a right circular cylinder is the diameter of its circular ends.

A *cone* resembles a pyramid. If the sides of the base of a pyramid be continually diminished in length and increased in number, the ultimate form of the boundary line of the base will be a curved line, and the ultimate form of the pyramid will be a cone.

RIGHT CIRCULAR CYLINDER.

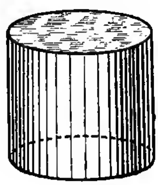


FIG. 327.

RIGHT CIRCULAR CONE.

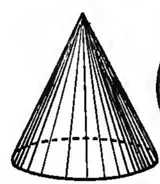


FIG. 328.

SPHERE.

FIG. 329.

A *right circular cone* has its axis at right angles to its base, which is a circle. A right circular cone may also be defined as a solid described by the revolution of a right angled triangle about one of the sides containing the right angle, which side remains stationary. The fixed line about which the triangle revolves is the axis, and the circle described by the other side containing the right angle is the base of the cone.

A *sphere* is a solid every point on the surface of which is at the same distance from a point within it called its centre. A sphere may also be defined as a solid described by the revolution of a semicircle about its diameter, which remains stationary. The middle point of the diameter of the semicircle is the centre of the sphere.

151. Projections of a Prism.—The projections of a prism are simplest when the solid is so placed that its ends are parallel to one of the planes of projection, and whatever projections of the prism may be

required it is generally necessary to first draw the projections of the solid when so situated.

When the ends of the prism are parallel to one of the planes of projection, their projections on that plane will show their true form, and as the form of the ends is supposed to be given these projections

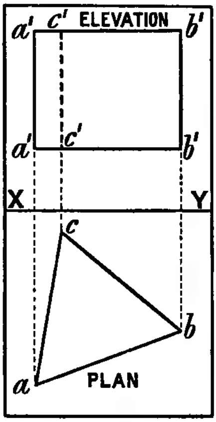


FIG. 330.

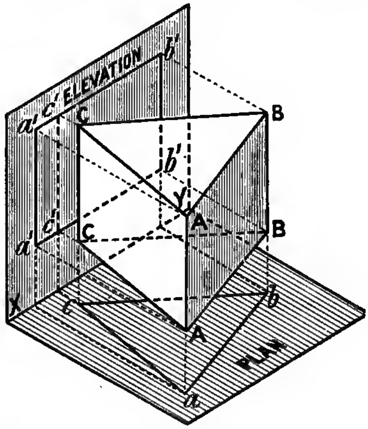


FIG. 331.

are drawn first. If the prism is a right prism the projections of the two ends on the plane of projection to which they are parallel will evidently coincide. The projections of the ends on the other plane of projection will be straight lines parallel to the ground line and at a distance apart equal to the altitude of the prism.

The foregoing observations are illustrated by Figs. 330, 331, and 332. Fig. 330 shows the plan and elevation of a right triangular prism when its ends are horizontal. The plan *abc* is first drawn, its position in relation to XY being arbitrarily chosen unless it is specified. The elevation is projected from the plan, as shown. The height of the elevation of the lower end above XY is equal to the height of that end above the horizontal plane, which may be arbitrarily chosen, unless it is specified. The distance between the horizontal lines which are the elevations of the ends is equal to the altitude of the prism, which would be given.

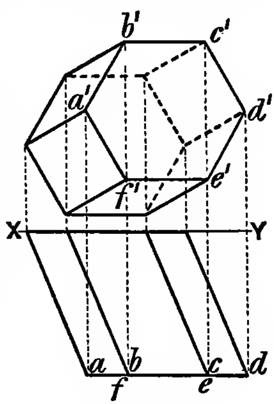


FIG. 332.

Fig. 331 is a pictorial projection of the prism and the planes of projection in their true positions.

Fig. 332 shows the plan and elevation of an oblique hexagonal prism when its ends are parallel to the vertical plane of projection, one end being in that plane. In this case the elevation is first drawn. The hexagons which are the elevations of the ends of the prism have the sides of the one respectively parallel to the sides of the other, their relative positions being otherwise determined from the specification of the prism. It will be observed that the hexagons have been placed with a diameter parallel to XY.

It will be noticed that in Fig. 330 the elevation of one of the edges of the prism is shown as a dotted line and that in Fig. 332 the elevations of five of the edges are so shown. The reason for this is that these edges are hidden by the solid when it is viewed from the front. In like manner if an edge of the solid is hidden when viewed from above the plan of that edge would be shown by a dotted line.

When a projection of an edge which should be a full line coincides with a projection of an edge which should be a dotted line only the full line can be shown.

152. Projections of a Pyramid.—The projections of a pyramid are simplest when the solid is so placed that its base is parallel to one of the planes of projection, and whatever projections of the pyramid may be required it is generally necessary to first draw the projections of the solid when so situated.

When the base of the pyramid is parallel to one of the planes of projection, its projection on that plane will show its true form, which

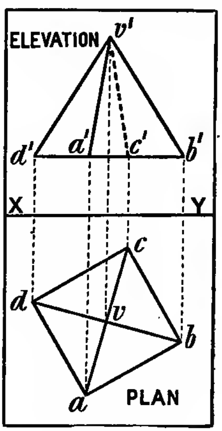


FIG. 333.

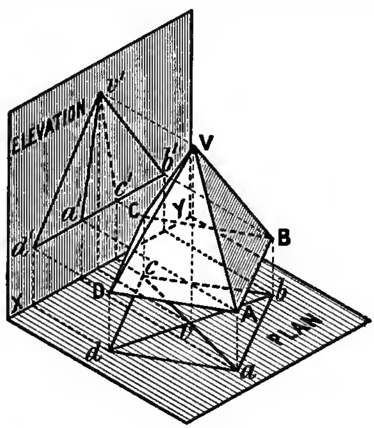


FIG. 334.

is supposed to be given. This projection is therefore drawn first. If the pyramid is a right pyramid the projection of its vertex on the plane of projection to which the base is parallel will be at the centre of the projection of the base on that plane. The projection of the base on the other plane of projection will be a straight line parallel

to the ground line, and the projection of the vertex will be at a distance from the projection of the base equal to the altitude of the pyramid.

These observations are illustrated by Figs. 333 to 336. Fig. 333 shows the plan and elevation of a right square pyramid when its base is horizontal. The plan is first drawn. This plan is a square *abcd* with its diagonals intersecting at *v*, the plan of the vertex of the pyramid. The position of the square *abcd* in relation to XY is arbitrarily chosen unless it is specified. The elevation is projected from the plan, as shown. The height of the elevation of the base above XY is equal to the height of the base above the horizontal plane, which may be arbitrarily chosen, unless it is specified. The distance of the elevation of the vertex from the elevation of the base is equal to the altitude of the pyramid, which would be given. Fig. 334 is a pictorial projection of the pyramid and the planes of projection in their true positions.

Fig. 335 shows the plan and elevation of an oblique pentagonal pyramid when its base is parallel to the vertical plane of projection.

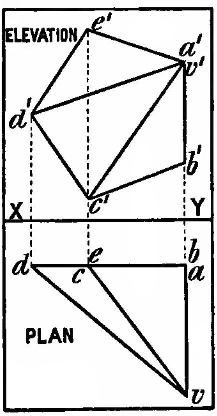


Fig. 335.

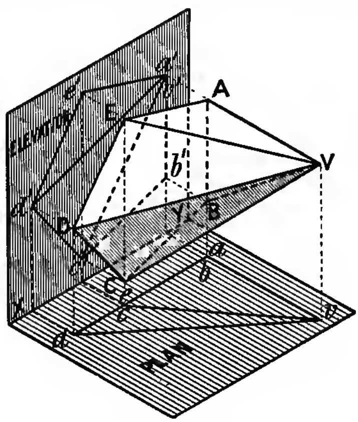


Fig. 336.

The vertex of the pyramid is in a straight line at right angles to the base and passing through one corner of the base. The elevation is first drawn. The elevation of the base is a regular pentagon *a'b'c'd'e'* which has been placed with one side *a'b'* at right angles to XY. The vertex has been placed in that perpendicular to the base which passes through the corner whose elevation is *a'*. The elevation *v'* of the vertex therefore coincides with *a'*. Joining *v'* to *c'* and *d'* completes the elevation. It will be noticed that the elevations of the edges VB and VE coincide with the elevations of the edges AB and AE of the base. The plan is projected from the elevation, as shown. Fig. 336 is a pictorial projection of the pyramid and the planes of projection in their true positions.

153. The Altitude of a Regular Tetrahedron.—Since the regular tetrahedron has four faces all equilateral triangles, this solid is evidently a right pyramid on a triangular base. Its projections may therefore be drawn as described in the preceding article if its base and altitude are known. The altitude will evidently depend on the size of the base and is found as follows. Fig. 337 shows the plan of a tetrahedron when one face ABC, which may be called its base, is on the horizontal plane. *av* is the plan of one of the sloping edges, and since all the faces are equilateral triangles, the true length of the edge of which *av* is the plan will be equal to *ab*. Hence, with *a* as centre and a radius *ab* draw an arc to cut *vv'*, which is perpendicular to *av*, at *v'*. *vv'* is the altitude required.

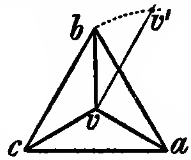


FIG. 337.

154. Projections of the Octahedron.—The lines joining the opposite angular points of the octahedron are its axes. There are three of these lines all equal in length, and bisecting one another at right angles at the centre of the solid.

It will be found on examination that the octahedron may be divided, in three different ways, into two right square pyramids having a common base, the triangular faces being equilateral triangles. Two of the axes of the octahedron are the diagonals of the square which forms the base of the two forementioned pyramids, while the third axis is the line joining their vertices.

If the octahedron be placed with an axis perpendicular to one of the planes of projection its projection on that plane will be a square with its two diagonals. Fig. 338 shows the plan and elevation of an octahedron when one axis is vertical. The plan is a square *abcd* with its two diagonals *ac* and *bd* and this projection is first drawn. The square *abcd* is the plan of the common base of two pyramids into which the octahedron may be divided, the point *v* where the diagonals *ac* and *bd* intersect being the plan of the vertices of the pyramids. *v'v'* the elevation of the vertical axis of the octahedron is at right angles to XY and has a length equal to *ac* or *bd*. *a'b'c'd'* the elevation of the common base of the two pyramids is a straight line parallel to XY and bisecting *v'v'*.

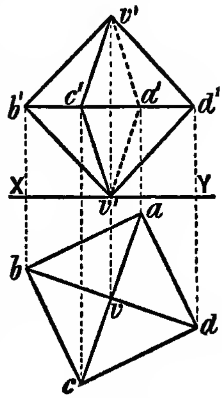


FIG. 338.

155. Projections of the Cylinder, Cone, and Sphere.—When the axis of a right circular cylinder is perpendicular to one of the planes of projection, the projection of the cylinder on that plane will be a circle having a diameter equal to that of the cylinder. The projection of the cylinder on the other plane of projection will be a rectangle, one side being parallel to the ground line and equal to the diameter of the circle, while an adjacent side is equal in length to the axis of the

cylinder. Fig. 339 shows the plan and elevation of a right circular cylinder when its axis is perpendicular to the vertical plane of projection.

When the axis of the cylinder is parallel to both planes of projections, each projection will be a rectangle equal to the one

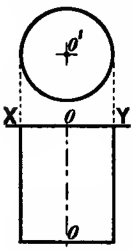


Fig. 339.

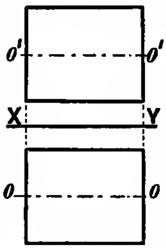


Fig. 340.

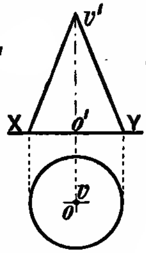


Fig. 341.

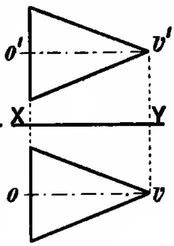


Fig. 342.

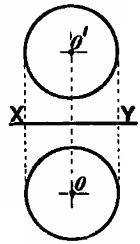


Fig. 343.

described above, but that side which is equal to the diameter of the cylinder will be perpendicular to the ground line. Fig. 340 shows the plan and elevation of a right circular cylinder when its axis is parallel to both planes of projection and therefore also parallel to the ground line.

When the axis of a right circular cone is perpendicular to one of the planes of projection, the projection of the cone on that plane will be a circle having a diameter equal to that of the base of the cone. The projection of the cone on the other plane of projection will be an isosceles triangle, its base being parallel to the ground line and equal to the diameter of the base of the cone, and having an altitude equal to that of the cone. Fig. 341 shows the plan and elevation of a right circular cone when its axis is vertical.

When the axis of the cone is parallel to both planes of projection, each projection will be a triangle equal to the one just described, but the base will be perpendicular to the ground line. Fig. 342 shows the plan and elevation of a right circular cone when its axis is parallel to both planes of projection, and therefore also parallel to the ground line.

All projections of a sphere are circles having a diameter equal to that of the sphere. The plan and elevation of a sphere have their centres in a straight line at right angles to the ground line as shown in Fig. 343.

Exercises XIII

1. The three rectangles (Fig. 344) are the plans of three right prisms the ends of which are vertical and 1 inch square. Each prism is 3 inches long. *ab*, *cd*, and *ef*, the plans of the axes of the prisms, bisect one another. The first, or lowest, prism rests on the ground, the second rests on top of the first, and the third rests on top of the second. Draw these plans and from them project the elevations of the prisms.

2. A pictorial projection of an angle bracket is given in Fig. 345. Draw the plan and elevation of this bracket when the edge AC is in the vertical plane of projection and the edge AB is in the horizontal plane of projection and inclined at 30° to XY.

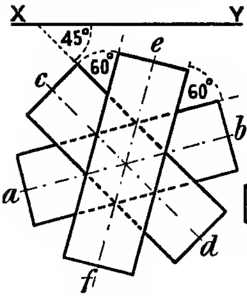


FIG. 344.

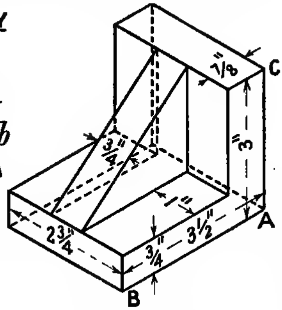


FIG. 345.

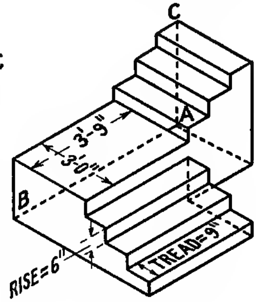


FIG. 346.

3. Fig. 346 shows, in pictorial projection, two flights of steps and an intermediate landing. Draw the plan and elevation of the steps and landing when the edge AC is in the vertical plane of projection, and the edge AB is on the ground and inclined at 30° to XY. Scale, half inch to one foot.

4. A cube of 1 inch edge has two faces horizontal, the lower one being 1 inch above the ground. The square which is the plan of the cube has one side inclined at 30° to XY, the corner which is nearest to XY being 1 inch distant from it. On each face of this cube there is fixed a cube. The corners of the bases of the added cubes are at the middle points of the edges of the original cube. Draw the plan and elevation of this built up solid.

5. A plan of an oblique triangular prism is given in Fig. 347. The base ABC is on the horizontal plane. The altitude of the prism is 1·5 inches. Draw this plan and from it project an elevation on XY. Draw also the plan and elevation of this prism when the base is on the ground and the edge AB is parallel to the ground line.

6. Fig. 348 shows the plan of an oblique prism when one end is on the horizontal plane. The altitude of the prism is 1·5 inches. Draw this plan and from it project an elevation on XY.

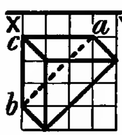


FIG. 347.

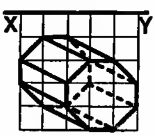


FIG. 348.

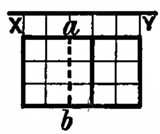


FIG. 349.

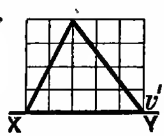


FIG. 350.

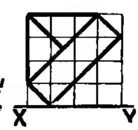


FIG. 351.

In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

7. A plan of a right square prism is given in Fig. 349, the edge AB of one end being on the ground. Draw this plan and from it project an elevation on XY.

8. An elevation of a right square pyramid is given in Fig. 350, v' being the elevation of the vertex. Draw this elevation and from it project the plan.

9. An elevation of a right square prism surmounted by a right square pyramid is given in Fig. 351. The corners of the base of the pyramid are at the middle points of the edges of one of the square faces of the prism. Draw this elevation and from it project the plan.

10. A cube, whose edges are 1·5 inches long, has two faces horizontal, the lower one being 1·25 inches above the ground. The square which is the plan of

this cube has one side inclined at 25° to the ground line. On each face of this cube is placed a right square pyramid, the corners of the square bases being at the middle points of the edges of the cube. The triangular faces of the pyramids are equilateral triangles. Draw the plan and elevation of this built up solid.

11. Draw the plan and elevation of a tetrahedron whose edges are 2 inches long when one face is in the vertical plane of projection and an edge of that face is inclined at 20° to XY.

12. Draw the plan and elevation of an octahedron whose edges are 1·5 inches long when one axis is perpendicular to the vertical plane of projection and the elevation of another axis is inclined at 60° to XY.

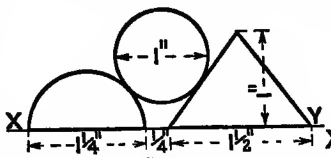


FIG. 352.

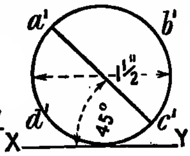


FIG. 353.

13. The semicircle (Fig. 352) is the elevation of a hemisphere whose base is on the ground and touching the ground line. The triangle is the elevation of a right circular cone whose base is on the ground and touching the ground line. The circle is the elevation of a right circular cylinder, 1·5 inches long, with one end in the vertical plane of projection and resting on the other two solids. Draw the plan of this group of solids.

14. The semicircle *a'b'c'* (Fig. 353) is the elevation of the half of a right circular cone, the base being in the vertical plane of projection. The semicircle *a'd'c'* is the elevation of the half of a right circular cylinder, one end being in the vertical plane of projection. The altitude of the cone is the same as that of the cylinder, namely 1·75 inches. Draw the given elevation and from it project the plan.

CHAPTER XIV

CHANGING THE PLANES OF PROJECTION

156. Auxiliary Projections.—In general the true form of an object is determined by a plan and an elevation, but there are cases where two projections are not sufficient, and in many other cases additional projections would make the form of the object much easier to understand. For example, in Fig. 354 are given a plan (a) and elevation (a') of a rectangular block having recesses in three of its faces. It is clear from the plan (a) and elevation (a') that the recess

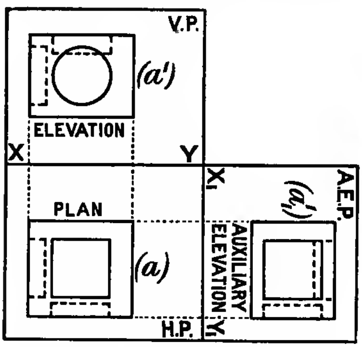


FIG. 354.

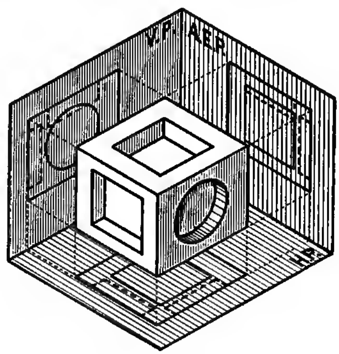


FIG. 355.

in the top is rectangular and that the recess in the front is cylindrical, but the recess in the left hand end might be either rectangular or cylindrical. The form of the end recess is however determined by an end view (a_1') which is a projection of the block on a plane parallel to the ends. This second vertical plane of projection intersects the horizontal plane in a line X_1Y_1 which is a second ground line. The relative positions of the solid and the three planes of projection are clearly shown in the pictorial projection, Fig. 355, where A.E.P. is the additional vertical plane of projection and may be called the *auxiliary elevation plane.*

157. Auxiliary Projections of a Point.—Referring to the pictorial projection, Fig. 356, A is a point in space, a is its projection

on the horizontal plane H.P. and a' is its projection on the vertical plane V.P. XY is the ground line in which these two planes of projection intersect. X_1Y_1 is another ground line and is the line of intersection of the vertical plane A.E.P. with the horizontal plane H.P. a_1' is the projection of the point A on the plane A.E.P. If these two vertical planes of projection with the projections on them be rabatted about their respective ground lines into the horizontal plane, Fig. 357 is obtained. An examination of Figs. 356 and 357 will show that the distance of the auxiliary elevation a_1' from X_1Y_1 is equal to the distance of the elevation a' from XY; and just as the straight line joining a and a' is perpendicular to XY so also is the straight line joining a and a_1' perpendicular to X_1Y_1.

Fig. 356.

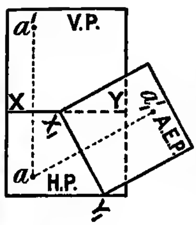


Fig. 357.

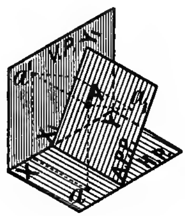


Fig. 358.

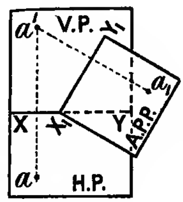


Fig. 359.

Referring next to Figs. 358 and 359, A, a, and a' and the planes of projection H.P. and V.P. are the same as before, but X_1Y_1 is now the line of intersection of a plane A.P.P. and the vertical plane of projection V.P., the plane A.P.P. being perpendicular to the plane V.P. When the planes H.P. and A.P.P. (Fig. 358) with the projections on them are rabatted about XY and X_1Y_1 respectively into the vertical plane V.P., Fig. 359 is obtained. An examination of Figs. 358 and 359 will show that the auxiliary plan a_1 is at a distance from X_1Y_1 equal to the distance of the plan a from XY, and just as the straight line aa' is at right angles to XY so also is the straight line $a'a_1$ at right angles to X_1Y_1.

It will be observed that the auxiliary plane of projection A.P.P. is not a horizontal plane, but it is usual to speak of the projection a_1 of the point A on this plane as an auxiliary plan.

The student must carefully study Figs. 356–359 and satisfy himself as to the correctness of the following rules which are used in the determination of auxiliary projections :—

(1) The plan and elevation of a point are in a straight line which is perpendicular to the ground line.

(2) When a number of elevations are projected from the same plan, the distances of all the elevations of the same point from their corresponding ground lines are the same.

(3) When a number of plans are projected from the same elevation, the distances of all the plans of the same point from their corresponding ground lines are the same.

158. Distance of a Point from the Ground Line.—The distance of a point aa' (Fig. 360) from the ground line XY is the length of the perpendicular from the point on to XY. As in general this perpendicular will be inclined to both planes of projection, neither its plan nor its elevation will show its true length; but by making a projection of the planes of projection and the perpendicular on a plane at right angles to XY the perpendicular will then be projected on a plane parallel to it and will therefore have its true length shown in this new projection. In Fig. 360, X_1Y_1 is at right angles to XY. a'_1 is the new elevation of the point A and $o'a'_1$ is the true distance of the point A from XY.

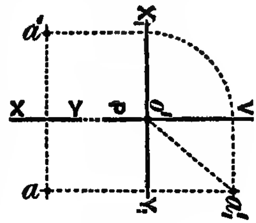


FIG. 360.

159. Auxiliary Projections of Solids.—Having mastered the rules for auxiliary projections of a point the student should have no difficulty in applying them to the drawing of auxiliary projections of solids. An example is illustrated by Fig. 361. The elevation (1) and plan (2) represent a right rectangular prism in a simple position. From the plan (2) an elevation (3) is projected on a ground line X_1Y_1. Considering the point R, the distance of r_1' from X_1Y_1 is equal to the distance of r' from XY; also the straight line rr_1' is perpendicular to X_1Y_1. From the elevation (3) a plan (4) is projected on a ground line X_2Y_2. Again considering the point R, the distance of r_2 from X_2Y_2 is equal to the distance of r from X_1Y_1; also the straight line $r_1'r_2$ is perpendicular to X_2Y_2.

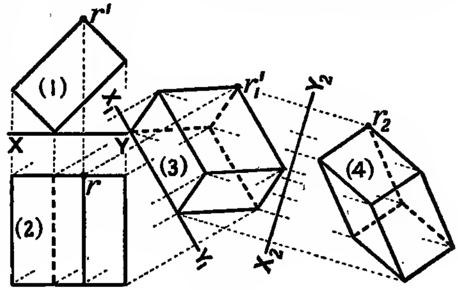


FIG. 361.

160. Projections of a Solid when a given Line in it is vertical.—A plan and an elevation of the solid are first drawn when it is in a simple position such that the given line is parallel to the vertical plane of projection. The plan of the given line will be parallel to the ground line. A new ground line is then taken at right angles to the elevation of the given line and a new plan of the solid is projected from the elevation. This new plan will be a plan of the solid when the given line is vertical. Two examples are illustrated by Figs. 362 and 363.

In the first example (Fig. 362) the plan of a right hexagonal pyramid is required when an edge containing the vertex is vertical. The solid is placed with its base on the horizontal plane and with a sloping edge VA parallel to the vertical plane of projection; the plan va of

this sloping edge is parallel to XY. Completing the plan (1) the elevation (2) is projected from it. X_1Y_1 is drawn at right angles to $v'a'$ and from the elevation (2) the plan (3) is projected. Applying the rules for auxiliary projections of a point it is found that the new plans v_1 and a_1 of the points V and A, coincide, which shows that the line VA is vertical when the plane of the plan (3) is considered to be a horizontal plane. An elevation (4) is also shown projected from the plan (3) on a ground line X_2Y_2.

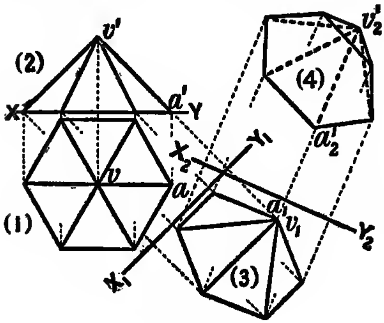


FIG. 362.

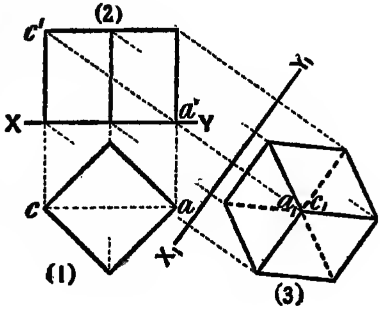


FIG. 363.

In the second example (Fig. 363) it is required to draw the plan of a cube when a diagonal of the solid is vertical. The cube is first placed with one face on the horizontal plane. The plan of the cube in this position is a square which is drawn with one diagonal ac parallel to XY as shown at (1). The elevation (2) is projected from the plan (1). ac is the plan and $a'c'$ is the elevation of a diagonal of the cube. X_1Y_1 is drawn at right angles to $a'c'$ and the plan (3) is projected from the elevation (2) as shown. The plan (3) is the one required. The outline of the plan (3) is a regular hexagon.

The plan of a solid may be drawn when a given line in it is inclined at *any* given angle to the horizontal plane by proceeding in the manner just explained, but making the new ground line inclined at the given angle to the elevation of the given line instead of perpendicular to it. But as the solid in this case can occupy any number of positions with reference to the horizontal plane and still fulfil the given condition, the problem is indefinite. In the case where the given line is vertical, however, the plan is always the same. These remarks will be understood if the solid be imagined to turn round the given line as an axis.

161. Projections of a Solid when a given Face is inclined at a given Angle, the Base of that Face being horizontal.—The solid is first placed in a simple position and having the plan of the base of the given face at right angles to the ground line. The elevation of the given face will then be a straight line. A new ground line is then drawn, making with the elevation of the given face an angle

equal to the given inclination. A plan on this new ground line is the plan required.

Two examples are illustrated by Figs. 364 and 365. In the first example (Fig. 364) projections of a right hexagonal pyramid are determined when the face VAC is inclined at a given angle θ to the plane upon which the new plan is to be projected. The plan (1) and elevation (2) are drawn representing the pyramid with its base on the horizontal plane and with *ac*, the plan of the base of the face VAC, at right angles to XY. It will be seen that the elevation of the face VAC is the straight line $v'a'$. X_1Y_1 is drawn at the given inclination θ to $v'a'$, and the plan (3) is projected as shown.

In the second example (Fig. 365) projections of a regular octahedron are determined when one face VAC is horizontal. The various projections are drawn in the order in which they are numbered. From the plan (3) an extra elevation has been projected on the ground line

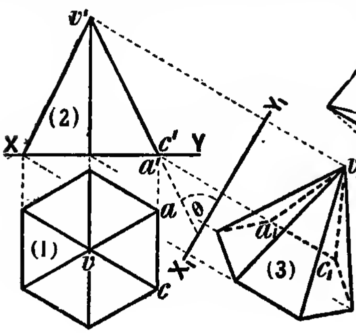


FIG. 364.

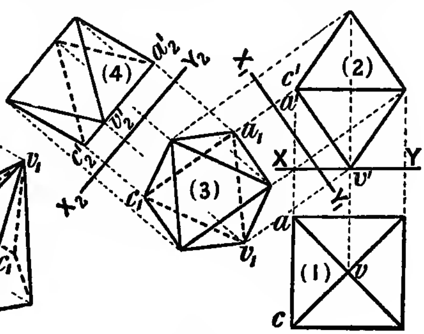


FIG. 365.

X_2Y_2. It is of interest to observe that the boundary line of the plan (3) is a regular hexagon, and this plan may evidently be drawn directly without the aid of any other projections.

Exercises XIV

1. Draw the elevations of the points given in Fig. 366 on a ground line inclined at 45° to XY.

2. Find the true distances of the points given in Fig. 366 from XY.

3. A triangle ABC is given in Fig 367. Draw an elevation of this triangle on a ground line parallel to *ac*. Also, project from the given elevation a new plan on a ground line parallel to $a'c'$.

4. A figure ABCDEF (not a plane figure) is given in Fig. 368. Draw an elevation of this figure on a ground line parallel to *bc*. Also, project from the given elevation a new plan on a ground line parallel to $a'b'$.

5. A plan and an elevation of a curved line are given in Fig. 369. Draw an elevation of this curve on a ground line inclined at 45° to XY.

6. Show the plans and elevations of the following points:—A, 1·5 inches in front of the V.P., above the H.P., and 2 inches from XY. B, 1·2 inches above

the H.P., behind the V.P., and 1·8 inches from XY. C, 1·6 inches behind the V.P., below the H.P., and 2 inches from XY. D, 2 inches below the H.P., in front of the V.P., and 2·5 inches from XY.

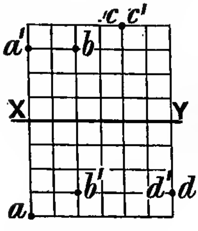


FIG. 366.

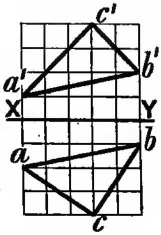


FIG. 367.

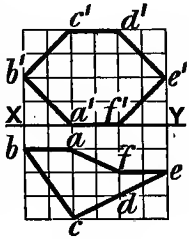


FIG. 368.

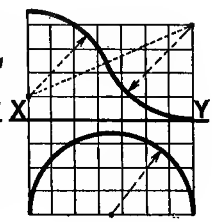


FIG. 369.

In reproducing the above diagrams take the small squares as of 0·3 *inch side.*

7. The ends of a right prism are regular pentagons of 1·25 inches side, and the altitude of the prism is 1·5 inches. This prism rests with one of its rectangular faces on the ground and with its ends inclined at 45° to the vertical plane of projection. Draw the plan and elevation.

8. Draw the plan of a right square prism when a diagonal of the solid is vertical. Side of base, 1·25 inches. Altitude, 2 inches.

9. A front elevation of a Maltese cross is given in Fig. 370. The thickness of the cross is 0·7 inch. Draw the plan and from it project an elevation on a ground line making 50° with XY.

10. Draw an elevation of the solid given in Fig. 371 on a ground line inclined at 60° to XY.

11. Draw a plan of the solid given in Fig. 371 when the line RS is vertical.

12. Draw an elevation of the solid given in Fig. 372 on a ground line inclined at 60° to XY.

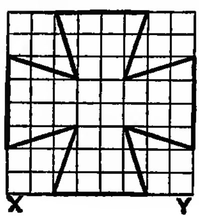


FIG. 370.

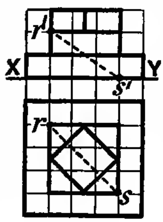


FIG. 371.

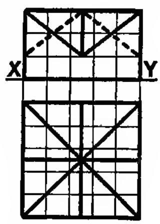


FIG. 372.

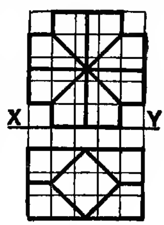


FIG. 373.

In reproducing the above diagrams take the small squares as of 0·4 *inch side.*

13. Draw a plan of the solid given in Fig. 372 when it is tilted about one edge of its base through an angle of 30°.

14. The solid given in Fig. 373 is formed out of two right square prisms. Draw an elevation of this solid on a ground line inclined at 60° to XY.

15. Draw the plan of the solid given in Fig. 373 when it is tilted about one edge of its base through an angle of 30°.

16. Draw the plan of a right hexagonal pyramid when it rests with one triangular face on the ground. From this plan project an elevation on a ground line inclined at 45° to the plan of the axis of the pyramid. Side of base, 1·25 inches. Altitude, 2 inches.

17. An isosceles triangle, base 1 inch, sides 1·5 inches, has its base parallel to XY. This triangle is the elevation of one of the triangular faces of a right

hexagonal pyramid. Side of base of pyramid, 1 inch. Altitude, 2 inches. Complete the elevation of the pyramid and from it project the plan.

18. Two elevations of an inkstand are given in Fig. 374. From the right-hand elevation project a plan on XY and from this plan project an elevation on a ground line inclined at 50° to XY.

19. Draw the plan of the inkstand (Fig. 374) when the edge AB is vertical.

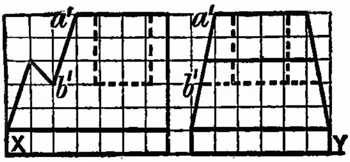


FIG. 374.

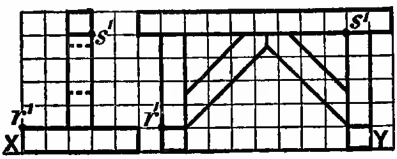


FIG. 375.

In reproducing the above diagrams take the small squares as of 0·3 inch side.

20. Two elevations of a trestle are given in Fig. 375. From the right-hand elevation project a plan on XY and from this plan project an elevation on a ground line inclined at 60° to XY.

21. Draw a plan of the trestle (Fig. 375) when the line joining the points R and S is vertical.

CHAPTER XV

PLANES OTHER THAN THE CO-ORDINATE PLANES

162. Representation of Planes—Traces of a Plane.—Planes other than the co-ordinate planes are represented by the lines in which they meet the latter. The lines in which a plane meets the co-ordinate planes or planes of projection are called the *traces* of that plane, the intersection with the vertical plane being called the *vertical trace*, and that with the horizontal plane the *horizontal trace*.

The line in which one plane meets another is also called the trace of the one plane on the other, but when the traces of a plane are mentioned its traces on the planes of projection are generally understood.

Planes occupying various positions in relation to the planes of projection are shown in pictorial projection in the lower parts of Figs. 376 to 384. The upper parts of the same Figs. show how the same planes are represented by means of their horizontal and vertical traces when the planes of projection are made to coincide as explained in Art. 135, p. 166.

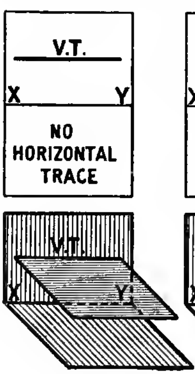


Fig. 376.

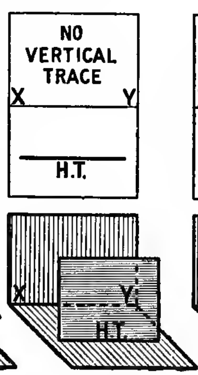


Fig. 377.

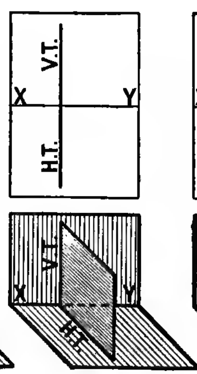


Fig. 378.

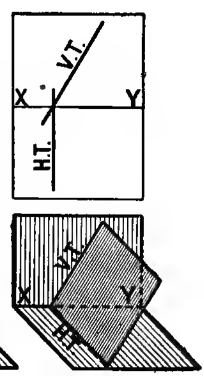


Fig. 379.

Referring to Figs. 376 to 384 separately. Fig. 376 shows a horizontal plane. Fig. 377 shows a plane which is parallel to the vertical plane of projection and is therefore also a vertical plane. Fig. 378 shows a plane which is perpendicular to both planes of projection and

is therefore also a vertical plane, and it is also perpendicular to the ground line. Fig. 379 shows a plane which is perpendicular to the vertical plane of projection and inclined to the horizontal plane. Such a plane is generally called an *inclined plane*. Fig. 380 shows a

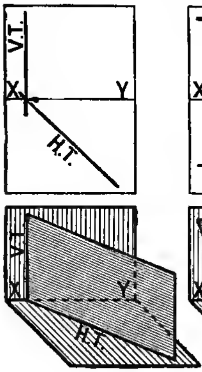


FIG. 380.

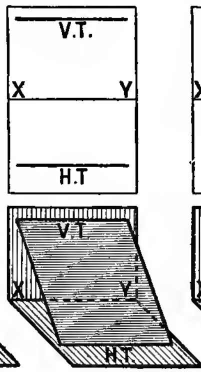


FIG. 381.

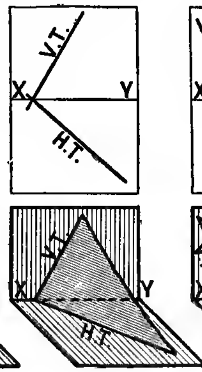


FIG. 382.

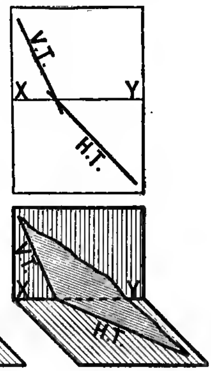


FIG. 383.

plane which is inclined to the vertical plane of projection and is perpendicular to the horizontal plane and is therefore a vertical plane. Fig. 381 shows a plane which is inclined to both planes of projection but is parallel to the ground line. Figs. 382 and 383 show planes which are inclined to both planes of projection and to the ground line.

Fig. 384 shows a plane containing the ground line XY and inclined to both planes of projection. The horizontal and vertical traces coincide with XY and these traces do not fix definitely the position of the plane. In this case a trace on another plane of projection is necessary, say on a vertical plane of which X_1Y_1 is the ground line.

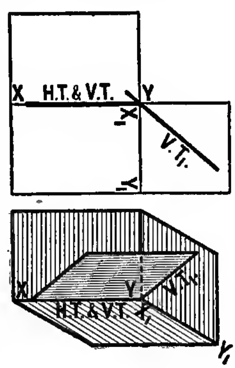


FIG. 384.

A *perpendicular plane* is one which is at right angles to one or other or both of the planes of projection (Figs. 376, 377, 378, 379, and 380).

An *oblique plane* is one which is inclined to both planes of projection (Figs. 381, 382, and 383).

It is obvious that in the case of a perpendicular plane P, its trace T on the plane of projection Q, to which the plane is perpendicular, is an edge view of the plane, and the projections on Q of all points and lines on P will lie on T.

It is not difficult to see that if the traces of a plane meet one another their point of intersection is on the ground line, and if the

traces are parallel to one another they must be parallel to the ground line.

163. Angle between Two Planes.—The angle between two planes, or the inclination of one plane to another, is the angle between two straight lines drawn from any point in their line of intersection, at right angles to it, one in each plane.

Referring to Fig. 385, which is a pictorial projection, AB and CD are two planes and CE is their line of intersection. FH is a straight line in the plane AB and at right angles to CE. FK is a straight line in the plane CD and at right angles to CE. If HF be produced to L, then the angle between the planes AB and CD is either the angle HFK or its supplement the angle LFK.

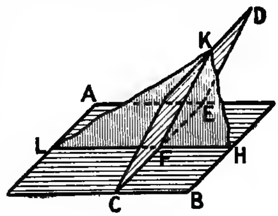


FIG. 385.

Of the two angles which one plane makes with another it is generally the acute angle which is taken as the angle between the planes. In the case of the angle between two adjacent plane faces of a solid it is generally the angle within the solid which is understood, whether it be acute or obtuse.

The angle between two planes is called a *dihedral angle.*

164. Inclinations of a Plane to the Planes of Projection.—A plane is given by its traces in Fig. 387. The same plane and the planes of projection in their natural positions are shown in oblique projection in Fig. 386. Referring to Fig. 386, AB is a line in the

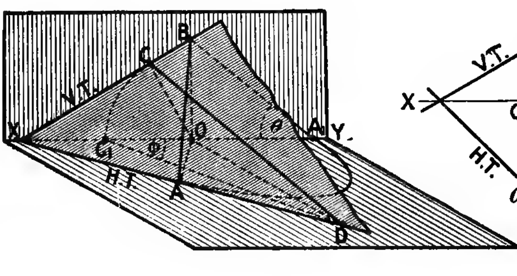


FIG. 386.

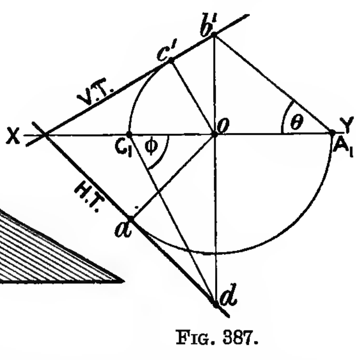


FIG. 387.

given plane and at right angles to its horizontal trace H.T. AO is a line in the horizontal plane and at right angles to H.T. By definition the angle OAB is the angle which the given plane makes with the horizontal plane. θ, the true value of this angle, is found by the following construction, shown in Fig. 387. Take a point o in XY and draw oa at right angles to H.T. With o as centre and oa as radius describe the arc $a\mathrm{A}_1$ to cut XY at A_1. Draw ob' perpendicular to XY to meet V.T. at b'. Join $b'\mathrm{A}_1$. The angle $o\mathrm{A}_1b'$ is the true value of θ.

It is obvious that the triangle oA_1b' in Fig. 387 is the true shape of the triangle OAB in Fig. 386. A similar construction gives the angle ϕ, the inclination of the given plane to the vertical plane of projection.

The arc aA_1 (Fig. 387) or the arc AA_1 (Fig. 386) may be looked upon as part of the outline of the base of a right circular cone whose axis is ob' (Fig. 387) or OB (Fig. 386). The base of this cone is on the horizontal plane, and the given plane is tangential to the slant surface of this cone. Hence θ which is the inclination of the slant surface of the cone to its base is also the inclination of the given plane to the horizontal plane. Similarly ϕ is the base angle of another right circular cone whose base is on the vertical plane of projection and whose axis is in the horizontal plane, and the given plane is tangential to the slant surface of this cone. Hence ϕ is the inclination of the given plane to the vertical plane of projection. It may be observed that it is not necessary that the axes of the two cones should spring from the same point O on XY.

The construction given above will apply whatever be the positions of the traces of the given plane; but when the traces are parallel it is not necessary to make a construction for finding ϕ after θ has been found because in this case $\phi = 90° - \theta$.

No construction is necessary for θ or ϕ when the given plane is situated as shown in Figs. 376, 377, 378, 379, or 380. In these cases θ, the inclination of the plane to the horizontal plane, is equal to the angle which the vertical trace makes with XY, and ϕ, the inclination of the plane to the vertical plane of projection, is equal to the angle which the horizontal trace makes with XY. When there is no horizontal trace (Fig. 376) $\phi = 90°$, and when there is no vertical trace (Fig. 377) $\theta = 90°$.

165. Plane having given Inclinations to the Planes of Projection.—It is required to draw the traces of a plane whose inclinations to the horizontal and vertical planes of projection are θ and ϕ respectively. The sum of the angles θ and ϕ must lie between 90° and 180°.

This problem is the converse of that discussed in the preceding Art. and two methods of solving it will be given.

In the first method the two right circular cones referred to in the preceding Art. are used. One cone has its base on the horizontal plane and its axis in the vertical plane of projection, and its base angle is equal to θ. The other cone has its base in the vertical plane of projection and its axis in the horizontal plane, and its base angle is equal to ϕ. The required plane is tangential to the slant surfaces of both cones. The important point to observe is that the relative dimensions and positions of the cones must be such that it is possible for the same plane to be tangential to the slant surfaces of both. The simplest way of ensuring that the two cones shall have a common tangent plane is to *arrange the cones so that they envelop the same sphere.*

Referring to Fig. 389 a circle with centre o on XY and of any convenient radius oe is first drawn. This circle is both plan and elevation of a sphere whose centre is on the ground line. Draw $b'A_1$ to touch

this circle and make with XY an angle equal to θ, cutting XY at A_1 and a line through o at right angles to XY at b'. Draw dC_1 to touch the same circle and make with XY an angle equal to ϕ cutting XY at C_1 and a line through o at right angles to XY at d. With centre o

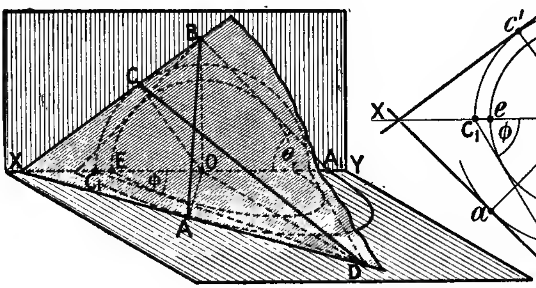


FIG. 388.

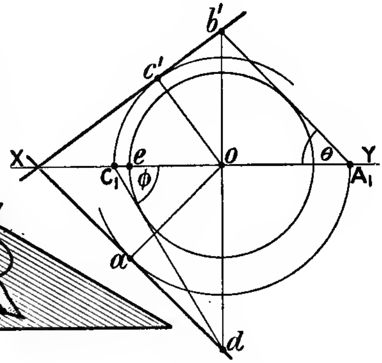


FIG. 389.

and radius oA_1 describe the arc A_1a. With centre o and radius oC_1 describe the arc C_1c'. A tangent da to the arc A_1a is the required horizontal trace, and a tangent $b'c'$ to the arc C_1c' is the required vertical trace. If these traces meet, their point of intersection must be on the ground line. The sphere and enveloping cones having been drawn there will, in general, be four planes which will be tangential to both cones. The student who has mastered the problem of the preceding Art. should now have little difficulty in understanding the solution just given if he carefully studies Figs. 388 and 389. It only remains to emphasize the point that the two cones are made to envelop the same sphere in order to ensure that they shall have a common tangent plane.

The second method of solving this problem depends on the following theorems. (1) The inclination of a plane to one of the planes of projection is the complement of the inclination of its normal to that plane of projection. (2) The trace of a plane on one of the planes of projection is perpendicular to the projection of its normal on that plane of projection. Hence the construction.—Draw (Art. 144, p. 176) the projections of a line inclined at $90^\circ - \theta$ to the horizontal plane and $90^\circ - \phi$ to the vertical plane of projection. The horizontal and vertical traces of the required plane will be perpendicular to the plan and elevation respectively of this line. If the traces meet, their point of intersection must be on the ground line. If the traces do not meet within a convenient distance, or if they are parallel to the ground line, care must be taken to ensure that the traces are the traces of the same plane.

166. Given one Trace of a Plane and the Inclination of the Plane to one of the Co-ordinate Planes, to determine the

other Trace.—In Figs. 390 and 391, θ the inclination of the plane to the horizontal plane is supposed to be given.

In Figs. 392 and 393, ϕ the inclination of the plane to the vertical plane of projection is supposed to be given.

CASE I. Figs. 390 and 391, and Figs. 392 and 393. LA is the given trace.

From any point O in XY draw OA at right angles to LA. With O as centre and radius OA describe the arc AC to cut XY at C. Draw CB making the angle OCB equal to the given inclination of the plane. Draw OB at right angles to XY to meet CB at B. Join LB. LB is the other trace of the plane.

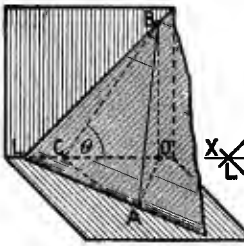


FIG. 390.

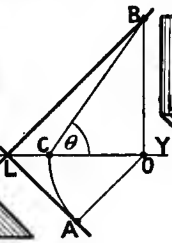


FIG. 391.

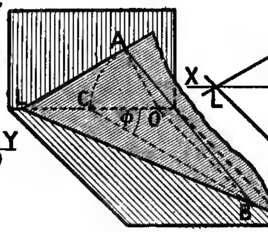


FIG. 392.

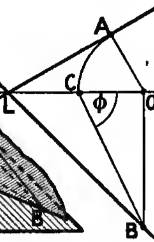


FIG. 393.

CASE II. Same figures. LB is the given trace.

From any point C in XY draw CB making the angle OCB equal to the given inclination of the plane, and meeting LB at B. Draw BO at right angles to XY. With O as centre and OC as radius describe the arc CA. Through L draw LA to touch the arc CA. LA is the other trace of the plane.

The correctness of the above constructions will be obvious, if, after the required trace has been found in each case, the construction for finding the given inclination be applied. It will be found that all the lines for the latter construction are already on the figures.

167. True Angle between the Traces of a Plane.—By the true angle between the traces of a plane is meant the angle between them when the plane and the planes of projection are in their natural positions as shown in Fig. 394 where the angle ALB is the angle referred to. This angle is determined by finding the true form of the triangle ABL in which the angle ABL is a right angle.

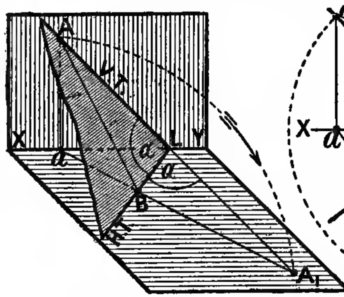


FIG. 394.

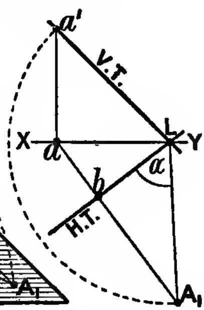


FIG. 395.

Referring to Fig. 395, from a point a in XY draw ab at right angles

to the horizontal trace of the plane and draw aa' at right angles to XY to meet the vertical trace of the plane at a'. With centre L, where the traces meet XY, and radius La', draw the arc $a'A_1$ to meet ab produced at A_1. Join A_1L. Then α, the angle A_1Lb, is the true angle between the traces of the plane.

The theory of the above construction is that the triangle ABL (Fig. 394) is supposed to turn about LB as an axis until it comes into the horizontal plane. The point A describes an arc of a circle whose plane is perpendicular to LB and whose centre is B.

Instead of taking the triangle ABL with the right angle at B, the triangle may be taken with the right angle at A and the triangle would then be supposed to turn about LA as an axis into the vertical plane of projection. The student should draw the figure for this case as well as for the other and show that the results are the same.

168. Given one Trace of a Plane and the true Angle between the Traces, to draw the other Trace.—Let LB (Figs. 396 and 397) be the given trace. Draw LA_1 making the angle BLA_1 equal to α, the given angle between the traces. Draw A_1B at right angles to LB and produce it to meet XY at O. Draw OA at right angles to XY. With centre L and radius LA_1 describe the arc A_1A to cut OA at A. Join LA. LA is the other trace of the plane.

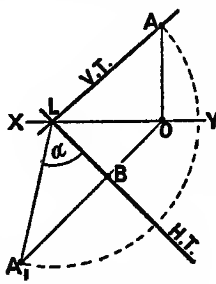


Fig. 396.

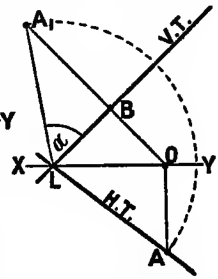


Fig. 397.

The correctness of the above construction will be obvious, if, after the required trace has been found, the construction for finding the given angle be applied. It will be found that all the lines for the latter construction are already on the figure.

169. Intersection of Two Planes.—The intersection of two

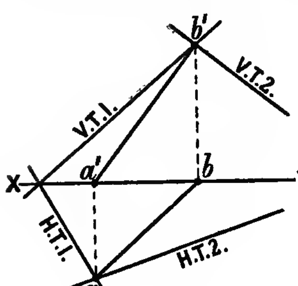


Fig. 398.

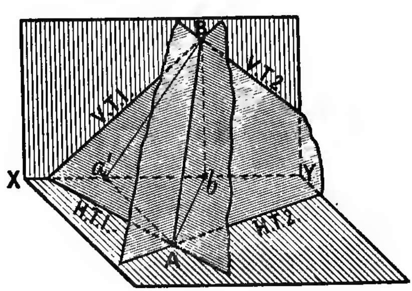


Fig. 399.

planes is a straight line which is determined when two points in it are known or when one point in it and the direction of the line are known.

If the vertical traces of the two planes meet one another and the horizontal traces almost meet one another and if the meeting points are within a convenient distance (Figs. 398 and 399) the intersection of the planes is readily found. It is clear that the point aa', where the horizontal traces intersect, is a point in the intersection of the two planes. It is also evident that the point bb', where the vertical traces intersect, is another point in the intersection of the two planes. Hence the required intersection is the line ab, $a'b'$.

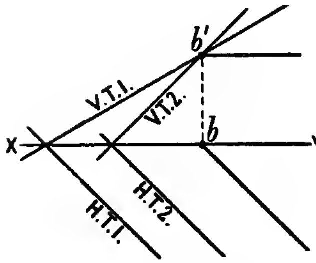


FIG. 400.

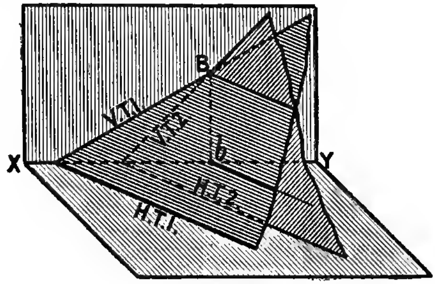


FIG. 401.

The case where the traces of the two planes on one of the planes of projection, the horizontal plane, are parallel, is shown in Figs. 400 and 401. The point bb' where the vertical traces meet is a point in the intersection of the two planes and the line of intersection is obviously a horizontal line, its plan being parallel to the horizontal traces of the planes, and its elevation parallel to XY.

Figs. 402 and 403 illustrate the case where the traces of the two planes intersect at the same point on the ground line. This point on the ground line is obviously one point on the line of intersection of the

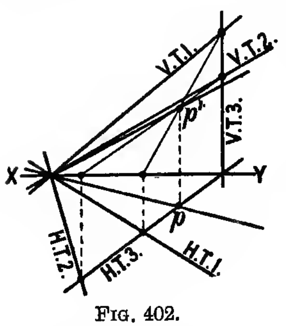


FIG. 402.

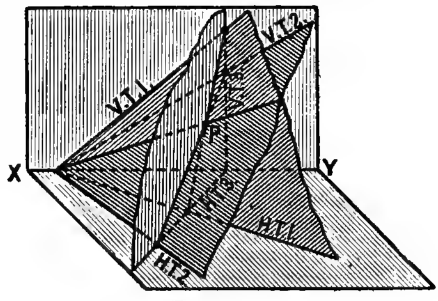


FIG. 403.

two planes. To find another point cut the given planes by a third plane. In Figs. 402 and 403 this third plane is perpendicular to the horizontal plane and inclined to the vertical plane of projection. The lines in which the third plane intersects the first and second planes

intersect at a point pp' which is another point in the intersection of the two given planes.

The case where the traces of the given planes are parallel to the ground line, and the planes themselves are therefore parallel to that line, is shown in Figs. 404 and 405. A point pp' in the line of intersection of the two planes is found by means of a third plane as before and lines through p and p' parallel to XY are the plan and elevation respectively of the intersection required.

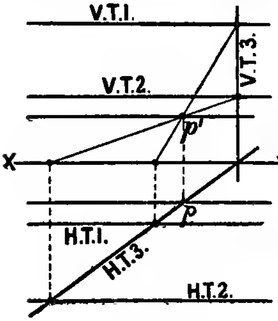


Fig. 404.

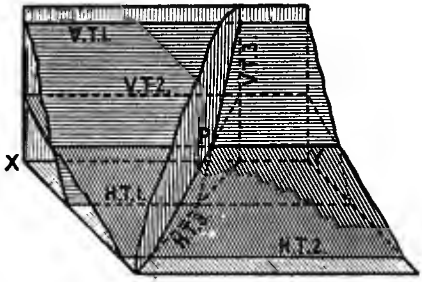


Fig. 405.

A plane may be fixed by the projections of three points in it, and the intersection of two planes fixed in this way may be found without using the traces of the planes on the planes of projection. Referring to Figs. 406 and 407, aa', bb' and cc' are the projections of three points in a plane. The triangle abc, $a'b'c'$ is obviously in this plane. dd', ee', and ff' are the projections of three points which are in another plane. The triangle def, $d'e'f'$ is obviously in this second plane. It is required to find the intersection of these two planes.

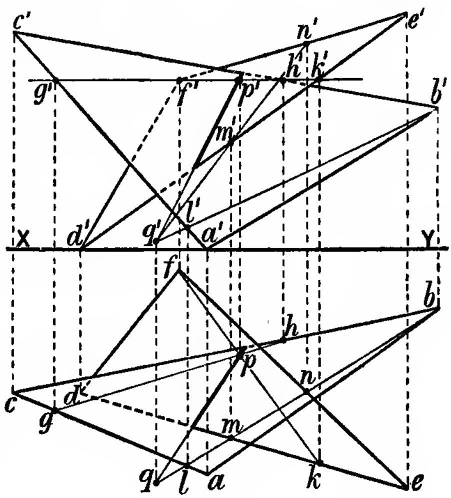


Fig. 406

Referring to Fig. 406, a horizontal plane at the level of the point ff' is taken. This plane cuts the plane of the triangle abc, $a'b'c'$ in the line gh, $g'h'$. This same plane cuts the plane of the triangle def, $d'e'f'$ in the line fk, $f'k'$. pp' the point of intersection of the lines gh,

$g'h'$, and fk, $f'k'$ is obviously a point in the intersection of the two given planes.

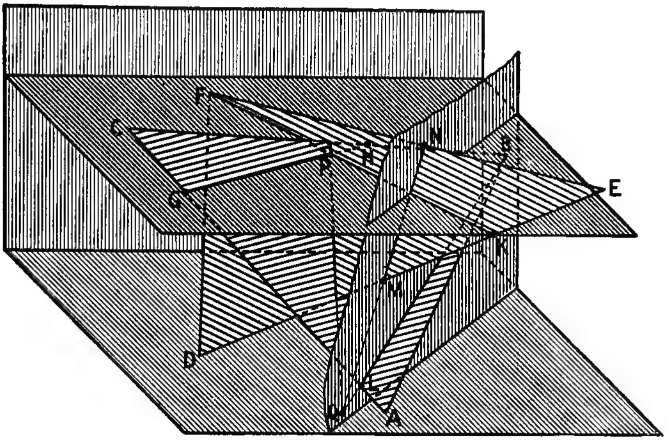


FIG. 407.

A vertical plane of which bl is the horizontal trace is taken. This plane cuts the plane of the triangle abc, $a'b'c'$ in the line bl, $b'l'$. This same plane cuts the plane of the triangle def, $d'e'f'$ in the line mn, $m'n'$. qq' the point of intersection of the lines bl, $b'l'$ and mn, $m'n'$ is obviously another point in the intersection of the two given planes. That part of the line of intersection of the two given planes which lies within both of the given triangles is shown thicker, and this thicker line is the intersection of the given triangles.

Fig. 407 shows the various planes and their intersections in oblique projection.

When an auxiliary cutting plane is required in order to find the intersection of two given planes, this auxiliary plane may be placed in any number of positions, and the student must exercise his judgment and ingenuity in determining a convenient position, that is, a position which will lead to points of intersection of lines which will fall within convenient distances and which can be found with accuracy. Lines intersecting at very acute angles should be avoided if possible.

170. Distance between two Parallel Planes.—Parallel planes have parallel traces, the vertical and horizontal traces being parallel to one another respectively.

In Figs. 408 and 409, V.T.1. and V.T.2. are the vertical traces and H.T.1. and H.T.2. are the horizontal traces of two parallel planes. It is required to find the distance between these planes. Cut the given planes by a vertical plane whose horizontal trace oc (Fig. 409) or OC (Fig. 408) is perpendicular to the given horizontal traces. This plane will cut the given planes in two parallel lines the true distance between which will be the required distance between the given planes.

Referring to Fig. 409, with o as centre and radii oa and oc describe arcs to cut XY at A_1 and C_1 respectively. Draw $ob'd'$ at right angles

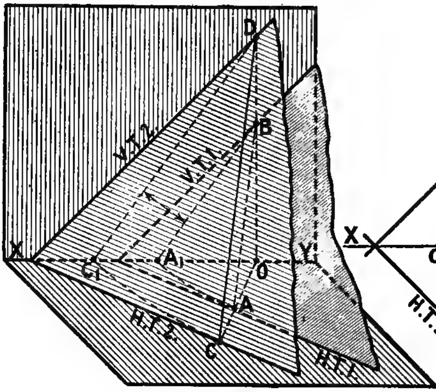


FIG. 408.

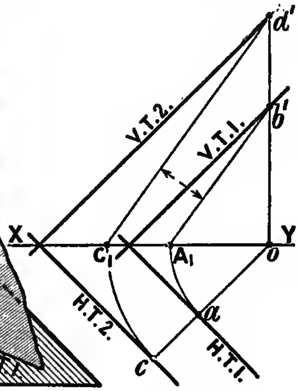


FIG. 409.

to XY cutting the given vertical traces at b' and d' as shown. Join A_1b' and C_1d'. The perpendicular distance between A_1b' and C_1d' is the distance between the two given planes.

The theory of the foregoing construction is that the vertical plane COD (Fig. 408) is supposed to turn about its vertical trace OD, which is perpendicular to XY, carrying with it the lines of its intersection with the given planes, until it is in the vertical plane of projection. The true distance between these lines can then be measured.

The auxiliary cutting plane may be, quite as conveniently, taken perpendicular to the given vertical traces, and the student should draw the figure with the cutting plane so taken.

171. To draw the Traces of a Plane which shall be parallel to a given Plane and at a given distance from it.—The solution of this problem is so obvious, if the preceding one has been understood, that no detailed description of it need be given. It may be pointed out however that two planes may be placed at a given distance from a given plane, one on each side of it.

Exercises XV

1. Planes are given by their traces at (*a*), (*b*), (*c*), (*d*), (*e*), and (*f*) in Fig. 410. Determine in each case the inclinations of the plane to the planes of projection.

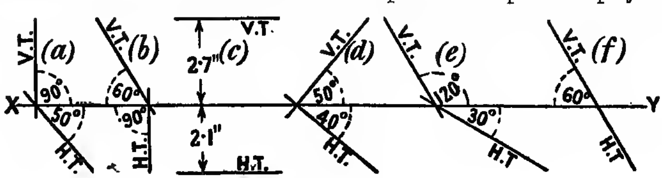


FIG. 410.

2. What are the true angles between the traces of the planes (*a*), (*b*), and (*c*), Fig 410?

3. Find the true angles between the traces of the planes (*d*), (*e*), and (*f*), Fig. 410.

4. The vertical trace of a plane is inclined at 45° to XY and the plane is inclined at 60° to the horizontal plane: draw both traces of the plane.

5. The vertical trace of a plane is inclined at 50° to XY and the plane is inclined at 50° to the vertical plane of projection: draw both traces of the plane.

6. The horizontal trace of a plane is inclined at 45° to XY and the plane is inclined at 60° to the horizontal plane: draw both traces of the plane.

7. The vertical trace of a plane is parallel to XY and at a distance of 2·5 inches from it. The plane is inclined at 55° to the horizontal plane. Draw the traces of this plane.

8. Draw the traces of a plane whose inclinations to the horizontal and vertical planes of projection are 55° and 60° respectively.

9. Draw the traces of a plane which is inclined at 50° to the horizontal plane and 40° to the vertical plane of projection.

10. The true angle between the traces of a plane is 60° and the vertical trace is inclined at 35° to XY. Draw both traces of this plane.

11. Draw the traces of a plane which is inclined to the horizontal plane at 50°, the true angle between the traces being 70°.

12. The true angle between the traces of a plane is 60° and the plane is equally inclined to the planes of projection. Represent this plane by its traces.

13. Pairs of planes are given by their traces at (*a*), (*b*), (*c*), and (*d*) Fig. 411. In each case show the plan and elevation of the line of intersection of the planes.

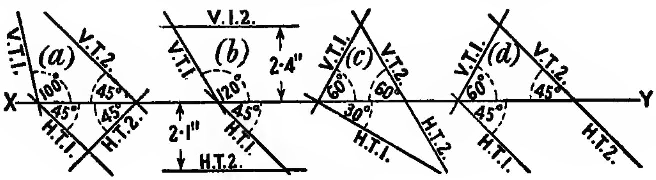


FIG. 411.

14. Find the plan and elevation of the point which is situated in each of the three planes given in Fig. 412.

15. Determine the intersection of the plane triangle *abc*, *a'b'c'* (Fig. 413) with the plane triangle *def*, *d'e'f'*. The dimensions given in Fig. 413 are in inches.

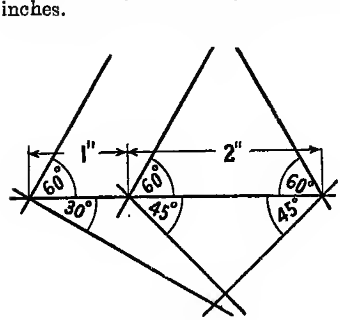


FIG. 412.

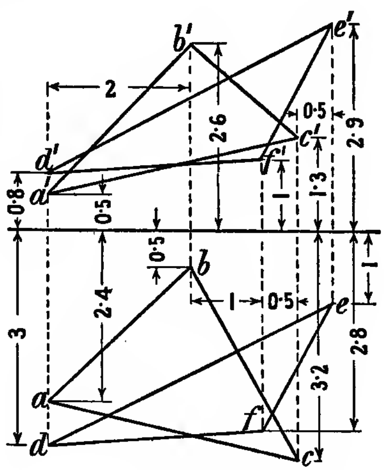


FIG. 413.

16. Two parallel planes are parallel to the ground line. The horizontal trace of the first is 1·5 inches below XY and that of the second 2 inches below XY. The vertical trace of the first is 2 inches above XY. Find the vertical trace of the second plane and the distance between the two planes.

17. Referring to Fig. 410, take each of the given planes and show two planes parallel to the given plane and 0·7 inch distant from it.

18. Two parallel planes have an inclination of 50°, and their horizontal traces are 1 inch apart. Determine a vertical plane which cuts these planes in lines which are 1·5 inches apart.

CHAPTER XVI

STRAIGHT LINE AND PLANE

172. Lines and Points in a Plane.—THEOREM I. *A straight line contained by a given plane has its traces on the traces of the plane.* The traces of the line are points in the planes of projection, but they are also points in the plane containing the line, therefore the traces of the line must lie on the intersection of the plane with the planes of projection, that is, on the traces of the plane.

Referring to the pictorial projection, Fig. 414, LA and LB are the traces of a plane and AB is a straight line contained by the plane ALB. The points A and B are the traces of AB.

THEOREM II. *A straight line which is parallel to one of the planes of projection and is contained by a given plane is parallel to the trace of that plane on the plane of projection to which the line is parallel.* Referring to the pictorial projection, Fig. 414, CD is a horizontal line lying in the plane LAB. CD is parallel to LA the horizontal trace of the plane. Hence *cd* the plan of CD is parallel to LA. *c'd'*, the elevation of CD, is parallel to XY. This line has only one trace C which is a vertical trace. EF is a line parallel to the vertical plane of projection and lying in the plane ALB. EF is parallel to LB the vertical trace of the plane. Hence *e'f'* the elevation of EF is parallel to LB. *ef*, the plan of EF, is parallel to XY. This line has only one trace E which is a horizontal trace.

Fig. 415 shows the various lines referred to above by ordinary plan and elevation.

It is obvious that the distance of CD from the horizontal plane of projection is equal to the distance of its elevation from XY, also the distance of EF from the vertical plane of projection is equal to the distance of its plan from XY.

A knowledge of the foregoing principles will enable the student to solve a number of problems.

For example.—Given the traces of a plane (Fig. 415) and the plan *p* of a point contained by the plane, to find the elevation of the point. Through *p* draw *ab* to cut the horizontal trace of the plane at *a* and XY at *b*. Consider *ab* as the plan of a line lying in the plane. The elevation *a'b'* of this line is found as shown, and *a'b'* must contain *p'* the elevation required. Instead of taking the line *ab*, *a'b'* lying in the

plane and containing the point P, the horizontal line *cd*, *c'd'* lying in the plane and containing the point P may be taken, or the line *ef*, *e'f'*

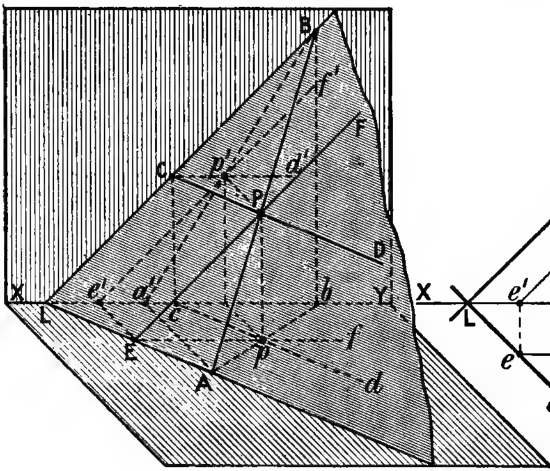


FIG. 414.

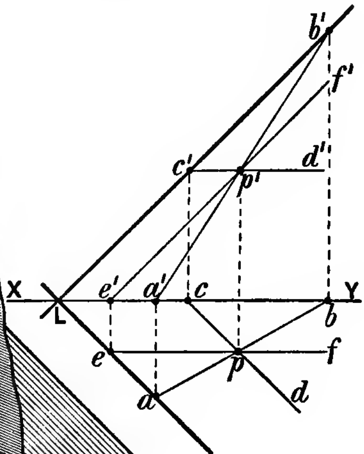


FIG. 415.

parallel to the vertical plane of projection may be used to find *p'*. If *p'* is given *p* may be found by working the foregoing construction backwards.

Again, given the traces of a plane and the distances of a point in the plane from the planes of projection to find its plan and elevation. Place in the plane (Fig. 415) a horizontal line *cd*, *c'd'* whose distance from the horizontal plane of projection is equal to the given distance of the point from the latter plane. Next place in the plane a line *ef*, *e'f'* whose distance from the vertical plane of projection is equal to the given distance of the point from the latter plane. *p* the intersection of *cd* and *ef* is the plan, and *p'* the intersection of *c'd'* and *e'f'* is the elevation required.

In another set of problems it is required to find the traces of a plane which contains given points or lines. Three points which are not in the same straight line will fix a plane, also two parallel straight lines or two intersecting straight lines, or one straight line and one point not in that line. These problems are solved by making use of the facts that if two points are in a plane the straight line joining them is in that plane and the traces of this line are on the traces of the plane.

173. To draw the Traces of a Plane which shall contain a given Line and have a given Inclination to one of the Planes of Projection.—Let *ab*, *a'b'* (Fig. 417) be the given line, and let the given inclination of the plane be to the horizontal plane.

Draw the projections of a cone having its vertex at the vertical

trace of the given line, its base on the horizontal plane of projection, and its slant side inclined to its base at the given angle (θ).

A line through the horizontal trace of the given line, to touch the base of the cone, will be the horizontal trace of the plane required, and a line through the point where this horizontal trace meets the ground line and through the vertical trace of the given line will be the vertical trace of the plane required. The foregoing construction will be readily understood by referring to the pictorial projection shown in Fig. 416.

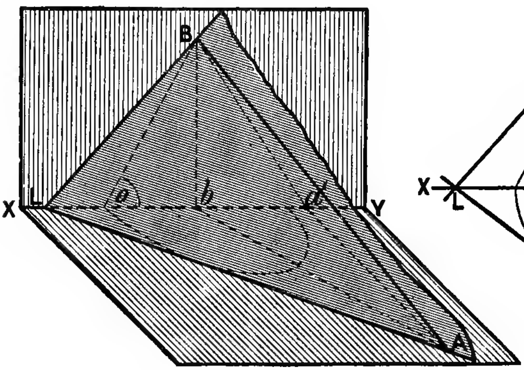


Fig. 416.

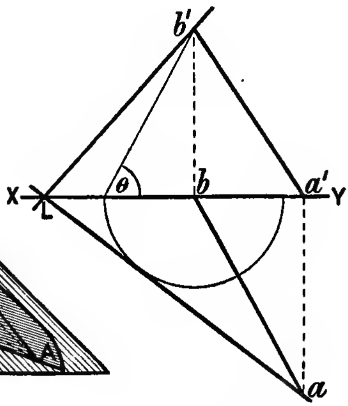


Fig. 417.

If the horizontal trace of the given line falls outside the base of the cone two tangents from this point to the base of the cone can be drawn and there will therefore be two planes which will satisfy the given conditions. In Figs. 416 and 417 only one of the two possible planes is shown. If the horizontal trace of the given line falls on the circumference of the base of the cone only one tangent from this point to the base of the cone can be drawn and there will then be only one plane which will satisfy the given conditions, and this plane will have the same inclination as the given line. If the horizontal trace of the given line falls inside the base of the cone no tangent from this point to the base of the cone can be drawn, which shows that the problem is impossible when the inclination of the plane is less than the inclination of the line.

In the construction just explained it has been assumed that the horizontal and vertical traces of the given line have been accessible, it now remains to be shown how the problem is to be dealt with when these points are inaccessible.

Referring to Fig. 419, the horizontal and vertical traces of the given line ab, $a'b'$ are supposed to be inaccessible.

Draw the projections of two cones having their vertices at points cc' and dd' in the given line, their bases on the horizontal plane,

and their slant sides inclined to their bases at the given angle (θ). The horizontal trace of the required plane is a common tangent to the bases of the two cones. The vertical trace of the plane may be found by taking a point ee' in the horizontal trace of the plane and joining this point with two points (say cc' and dd') in the given line. The line

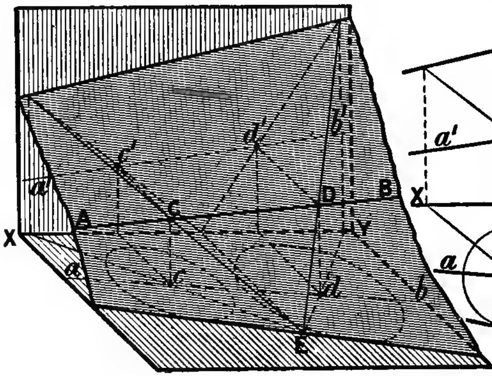


FIG. 418.

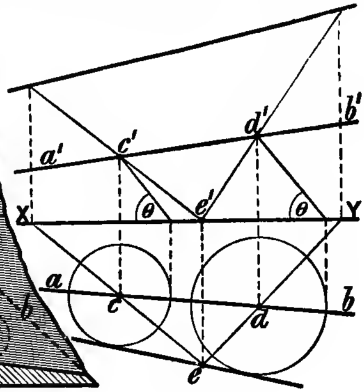


FIG. 419.

joining the vertical traces of these two lines is the vertical trace of the plane required. A reference to the pictorial projection (Fig. 418) will make the construction clear.

It should be noticed that although four tangents may be drawn to the bases of the cones shown in Figs. 418 and 419 only the outside tangents, and not the cross tangents, can be the horizontal traces of planes which are tangential to the two cones.

If the inclination of the required plane be given to the vertical plane of projection instead of to the horizontal plane the bases of the cones must be placed on the former plane instead of on the latter.

174. In a given Plane to place a Line having a given Inclination to one of the Planes of Projection.—The inclination of the line must not exceed the inclination of the plane.

Let the given inclination be to the horizontal plane.

Take a point C (Fig. 421) on the ground line and draw Cb' making the given angle θ with XY and meeting the vertical trace of the given plane at b'. Draw $b'b$ perpendicular to XY meeting the latter at b. With centre b and radius bC describe an arc of a circle to cut the horizontal trace of the given plane at a. Draw aa' perpendicular to XY meeting it at a'. ab is the plan and $a'b'$ the elevation of the line required.

The correctness of the construction is obvious, for AB (Fig. 420) is evidently on the given plane and it is also on the surface of a cone whose base is on the horizontal plane and whose base angle is θ.

The construction is applicable whatever be the positions of the traces of the given plane.

If the given inclination be to the vertical plane of projection instead of to the horizontal plane, the construction will present no difficulty to the student who has followed that just given.

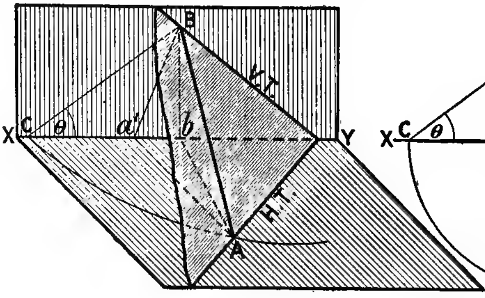


FIG. 420.

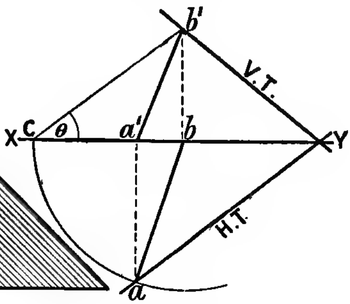


FIG. 421.

175. To draw the Projections of a Line which shall pass through a given Point, have a given Inclination, and be parallel to a given Plane.—By the construction of the preceding Art. place in the given plane a line having the given inclination. Through the plan and elevation of the given point draw lines parallel to the plan and elevation respectively of the line which has been placed in the given plane. These will be the projections required.

That the line whose projections have thus been found fulfils the given conditions is not difficult to see; for it is parallel to a line which has the given inclination, and must therefore have itself the given inclination. Also it can never meet the given plane, for if it did it could not be parallel to any line in that plane, therefore it is parallel to the given plane.

176. To draw the Traces of a Plane which shall contain a given Point and be parallel to a given Plane.—The horizontal and vertical traces of the plane required must be parallel to the horizontal and vertical traces respectively of the given plane.

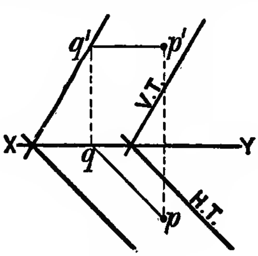


FIG. 422.

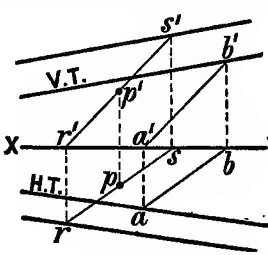


FIG. 423.

pp' (Figs. 422 and 423) is the given point and H.T. and V.T. are the traces of the given plane.

The projections of a line passing through pp' and parallel to the given plane are first drawn. The traces of this line are on the traces of the plane required.

In Fig. 422 a horizontal line pq, $p'q'$ parallel to the given plane is shown. The vertical trace of this line is a point on the vertical trace of the plane required, and the point where the vertical trace of the plane meets XY is a point on the horizontal trace of the plane.

In Fig. 423 a line ab, $a'b'$ is placed in the given plane and the projections rs, $r's'$ of a line parallel to ab, $a'b'$ are drawn through p and p' respectively. The traces of rs, $r's'$ are on the traces of the plane required.

The construction shown in Fig. 423 is used when the vertical trace of a horizontal line through pp' parallel to the given plane is inaccessible, or when the horizontal trace of a line through pp' parallel to the vertical plane of projection and parallel to the given plane is inaccessible.

177. In a given Plane to place a Line which shall be parallel to and at a given distance from another given Plane not parallel to the first.—Draw the traces of a plane parallel to the second given plane and at the given distance from it (Art. 171, p. 208). The line of intersection of this third plane with the first given plane is the line required.

178. To draw the Projections of a Line which shall pass through a given Point and be parallel to two given intersecting Planes.—The required projections will pass through the projections of the given point, and be parallel to the projections of the line of intersection of the two given planes.

179. To draw the Traces of a Plane which shall contain a given Line and be parallel to another given Line.—Draw the projections of a line to intersect the first given line and be parallel to the second. The plane containing these two intersecting lines is the plane required.

180. To draw the Traces of a Plane which shall contain a given Point and be parallel to two given Lines.—Draw the projections of two lines to pass through the given point and be parallel respectively to the two given lines. The plane containing the lines passing through the given point is the plane required.

181. Intersection of a Line and Plane.—Let LMN (Fig. 425) be a given plane and ab, $a'b'$ a given line. It is required to show the projections of the point of intersection of the line and plane.

Draw the traces LR and RN of a plane, preferably perpendicular to one of the planes of projection, to contain the given line. Find the line of intersection of this plane with the given plane. The points p and p' where ab and $a'b'$ meet the plan and elevation respectively of the intersection of the planes are the projections required.

Referring to the pictorial projection (Fig. 424) it will be seen that AB and LN being in the same plane LRN they must intersect at some point P unless they happen to be parallel. Again, since LN is the line of intersection of the planes LMN and LRN the line LN

is in the plane LMN, therefore the point P is in the plane LMN. But P is in AB, therefore the point P is the point of intersection of AB and the plane LMN. If AB is parallel to LN then AB is parallel to the plane LMN.

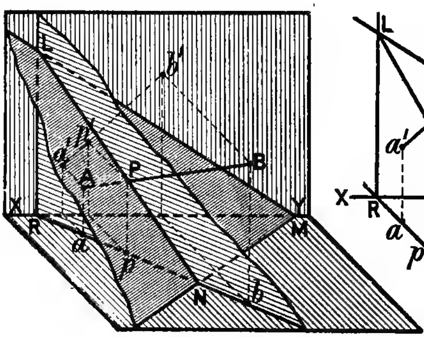


FIG. 424.

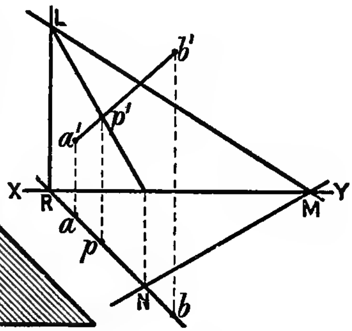


FIG. 425.

182. To draw the Projections of a Line which shall pass through a given Point and be perpendicular to a given Plane.—If a line is perpendicular to a plane the projections of the line are perpendicular to the traces of the plane, the plan to the horizontal and the elevation to the vertical trace.

The construction is therefore as follows. Through the plan of the point draw a line at right angles to the horizontal trace of the plane, and through the elevation of the point draw a line at right angles to the vertical trace of the plane; these will be the projections required.

183. To draw the Traces of a Plane which shall contain a given Point and be perpendicular to a given Line.—Let pp' (Fig. 426) be the given point and ab, $a'b'$ the given line.

Through p draw pq perpendicular to ab, meeting XY at q. Draw qq' perpendicular to XY, meeting a parallel to XY through p' at q'. Through q' draw LM perpendicular to $a'b'$, meeting XY at M. Through M draw MN perpendicular to ab. LM and MN are the vertical and horizontal traces respectively of the plane required.

In the foregoing construction it has been assumed that the line through p perpendicular to ab meets XY within the paper, but as this is not always the case another construction will now be given which will apply to any case.

Let pp' (Fig. 427) be the given point and ab, $a'b'$ the given line.

On ab as a ground line make another elevation, p_1', of the given point P, and $a_1'b_1'$ of the given line AB. Through p_1' draw $c_1'd$ perpendicular to $a_1'b_1'$, meeting ab at d. $c_1'd$ is the trace on the vertical plane, of which ab is the ground line, of the plane perpendicular to AB and containing the point P. It is obvious that the point d is a point

in the horizontal trace of the plane required, and that this trace will be determined by drawing a line through d perpendicular to ab.

Produce ab to meet XY at c and draw cc_1' perpendicular to ab meeting $c_1'd$ at c_1'. The point c_1' is an auxiliary elevation of a point

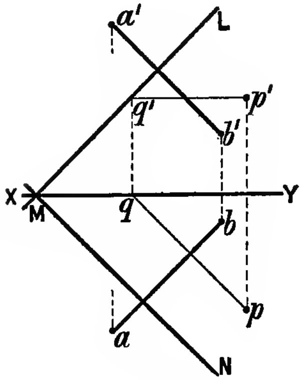


FIG. 426.

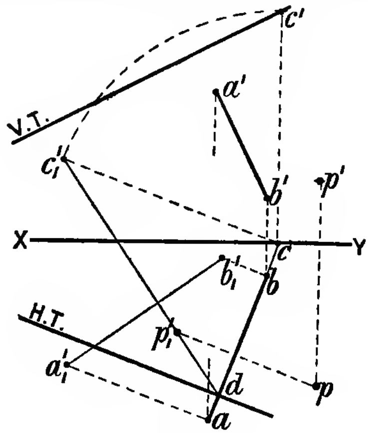


FIG. 427.

which is in the required plane and also in the vertical plane of which XY is the ground line. If therefore cc' be drawn perpendicular to XY and made equal to cc_1', c' will be a point in the vertical trace required which will be perpendicular to $a'b'$.

An examination of the figures will show that the plane determined satisfies the given conditions.

184. To draw the Traces of a Plane which shall contain a given Point and be perpendicular to two given Planes.— The plane whose traces are required will be perpendicular to the line of intersection of the two given planes. Hence one construction is— find the line of intersection of the given planes, and then by the construction of the preceding article determine the traces of the plane which contains the given point and which is perpendicular to this line of intersection.

Another construction is—determine the projections of two lines which shall pass through the given point and be perpendicular one to one plane and the other to the other. The plane containing these two intersecting lines is the plane whose traces are required.

185. To draw the Projections of a Line which shall pass through a given Point and meet a given Line at right angles. —Determine (Art. 183, p. 217) the traces of a plane which shall contain the given point and be perpendicular to the given line. Find (Art. 181, p. 216) the projections of the intersection of this plane and the given line. The line joining the plan of the given point with the plan of the intersection of the line and plane will be the plan, and the line joining the elevations of the same points will be the elevation required.

186. To draw the Traces of a Plane which shall contain a given Line and be perpendicular to a given Plane.—Determine the projections of a line to intersect the given line and be perpendicular to the given plane. Find the traces of a plane to contain these two intersecting lines. These will be the traces required.

187. Given the Inclinations of two intersecting Lines and the Angle between them, to draw their Projections and the Traces of the Plane containing them.—Draw C_1A and C_1B (Fig. 428) making the angle AC_1B equal to the given angle between the lines. From a point A in C_1A draw AD making the angle C_1AD equal to α, the given inclination of one of the lines. Draw C_1D perpendicular to AD. With centre C_1 and radius C_1D describe an arc DEF and draw BF to touch this arc and make the angle C_1BF equal to β, the given inclination of the other line. Join AB. Consider the triangle AC_1B to be on the horizontal plane and imagine this triangle to rotate about the side AB until the point C_1 is at a distance equal to C_1D or C_1F from the horizontal plane. Denoting the new position of the point C_1 by C (see the pictorial projection, Fig. 429), the lines

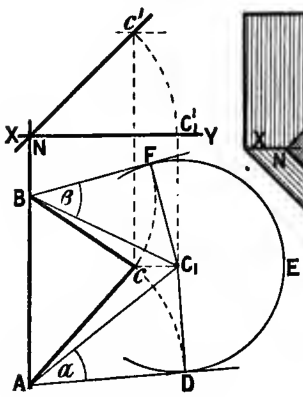


FIG. 428.

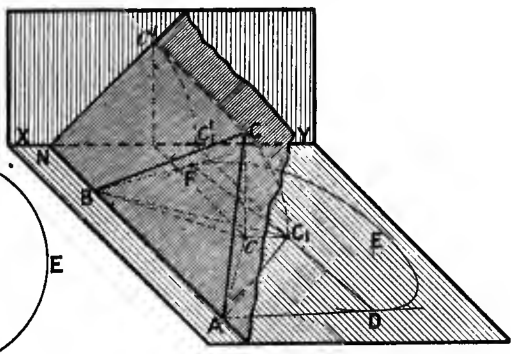


FIG. 429.

CA and CB will be inclined to the horizontal plane at angles equal to α and β respectively, and their plans will be equal in length to AD and BF respectively. Hence, if with centre A and radius AD the arc D*c* be described, and if with centre B and radius BF the arc B*c* be described, meeting the former arc at *c*, A*c* and B*c* will be the plans of the lines required.

AB will be the horizontal trace of the plane containing the lines CA and CB.

An elevation of the lines on any vertical plane can easily be obtained, since A and B are on the horizontal plane and the distance of C from the horizontal plane is known. In Fig. 428 an elevation is shown on a ground line perpendicular to AB. The distance of c' from

XY is equal to C_1D or C_1F. Nc' is the vertical trace of the plane containing the lines CA and CB.

Instead of using the arcs Dc and Fc to find the point c, this point may be determined as follows: Draw XY perpendicular to AB and produce AB, if necessary, to meet XY at N. Draw C_1C_1' perpendicular to XY. With centre N and radius NC_1' draw the arc $C_1'c'$ to meet at c' a parallel to XY at a distance from XY equal to C_1D or C_1F. A perpendicular to XY from c' to meet a parallel to XY from C_1 determines the point c.

The student should after drawing the figure $ABFC_1DA$ on a piece of paper cut it out and fold it along the lines AC_1 and BC_1 until the edges C_1D and C_1F coincide. Placing this model so that the edges AB, BF, and DA are on the table, he should then have no difficulty in fully understanding this important problem.

188. Given the Traces of a Plane, to determine the Trace of this Plane on a new Vertical Plane.—Let MN (Fig. 431) be the horizontal trace of a plane, and let LM be the trace of this plane on a vertical plane of which XY is the ground line; it is required to find the trace of this plane on a vertical plane of which X_1Y_1 is the ground line. Let MN meet X_1Y_1 at S.

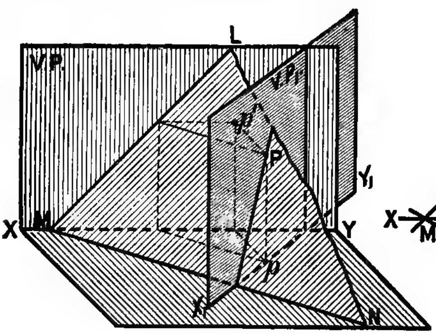


FIG. 430.

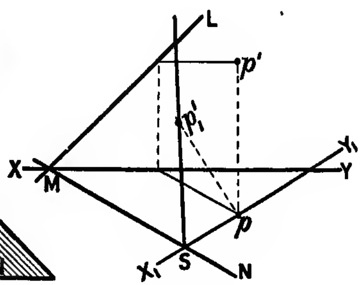


FIG. 431.

Take a point p in the new ground line and consider this as the plan of a point lying in the given plane LMN. Find the elevation p' of this point on the ground line XY. Draw pp_1' perpendicular to X_1Y_1 and make pp_1' equal to the distance of p' from XY. Sp_1' is the vertical trace required. For, P is a point in the given plane and also a point in the new vertical plane, therefore p_1' must be a point in the new vertical trace. Also the point S, where the horizontal trace meets X_1Y_1, is evidently a point in the new vertical trace, therefore the line Sp_1' is the trace required.

In the case where the new ground line does not meet the horizontal trace of the plane within a convenient distance, a second point in the required vertical trace may be found in the same way as the first point p_1'.

189. To rabat a given oblique Plane, with any Points or Lines on it, into the horizontal Plane.—Let LMN (Fig. 432) be the given oblique plane, and let abc, $a'b'c'$ be a triangle lying on this plane. Take a new ground line X_1Y_1 at right angles to MN, the horizontal trace of the given oblique plane. Find (Art. 188, p. 220) ST the vertical trace of the given plane on the new vertical plane of which X_1Y_1 is the ground line. This line ST will be the edge view of the given plane and $a_1'c_1'b_1'$ the elevation of the triangle ACB on the new vertical plane will be on ST as shown.

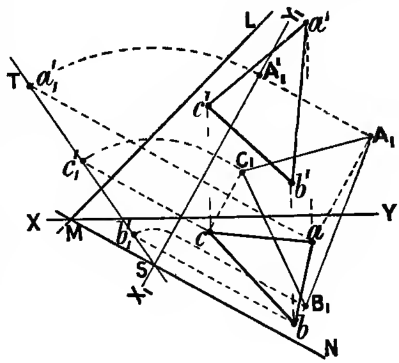


FIG. 432.

Now imagine the plane TSN to turn about SN as an axis. The point A will describe an arc of a circle whose plane is perpendicular to SN, and will therefore be vertical, and its plan will be a straight line aA_1 perpendicular to SN. The elevation (on X_1Y_1) of this arc will be an arc of a circle $a_1'A_1'$ whose centre is S. The point A_1' on X_1Y_1 is the elevation of the point A when the plane with A on it has been brought into the horizontal plane. Hence a perpendicular from A_1' to X_1Y_1 to meet aA_1 determines A_1 the plan of A in the new position. In like manner the points B_1 and C_1 may be determined, and these points A_1, B_1, and C_1 being joined a triangle is formed which is the triangle ABC brought into the horizontal plane by turning it about SN.

It is evident that the triangle $A_1B_1C_1$ is the true form of the triangle ABC.

The construction given above will be found of great use in solving numerous problems.

If any line be drawn in the triangle $A_1B_1C_1$, or if any figure be built up on that triangle, it is obvious how this line or figure may be turned about SN into the plane TSN and their projections in the new position found.

Exercises XVIa

In reproducing the figures which are shown on a squared ground take the small squares as of half-inch side.

1. The traces of a plane are given in Fig. 433, also one projection of each of three points A, B, and C contained by the plane. Find the true form of the triangle ABC.

2. A plane is given by its traces in Fig. 434, the plan abc of a triangle lying on the plane is also given. Draw the elevation of the triangle. Show also the

plan and elevation of a point which is in the given plane and 1·25 inches from each of the planes of projection.

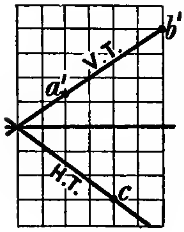


FIG. 433.

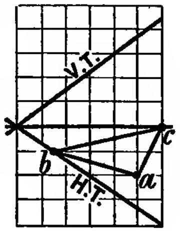


FIG. 434.

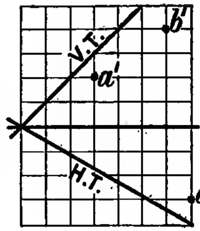


FIG. 435.

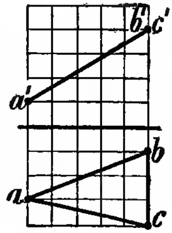


FIG. 436.

3. a' and b' (Fig. 435) are the elevations of two of the angular points of a triangle lying on the plane whose traces are given, and c is the plan of the other angular point. Draw the plan and elevation of the triangle and find its true form.

4. Draw the traces of the plane which contains the given triangle abc, $a'b'c'$ (Fig. 436). Show also the traces of the plane which contains the line ab, $a'b'$ and is parallel to the ground line.

5. The horizontal trace of a plane is given in Fig. 437, also the plan and elevation of a point P in the plane. Draw the vertical trace of the plane. Show also the plan and elevation of a point Q which is in this plane, and which is 2 inches from P and 1 inch above the horizontal plane. Assume that the point where the given horizontal trace meets the ground line is inaccessible.

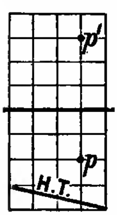


FIG. 437.

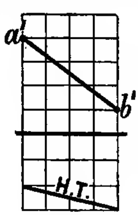


FIG. 438.

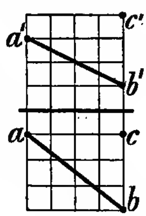


FIG. 439.

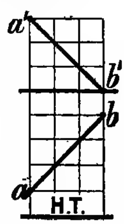


FIG. 440.

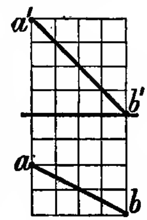


FIG. 441.

6. The horizontal trace of a plane is given in Fig. 438, also the elevation $a'b'$ of a line AB contained by the plane. The length of AB is 3 inches. Draw the plan of AB and the vertical trace of the plane. Assume that the point where the horizontal trace of the plane meets the ground line is inaccessible.

7. Draw the traces of the plane which contains the line ab, $a'b'$, and the point cc' given in Fig. 439.

8. The plan and elevation of a line AB are given in Fig. 440. This line cuts a plane whose horizontal trace is given at a point 1·5 inches from B. Draw the vertical trace of the plane.

9. Draw the traces of the two planes which contain the line ab, $a'b'$ (Fig. 441) and which are inclined at 60° to the horizontal plane. Assume that points to the left of aa' and points to the right of bb' are inaccessible.

10. The horizontal trace of a vertical plane makes an angle of 45° with XY. Draw the elevation of two lines contained by this plane, one line being inclined at 30° to the vertical plane of projection and the other inclined at 60° to the horizontal plane. The first line intersects the ground line and the second intersects the first at a point 1·5 inches from the vertical plane of projection.

11. The vertical and horizontal traces of a plane make angles of 60° and 45°

respectively with XY. Draw the projections of a line lying in this plane, inclined at 30° to the horizontal plane and passing through a point 1 inch from each of the planes of projection.

12. Taking the plane given in Fig. 435 place in this plane a line inclined at 30° to the horizontal plane, the portion of the line between the traces of the plane to be 2·5 inches long.

13. Draw the projections of a line passing through the given point *pp'* (Fig. 442) having an inclination of 45° to the horizontal plane and parallel to the given plane.

14. Draw the traces of a plane which shall contain the given point *pp'* (Fig. 442) and be parallel to the given plane.

15. In the given plane LMN (Fig. 443) place a line which shall be parallel to the plane PQR and at a distance of 0·5 inch from it.

16. Draw the projections of a line which shall pass through the point *aa'* (Fig. 443) and be parallel to each of the given planes.

17. The vertical and horizontal traces of a plane make angles of 60° and 45° respectively with XY. Draw the plan and elevation of a line perpendicular to this plane, meeting it at a point A, and intersecting XY at B so that AB is 1·5 inches long.

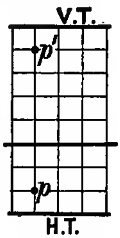


FIG. 442.

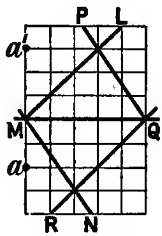


FIG. 443.

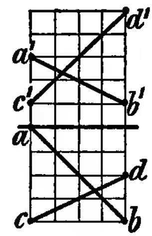


FIG. 444.

18. A line inclined at 35° to XY is both horizontal and vertical trace of a plane. A point P in XY is 2·5 inches from the point where the plane cuts XY. Find the perpendicular distance of P from the plane.

19. The projections of two non-intersecting lines AB and CD are given in Fig. 444. Draw the traces of the plane which contains the line CD and is parallel to AB. Show also the plan and elevation of the projection of AB on this plane.

20. Draw the traces of the plane which bisects the line *ab*, *a'b'* (Fig. 440) at right angles.

21. Draw the traces of the plane which shall contain the point *aa'* (Fig. 443) and be perpendicular to each of the given planes.

22. Draw the projections of a line which shall pass through the point *cc'* Fig. 439) and meet the line *ab*, *a'b'* at right angles.

23. Show by its traces a plane perpendicular to the plane of the triangle *abc*, *a'b'c'* (Fig. 436) and bisecting the sides AC and BC.

24. Two intersecting straight lines containing an angle of 60° are placed so that one of them is inclined at 35° while the other is inclined at 45° to the horizontal plane. Find the inclination of the plane containing the two lines.

25. A person using a theodolite at a station point A observes the angular elevations of two objects P and Q above the horizon to be 35° and 65° respectively. The horizontal angle (or azimuth) between their directions, that is between vertical planes containing them, measures 75°. Find the angle PAQ subtended by the two objects at the place A, as would be measured by a sextant. [B.E.]

190. Inclination of a Line to a Plane.—The inclination of a given line to a given plane is the angle which the line makes with its projection on the plane. Let PQ be the given line and let it intersect the plane at Q. Let PR be a normal to the plane, intersecting the plane at R. If Q and R be joined, then the line RQ is the projection of PQ on the plane and the angle PQR is the inclination of the line PQ to the plane. But since the angle PRQ is a right angle the

angle QPR is the complement of the angle PQR. Hence, if from a point P in the given line a normal PR be drawn to the given plane (Art. 182, p. 217) the true angle between this normal and the given line may be found as explained in Art. 147, p. 177, and the complement of this angle is the inclination required.

It is obviously unnecessary to find the points in which the given line PQ and the normal PR intersect the given plane.

191. To find the Angle between two Planes.—Referring to the pictorial projection, Fig. 445, AB and CD are two planes intersecting in the line CE. FH is a third plane which is at right angles to the line CE. The plane FH intersects the planes AB and CD in the lines KH and KL respectively. Since CE is at right angles to the plane FH it follows that CE is at right angles to each of the lines KH and KL. Hence the angle between the planes AB and CD is the angle between the lines KH and KL. If from a point P in the plane FH perpendiculars PR and PQ be drawn to KH and KL, then, the angle QKR is the supplement of the angle QPR, and if RP be produced to S the angle QKR is therefore equal to the angle QPS. It may be proved that the lines PR and PQ are perpendicular to the planes AB and CD respectively.

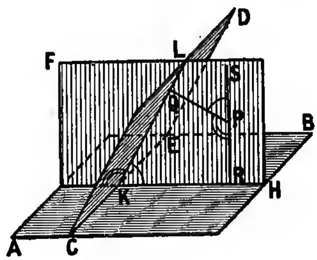


FIG. 445.

Hence, to find the angle between two given planes, draw, from any point, two perpendiculars, one to each plane, and find the angle between these perpendiculars. The acute angle between these perpendiculars will be equal to the acute angle between the planes.

Another method is as follows. Let LMN and LRN (Fig. 447) be the two given planes. Draw lN, the plan of the line of intersection

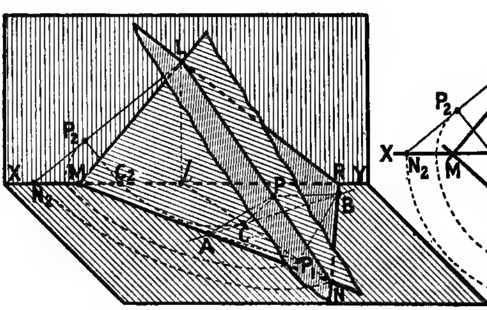


FIG. 446.

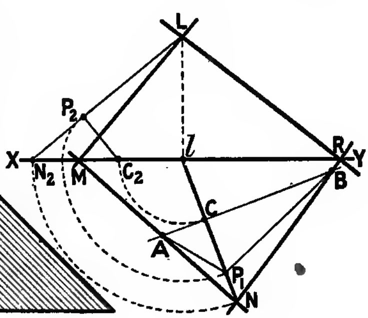


FIG. 447.

of the planes. Draw ACB at right angles to lN intersecting MN, lN, and RN at A, C, and B respectively. With centre l and radii lC and

LN describe arcs CC_2 and NN_2 cutting XY at C_2 and N_2. Join LN_2 and draw C_2P_2 perpendicular to LN_2. On lN make CP_1 equal to C_2P_2. Join AP_1 and BP_1. The acute angle between the lines AP_1 and BP_1 is the acute angle between the planes LMN and LRN. In this construction AB is the horizontal trace of a plane which is perpendicular to the line of intersection of the given planes. This plane intersects the given planes in two lines AP and BP (see the pictorial projection, Fig. 446) and the acute angle between these lines is the angle between the given planes. The lines AP and BP together with the line AB form a triangle of which AP_1B is the true form.

The points A and B where the line perpendicular to lN meets the horizontal traces of the given planes may be on the same side of lN instead of on opposite sides as in Fig. 447. In particular cases AB may be parallel to the horizontal trace of one of the given planes and the point A *or* the point B will then be at an infinite distance from C, and AP_1 *or* BP_1 will be perpendicular to lN.

192. To draw the Traces of a Plane which shall make a given Angle with a given Plane and contain a given Line in it.—Let LMN (Fig. 447) be the given plane, and lN the plan of the line in it which the required plane is to contain.

Draw AC perpendicular to lN, meeting the latter at C and the horizontal trace of the given plane at A. With centre l and radii lC and lN describe the arcs CC_2 and NN_2 cutting XY at C_2 and N_2. Join LN_2 and draw C_2P_2 perpendicular to LN_2. On lN make CP_1 equal to C_2P_2. Join AP_1. Draw P_1B making the angle AP_1B equal to the given angle. Let P_1B meet AC or AC produced at B. Join NB and produce it to meet XY at R. Join LR. NR and RL are the traces of the plane required.

If the student has understood the construction of the preceding Art. referring to Fig. 447, he should have no difficulty in satisfying himself as to the correctness of the construction given above for finding the plane LRN.

193. To draw the Traces of a Plane which shall make a given angle with a given Plane, and contain a given Line not contained by the given Plane.—Let LMN (Fig. 448) be the given plane. This plane is shown as an *inclined plane*, that is a plane inclined to the horizontal plane but perpendicular to the vertical plane of projection. If the plane is given as an *oblique plane* it should be converted into an inclined plane by the construction of Art. 188, p. 220.

ab, *a'b'* is the given line.

Take two points in the given line and consider them to be the vertices of two cones whose bases are on the given plane, and whose slant sides are inclined to their bases at the given angle (β).

It is evident that a plane which touches these two cones will fulfil the conditions of the required plane.

Turn the given plane about its horizontal trace until the plane, with the circles which are the bases of the cones, comes into the horizontal plane. Draw the tangent C_1N to these circles when in this position. Now turn the plane with the tangent on it back to its former position,

The plane which contains the lines AB and CN is the plane required; its traces are NR and ST. There are two planes which satisfy the given conditions but only one of them is shown in Fig. 448.

194. To draw the Traces of a Plane which shall contain a given Point, have a given Inclination, and make a given Angle with a given Plane.—Conceive a cone having its vertex at the given point, its base on the horizontal plane, and having a base angle equal to the angle which the required plane is to make with the horizontal plane. Conceive also a second cone having its vertex at the given point, its base on the given plane, and having a base angle equal to the angle which the required plane is to make with the given plane. It is evident that a plane which touches these two cones will fulfil the conditions of the problem.

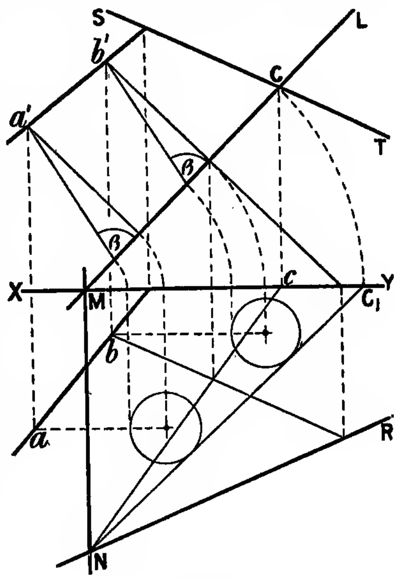


FIG. 448.

Referring to Fig. 449, vv' is the given point, and LMN is the given plane, which is assumed to be perpendicular to the vertical plane of projection. $v'a'b'$ is the elevation of the cone whose base is on the horizontal plane and whose base angle α is equal to the angle which the required plane is to make with the horizontal plane. $v'c'd'$ is the elevation of the cone whose base is on the given plane and whose base angle β is equal to the angle which the required plane is to make with the given plane.

Let the surface of the second cone be produced to cut the horizontal plane of projection. The curve of section, or horizontal trace of the cone, will be a "conic section" and it may be found as follows. Draw $t'r'$ perpendicular to $v'e'$, the elevation of the axis of the cone, meeting $v'e'$ produced at t', XY at s' and $v'd'$ produced at r'. With t' as centre and $t'r'$ as radius describe an arc, and through s' draw $s'H_2$ perpendicular to $t'r'$ to meet the arc at H_2. Draw $s's$ perpendicular to XY to meet a parallel to XY through v at s. On ss' make sh equal to $s'H_2$. h is a point on the horizontal trace of the cone. Another point will be at a distance below s equal to sh. By taking different positions for the line $t'r'$ any number of points on the horizontal trace of the cone may be found.

The base of the cone whose axis is vertical will be its horizontal trace and will be a circle whose centre is v and diameter equal to $a'b'$.

The horizontal traces of all planes which touch the two cones will

be tangents to the horizontal traces of these cones. OQ is one of these tangents meeting XY at Q. Through v draw vp parallel to OQ to

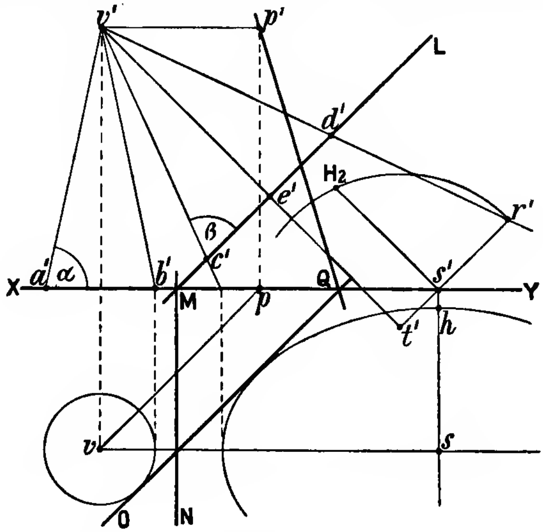


FIG. 449.

meet XY at p. From p draw pp' perpendicular to XY to meet a line through v' parallel to XY at p'. Join p'Q. The plane p'QO will satisfy the conditions of the problem. If the horizontal trace OQ does not meet XY within a convenient distance, the vertical trace may be found by the construction shown in Fig. 419, p. 214.

The drawing of the conic section which is the horizontal trace of the cone whose base is on the given plane may be avoided by the following construction. Conceive a sphere to be inscribed in this cone. The elevation of this sphere will be a circle touching the lines $v'c'$ and $v'd'$, as shown in Fig. 450, its centre being on $v'e'$. Next conceive a cone to envelop this sphere and have its axis vertical and its base angle equal to α. u' is the elevation of the vertex of such a cone and the circle shown whose centre is u is its horizontal trace.

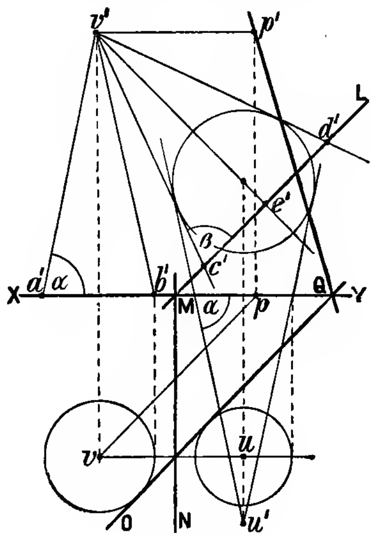


FIG. 450.

A plane which touches the cones whose vertices are at vv' will also touch the cone whose vertex is at uu', and the horizontal trace of the plane will be a tangent to the two circles shown in the plan. A little consideration will show that if the horizontal trace of the required plane is be a cross tangent to the circles the vertices vv' and uu' must be on opposite sides of the horizontal plane of projection as shown in Fig. 450.

In the particular case where the required plane is to be perpendicular to the given plane the cone whose base is on the given plane will become a straight line perpendicular to the given plane, and the horizontal trace of the required plane will pass through the horizontal trace of this line and touch the base of the other cone.

195. To draw the Projections of a Line which shall pass through a given Point and meet a given Line at a given Angle.—Let pp' (Fig. 451) be the given point and ab, $a'b'$ the given line. Determine (Art. 172, p. 211) the plane LMN which contains the point P and the line AB. Rabat this plane with the point P and line AB into the horizontal plane (Art. 189, p. 221). P_1 and A_1b are the rabatted positions of P and AB respectively. Through P_1 draw P_1C_1 making with A_1b an angle equal to the given angle. Through C_1 draw C_1c perpendicular to MN to meet ab at c. Draw cc' perpendicular to XY to meet $a'b'$ at c'. The lines pc and $p'c'$ are the projections of the line required.

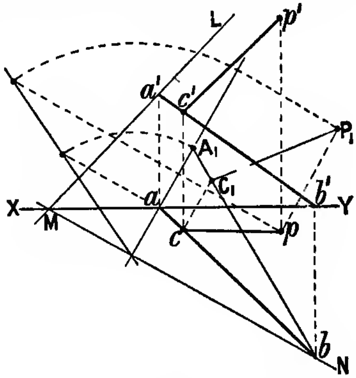


FIG. 451.

If the required line is to be perpendicular or parallel to the given line there is only one solution, in other cases there will be two lines which will satisfy the given conditions. In Fig. 451 only one of the two lines which may be drawn to satisfy the given conditions is shown.

196. To find the Traces of a Plane which shall contain one given Line and make a given Angle with another given Line.—Let AB denote the first given line and CD the second.

Determine the projections of a line EF which shall be parallel to CD and meet AB at a point E.

Take E as the vertex and EF as the axis of a cone whose semi-vertical angle is equal to the given angle. Find as in Art. 194 the horizontal trace of this cone. A tangent from the horizontal trace of AB to the horizontal trace of the cone will be the horizontal trace of the plane required. Having found the horizontal trace of the plane, its vertical trace may be determined from the condition that the plane is to contain the line AB.

If the horizontal trace of AB falls outside the curve which is the

horizontal trace of the cone, there will be two solutions of the problem; if it falls on the curve there will be one solution only, and if it falls inside the curve there can be no solution.

197. To draw the Traces of two Planes which shall be perpendicular to one another and have given Inclinations.—Let the given inclinations be to the horizontal plane and let them be α and β. Draw MN (Fig. 452), the horizontal trace of one of the planes, at any angle to XY (preferably a right angle), and determine its vertical trace LM from the condition that the inclination of the plane is the given angle α. Take any point pp' in this plane (preferably in its vertical trace) and determine a line pq, $p'q'$ perpendicular to the plane. Any plane which contains this perpendicular will be perpendicular to the plane LMN. Determine also a cone having its vertex at pp', its base on the horizontal plane and having a base angle β the given inclination of the second plane. A line through the horizontal trace of pq, $p'q'$ to touch the base of this cone will be the horizontal trace ST of the second plane. The vertical trace RS of the second plane is easily found from the condition that the plane is to contain the line pq, $p'q'$.

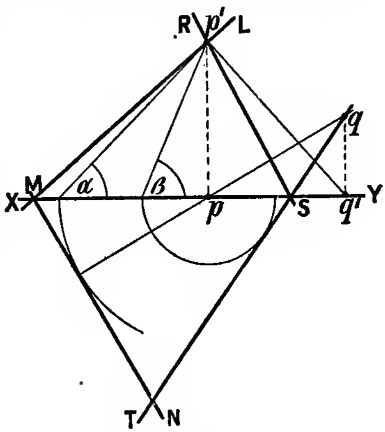


FIG. 452

198. To draw the Traces of three Planes which are each perpendicular to the other two, having given the Inclinations of two of the Planes.—Determine by the preceding Article the traces of the two planes whose inclinations are given. Next determine the line of intersection of these two planes. The third plane will be perpendicular to this line and its horizontal and vertical traces will be perpendicular to the plan and elevation of the line respectively.

199. Projections of a Solid Right Angle.—A solid right angle is formed by three planes which are mutually perpendicular. These three planes will intersect in three straight lines which are each perpendicular to the other two, and the solid angle is represented on paper by the projections of these three mutually perpendicular straight lines.

Any three intersecting straight lines on the paper may be taken as a projection of three lines which are mutually perpendicular. Let *oa*, *ob*, and *oc* (Fig. 453) be the plans of three lines which are mutually perpendicular.

Since each line is perpendicular to the other two it will be perpendicular to the plane containing the other two, and therefore the

plan of each will be perpendicular to the horizontal trace of the plane of the other two. Also the horizontal trace of the plane containing any two of the lines will pass through the horizontal traces of these lines.

It is evident that the form of any elevation of the lines will not be affected by the position of the horizontal plane of projection: the horizontal plane of projection may therefore be taken at any level.

Take any line *ab*, perpendicular to *oc*, as the horizontal trace of the plane containing the lines OA and OB. A perpendicular from *b* to *oa* to meet *oc* at *c* will be the horizontal trace of the plane containing OB and OC, and *ac* will be the horizontal trace of the plane of OA and OC.

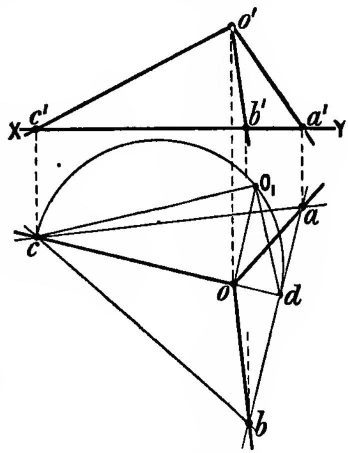


FIG. 453.

Produce *co* to meet *ab* at *d*, and consider *od* as the plan of a line lying in the plane AOB. Since OC is perpendicular to the plane AOB, the angle COD will be a right angle.

Conceive the triangle COD to revolve about CD as an axis until it comes into the horizontal plane. The point O will describe an arc of a circle whose plan will be a line through *o* perpendicular to *cd*, and the point O will, when brought into the horizontal plane, occupy such a position that the lines drawn from it to c and *d* will contain a right angle. Hence on *cd* describe a semicircle to cut the line oO_1 perpendicular to *cd* at O_1. oO_1 will be the distance of the point O from the horizontal plane containing the points A, B, and C. Having found this distance, an elevation of the lines on any vertical plane can be drawn. In the figure an elevation is shown on XY.

This problem is one of great importance in connection with isometric and axometric projections and it is further considered in Chap. XXI.

200. To find a Point on a given Plane such that the sum of its Distances from two given Points shall be the least possible.—Let LMN (Fig. 454) be the given plane and *aa'*, *bb'* the given points. These given points are on the same side of the given plane. Determine the projections of a line to pass through *aa'* and be perpendicular to the plane LMN. Find the point *cc'* where this perpendicular meets the plane LMN. Determine a point *dd'* in *ac*, *a'c'* produced such that CD is equal to AC. The projections *d* and *d'* of D are such that $cd = ac$, and $c'd' = a'c'$. Join *bd* and *b'd'*. Find the point *ee'* where the line *bd*, *b'd'* intersects the plane LMN. *ee'* is the point required.

The student should have no difficulty in proving the correctness of the above construction after referring to the corresponding problem in plane geometry discussed in Art. 9, p. 8.

It is easy to show that the lines ae, $a'e'$ and be, $b'e'$ are equally inclined to the plane LMN, and hence if a ray of light from aa' impinges upon the plane LMN and on reflection from the plane passes through bb', the incident ray must strike the plane at ee'.

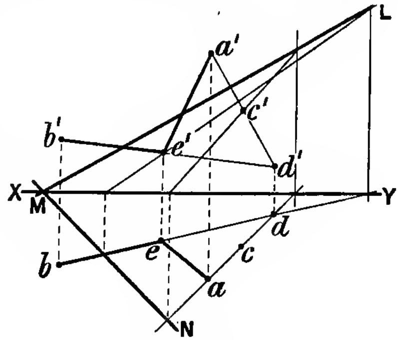


FIG. 454.

201. To find the Locus of a Point which moves on a given Plane so that the Ratio of its Distances from two given Points shall be equal to a given Ratio.—Let LMN (Fig. 455) be the given plane and aa' and bb' the given points.

Draw a new elevation $a_1'b_1'$ of the line AB on the plan ab as a ground line. Determine by the construction of Art. 16, p. 20, the circle whose centre is o_1' and radius $o_1'd_1'$ which is the locus of a point which moves in the plane of the new elevation and whose distances from a_1' and b_1' are to one another in the given ratio. In Fig. 455 this ratio is 2 : 1. The point o_1' is in $a_1'b_1'$ produced. Find o the plan and o' the elevation on XY of the point of which o_1' is the elevation on ab as a ground line. The surface of a sphere whose centre is at oo' and whose radius is $o_1'd_1'$ will contain all points in space whose distances from aa' and bb' are to one another in the given ratio. The circle which is the section of this sphere by the plane LMN is the locus required. In Fig. 455 the plane LMN being perpendicular to the vertical plane of projection, the circle which is the section of the sphere by the plane LMN will have the straight line $c'd'$ for its elevation and the ellipse cd for its plan.

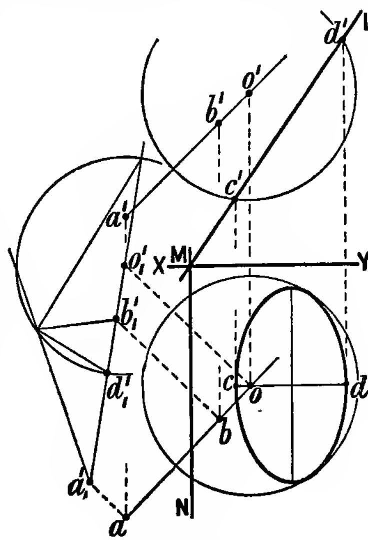


FIG. 455.

202. To determine a Line which shall pass through a given

Point and intersect two given Lines.—Let P be the given point and AB and CD the given lines.

First method. Determine the traces of the plane containing the point P and the line AB. Determine also the traces of the plane containing the point P and the line CD. The line of intersection of these two planes is the line required.

Second method. Fig. 456. Determine the projections of two lines, PKL and PMN, intersecting AB at points K and M. Let these lines meet the vertical plane containing CD at the points L and N. The line LN will be the intersection of the plane containing P and AB with the vertical plane containing CD. Let LN meet CD at Q. PQ is the line required.

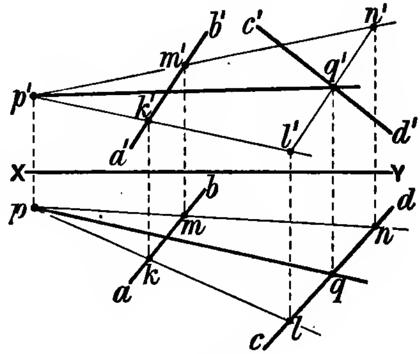


Fig. 456.

203. To determine the common Perpendicular to two given Lines which are not in the same Plane.—Let AB and CD (Figs. 457 and 458) be the two given lines. Determine a line FEH

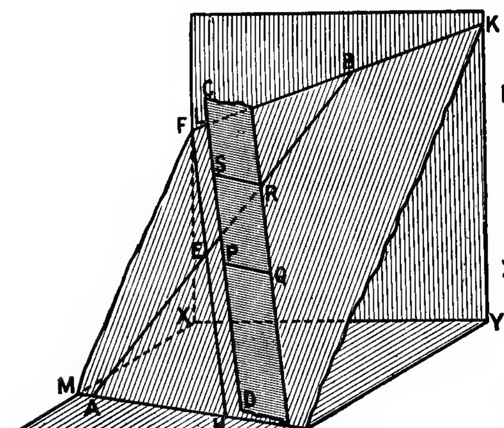


Fig. 457.

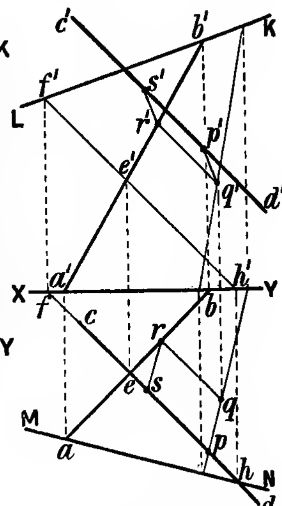


Fig. 458.

to intersect AB at E and be parallel to CD. The plane containing AB and FEH will be parallel to CD. The traces of this plane are KL and MN.

Take any point P in CD and determine the normal PQ to the plane KLMN, meeting the plane at Q. The length of PQ is the shortest distance between the lines AB and CD. Determine a line through Q parallel to CD to meet AB at R. This line will lie in the plane KLMN. A line RS parallel to QP and meeting CD at S will be the common perpendicular to the lines AB and CD.

It is evident that the line QR, produced both ways, is the projection of CD on the plane KLMN.

204. To determine the Centre and Radius of a Sphere the Surface of which shall contain four given Points.—It should first be observed that no three of the given points must be in the same straight line, and if all four lie in the same plane they must lie on the circumference of a circle. In the latter case the problem is indeterminate, as any number of spheres can be found on the surface of which the points will lie.

Let A, B, C, and D be the given points. Determine by the construction of Art. 183 a plane to bisect the line AB at right angles. Since every point in this plane is equidistant from the points A and B it must contain the centre of the required sphere. In like manner the centre of the sphere must lie on the planes bisecting BC and CD at right angles. Therefore the point of intersection of these three planes must be the centre required, and the distance of this point from any one of the given points will be the radius of the sphere.

As there are four given points, six lines can be got by joining them, and the point of intersection of three planes bisecting at right angles *any three* of these six lines *not lying in the same plane* will be the centre of the sphere.

Another method of finding the centre of the sphere is to determine the centre of the circle containing three of the given points and then determine the normal to the plane of this circle through its centre. This normal will contain the centre of the required sphere. Taking another three of the points and proceeding in the same way another normal is found which contains the centre of the sphere, and the point of intersection of these two normals is the centre required.

205. Trihedral Angles—Spherical Triangles.—The angle formed by three planes meeting at a point is called a *trihedral angle*. The angles between the lines in which the three planes intersect are the *sides* or *faces*, and the dihedral angles between the planes are the *angles* of the trihedral angle. Referring to Fig. 459, the three planes OAB, OBC, and OCA form at O a trihedral angle. The plane angles AOB, BOC, and COA are the sides or faces of the trihedral angle referred to, and the dihedral angles between these faces are its angles.

If a sphere be placed with its centre at O, the three planes which form the trihedral angle will intersect the surface of the sphere in three great circles, which will intersect one another and form what is called a *spherical triangle*. In Fig. 459, ABC is a projection of a spherical triangle.

The sides and angles of a spherical triangle are the sides and

angles of the trihedral angle which it subtends at the centre of the sphere on the surface of which it is drawn. Hence the solution of a spherical triangle is the solution of a trihedral angle.

If from a point S within the trihedral angle (Fig. 459), perpendiculars SP, SQ, and SR, be let fall on the faces OBC, OCA, and OAB respectively of the trihedral angle, these perpendiculars will form the edges of another trihedral angle S, which has the remarkable properties that its angles are the supplements of the sides of the trihedral angle O, and its sides are the supplements of the angles of the trihedral angle O. The trihedral angles O and S are therefore called *supplementary trihedral angles*, either being the supplementary trihedral angle of the other.

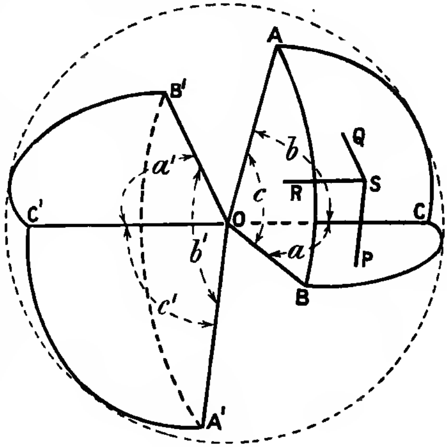


FIG. 459.

If from the point O (Fig. 459) perpendiculars OA', OB', and OC' be drawn outwards from and to the faces OBC, OCA, and OAB respectively to meet the sphere at A', B', and C', then A', B', and C' are poles of the great circles BC, CA, and AB respectively, and great circles of the sphere through A' and B', B' and C', and C' and A' form another spherical triangle A'B'C' which is called the *polar triangle* of the triangle ABC. It is obvious that the trihedral angle subtended by A'B'C' at the centre of the sphere is equal in every respect to the trihedral angle at S. Hence the polar triangle A'B'C' and the triangle ABC are supplementary.

The following notation is used to denote the elements of a trihedral angle or spherical triangle. The three angles are denoted by the capital letters A, B, and C, and the three sides by the italic letters *a*, *b*, and *c*, *the sides a, b, and c being opposite to the angles* A, B, *and* C *respectively*. The elements of the supplementary trihedral angle or the polar triangle are denoted by the same letters accented; thus the angles are A', B', and C', and the sides *a'*, *b'*, and *c'*.

The relations between the elements of two trihedral angles which are supplementary or between a spherical triangle and its polar triangle are expressed by the following equations, all angles being measured in degrees—

$$\mathrm{A}' = 180 - a \qquad a' = 180 - \mathrm{A}$$
$$\mathrm{B}' = 180 - b \qquad b' = 180 - \mathrm{B}$$
$$\mathrm{C}' = 180 - c \qquad c' = 180 - \mathrm{C}$$

From the above equations the following are obtained—

$a = 180 - A'$ $\qquad$ $A = 180 - a'$
$b = 180 - B'$ $\qquad$ $B = 180 - b'$
$c = 180 - C'$ $\qquad$ $C = 180 - c'$

Any three of the six elements of a trihedral angle or spherical triangle being given, the other three can be found. There are in all six cases to consider.—I. Given the three sides. II. Given two sides and the angle between them. III. Given two sides and the angle opposite to one of them. IV. Given the three angles. V. Given one side and the two adjacent angles. VI. Given one side, an adjacent angle, and an opposite angle.

By means of the properties of the supplementary trihedral angle, cases IV, V, and VI may be reduced to cases I, II, and III respectively. Thus, in solving case IV, find by the solution of case I the angles of the trihedral angle whose sides are the supplements of the given angles, then the supplements of the angles found will be the sides required. Or, using symbols, given A, B, and C, then $a' = 180 - A$, $b' = 180 - B$, and $c' = 180 - C$; now find, by the construction for case I, A′, B′, and C′, then $a = 180 - A'$, $b = 180 - B'$, and $c = 180 - C'$.

In case V suppose a, B, and C given, then $A' = 180 - a$, $b' = 180 - B$, and $c' = 180 - C$. Now, by the construction for case II, find a', B′, and C′, then $A = 180 - a'$, $b = 180 - B'$, and $c = 180 - C'$. In like manner case VI is reduced to case III.

The solutions of cases I, II, and III will now be given.

Case I. *Given the three sides* a, b, *and* c *of a trihedral angle to determine the three angles* A, B, *and* C. Construct the three sides a, b, and c, as shown in Fig. 460, forming the development of the faces of

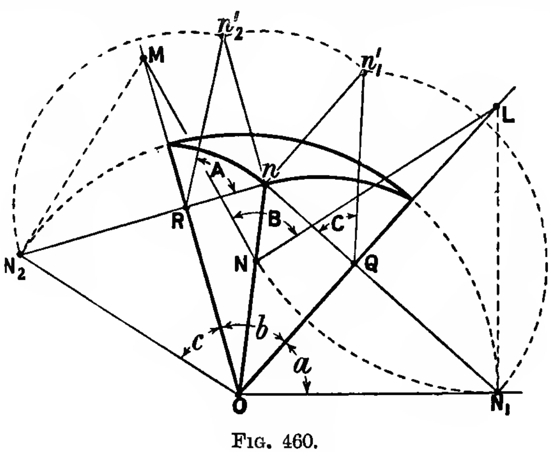


Fig. 460.

the trihedral angle. Consider this development to be lying on the horizontal plane. Mark off equal lengths ON_1 and ON_2 on the outer lines of the development. Draw N_1n perpendicular to OL, and N_2n

perpendicular to OM to meet N_1n at n. Join On. The three lines OL, OM, and On will form the plan of the trihedral angle, the face b being on the horizontal plane. In finding the point n the faces a and c have been supposed to turn about the edges OL, and OM respectively until the lines ON_1 and ON_2 coincide. It is evident that the point N_1 describes an arc of a circle whose plan is the straight line N_1n; also, the point N_2 describes an arc of a circle whose plan is the straight line N_2n; and since N_1 and N_2 are equidistant from O, n must be the plan of the point where they meet, and On is the plan of the edge between the faces a and c.

To determine the angles A and C, draw nn_1' perpendicular to N_1n, and nn_2' perpendicular to N_2n. With Q as centre, and QN_1 as radius, describe an arc to cut nn_1' at n_1', and with R as centre, and RN_2 as radius, describe an arc to cut nn_2' at n_2'. Join $n_1'Q$ and $n_2'R$. $n_1'Qn$ is the angle C, and $n_2'Rn$ is the angle A. It is evident that when the face a is brought into its natural position, the line N_1Q on that face will remain perpendicular to OL; also the line nQ on the face b is perpendicular to OL; therefore the angle $n_1'Qn$, which is the angle between these lines, will be equal to the angle between the faces a and b, that is, the angle C. In like manner the angle $n_2'Rn$ is equal to the angle A.

To find the angle B, draw N_1L perpendicular to ON_1 to meet OL at L, and draw N_2M perpendicular to ON_2 to meet OM at M. With centre L, and radius LN_1, describe an arc to cut On at N. Join N to L and M. LNM is equal to the angle B. It is evident that when the faces a and c are brought into their natural positions, the lines LN_1 and MN_2 lying on these faces will be perpendicular to their line of intersection, and therefore the angle between them, which is obviously equal to the angle LNM, will be equal to the angle B.

The student will more clearly understand the foregoing constructions if, after drawing the figure shown, he cuts it out along the lines LN_1, N_1O, ON_2, and N_2M, and then folds the triangles LN_1O and MN_2O about OL and OM respectively until the lines ON_1 and ON_2 meet one another.

CASE II. *Given two sides of a trihedral angle, and the angle between them, to determine the third side and the other angles.* Let a and b be the given sides, and C the angle between them. Construct the two sides a and b on the horizontal plane as shown in Fig. 461. From a point N_1 in ON_1 draw N_1Qn perpendicular to OL and intersecting it at Q. Draw Qn_1' making the angle $n_1'Qn$ equal to C. Make Qn_1' equal to QN_1, and draw $n_1'n$ perpendicular to N_1n to meet the latter at n.

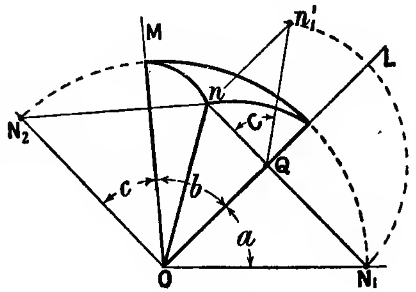


FIG. 461.

Join On. The lines OL, OM, and On will form the plan of the trihedral angle, the face b being on the horizontal plane.

To determine the side c, draw nN_2 perpendicular to OM, and with centre O and radius ON_1 describe an arc to cut nN_2 at N_2. Join ON_2. N_2OM is the side c brought into the horizontal plane.

The angles A and B may now be determined exactly as in case I. If the student has understood the constructions in case I, he should have no difficulty in demonstrating the above constructions for case II.

CASE III. *Given two sides of a trihedral angle, and the angle opposite to one of them, to determine the third side and the other angles.* Let a and b be the given sides and A the given angle. As before, construct the given sides on the horizontal plane as shown in Fig. 462. Take a ground line XY at right angles to OL, cutting OL at Q, ON_1 at N_1 and OM at M. From Q draw QT at right angles to OM meeting it at T. With Q as centre, and QT as radius, describe the arc TT_2 to meet XY at T_2. Draw T_2V making the

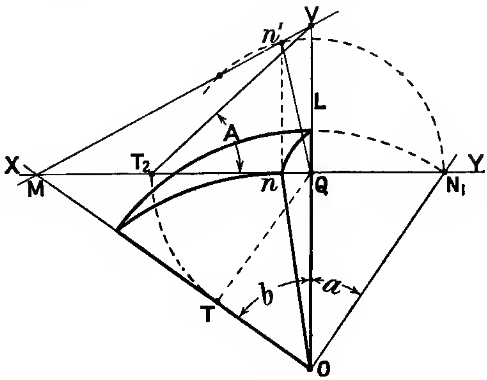


FIG. 462.

angle QT_2V equal to the angle A and meeting OL at V. Join MV. MV is the vertical trace of the face c, OM being its horizontal trace. Now let the face a revolve about OL. Since OL is perpendicular to XY the elevation of the circle described by N_1 will be a circle having a radius equal to QN_1, with Q as centre. Let this circle cut MV at n'. Join $n'Q$. The line $n'Q$ is the vertical trace of the face a when it is in its natural position, and OQ is its horizontal trace; therefore, if a perpendicular be let fall from n' to XY to meet XY at n, the line On will be the plan of the intersection of the faces a and c. Hence the lines OL, OM, and On form the plan of the trihedral angle, the face b being on the horizontal plane. The third face c may now be determined as in case II and the angles B and C may be found as in case I.

The arc N_1n' may meet the line MV at two points (it does so in Fig. 462), and there are then two solutions, that is, there are two trihedral angles which satisfy the conditions of the problem. This is then called the *ambiguous case.*

206. Circumscribed and Inscribed Circles of Spherical Triangles.—A plane containing the three angular points A, B, and C of a spherical triangle (Fig. 463) will intersect the surface of the sphere in a circle which is the circumscribed circle of the spherical

triangle. A right circular cone, having its vertex at O, the centre of the sphere, and having its curved surface touching the three faces of the trihedral angle subtending the spherical triangle at O, will intersect the surface of the sphere in a circle which is the inscribed circle of the spherical triangle. The inscribed cone is determined by first drawing a sphere inscribed in the trihedral angle. The cone envelops this sphere.

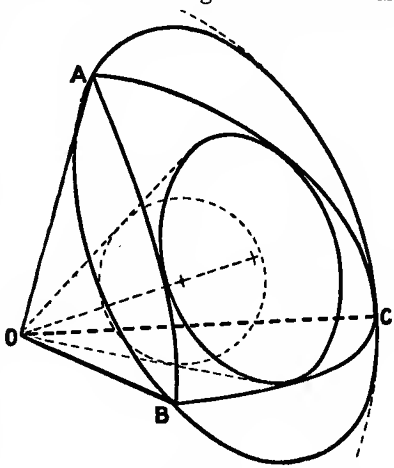


FIG. 463.

Exercises XVIb

1. Find the angle between the pairs of planes shown at (*a*), (*b*), (*c*), and (*d*), Fig. 464. Show also in each case the traces of the plane which bisects the angle between the given planes.

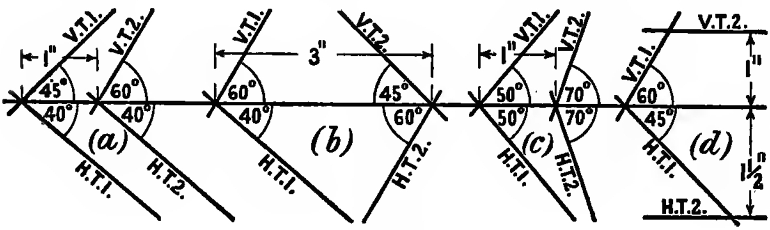


FIG. 464.

2. The horizontal and vertical traces of a plane (1) make angles of 90° and 45° respectively with XY. A line inclined at 40° to the vertical plane of projection lies in this plane and the distance between its horizontal and vertical traces is 3 inches. This line is the intersection of a plane (2) with the plane (1), the inclination of the plane (2) to the horizontal plane being 60°. Draw the traces of the planes (1) and (2) and find the angle between the planes.

3. OA and OB (Fig. 465) are two straight lines lying on the horizontal plane. OC is another line of which O*c* is the plan. The inclination of OC to the horizontal plane is 50°. Determine the angle between the planes AOC and BOC, also the angle between the planes AOC and AOB, and the angle between the lines OA and OC.

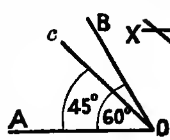


FIG. 465.

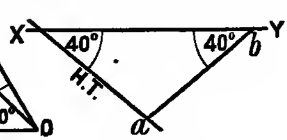


FIG. 466.

4. *ab* (Fig. 466) is the plan of a line lying in a plane of which H.T. is the horizontal trace. The inclination of AB to the horizontal plane is 50°. Draw the traces of second plane to contain AB and make an angle of 60° with the first plane.

5. The vertical and horizontal traces of a plane (1) make angles of 40° and 45° respectively with XY. A point in the vertical trace of this plane and 1·5 inches

above XY is the elevation of a line AB. Draw the traces of a plane (2) to contain the line AB and make an angle of 75° with the plane (1).

6. Draw the traces of a plane (1) which is inclined at 45° to the horizontal plane and is perpendicular to the vertical plane of projection. Then draw the traces of a plane (2) which is inclined at 65° to plane (1) and inclined at 75° to the horizontal plane.

7. The traces of a plane LMN are both inclined at 45° to XY. A point P in this plane is 2 inches above the horizontal plane and 1 inch in front of the vertical plane. Draw the projections of a line passing through the point P, inclined at 30° to the horizontal plane and inclined to the plane LMN at 60°.

8. *abc* is the plan of a triangle. $ab = 2{\cdot}1$ inches, $bc = 2{\cdot}9$ inches, and $ca = 1{\cdot}4$ inches. The heights of the points A, B, and C above the horizontal plane are, 0, 2·5, and 1·25 inches respectively. Find the plan of a point D in AB such that the angle ADC is 50°.

9. A line AB of indefinite length passes through a point C which is in the vertical plane of projection and 2 inches above XY. AB is inclined at 40° to the horizontal plane and its plan makes 30° with XY. A point P is 1·5 inches in front of the vertical plane and 1 inch above the horizontal plane, and its plan is at a perpendicular distance of 1·8 inches from the plan of AB. Draw the plan and elevation of an equilateral triangle which has one angular point at P and the opposite side on AB.

10. *a*MN*b* is a quadrilateral, the angles at M and N being right angles. MN = 1·5 inches, *a*M = 1 inch, and *b*N = 2·5 inches. *ab* is the plan of a horizontal line which is 0·8 inch above the horizontal plane. MN is the horizontal trace of a plane which contains the point A. Draw the plan of a line which lies in the plane AMN and makes an angle of 75° with AB.

11. Draw the traces of a plane which shall contain the line AB (Exercise 17) and be inclined to the line CD at an angle of 15°.

12. One plane is inclined at 60° to the horizontal plane and 45° to the vertical plane. Another plane is perpendicular to the first and inclined at 70° to the horizontal plane. Draw the traces of the two planes.

13. The horizontal and vertical traces of a plane are in one straight line inclined at 40° to XY. A second plane is perpendicular to the first and inclined at 60° to the horizontal plane. Draw the traces of the two planes.

14. Three planes are mutually perpendicular. One is inclined at 50° and another at 45°. Find the inclination of the third plane.

15. H.T. (Fig. 467) is the horizontal trace of a plane which contains the point whose plan is *c*. The height of C above the horizontal plane is 1 inch. *a* and *b* are the plans of two points whose heights above the horizontal plane are 1·5 inches and 2 inches respectively. Find the projections of a point P on the given plane such that AP + BP shall be a minimum.

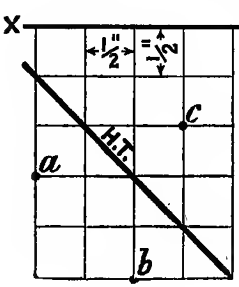


FIG. 467.

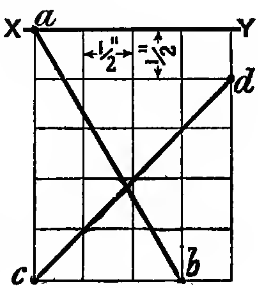


FIG. 468.

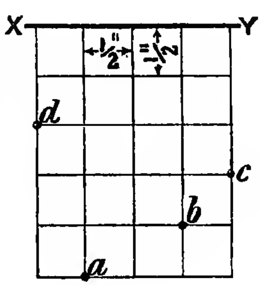


FIG. 469.

16. Taking the same plane and the same points as in the preceding exercise, draw the projections of the locus of a point P which moves on the plane in such a manner that the ratio of AP to BP shall be equal to the ratio of 3 to 2.

17. *ab* and *cd* (Fig. 468) are the plans of two straight lines. The heights of the points A, B, C, and D above the horizontal plane are, 1·5, 0·5, 1·0, and 2·5 inches respectively. Draw the plan and elevation of the common perpendicular to the lines AB and CD and find its true length.

18. *a*, *b*, *c*, and *d* (Fig. 469) are the plans of four points. The heights of A, B, C, and D above the horizontal plane are, 2·5, 0·25, 2·6, and 0·6 inches respectively. Determine the plan and elevation of the sphere whose surface contains the points A, B, C, and D.

19. Taking the lines given in exercise 17 and Fig. 468, determine the locus of a point which moves on the horizontal plane so as always to be equally distant from these lines.

20. The common perpendicular EF to two straight lines AB and CD is 1 inch long and its inclination is 40°. The inclination of AB is 35° and the angle between the plans of AB and CD is 90°. Draw the plans of AB, CD, and EF, and find the inclination of CD.

21. A person standing in the corner of a quarry notices that on one vertical face the line of a particular bed slopes downwards from the corner at an angle of 30° with the horizontal, whilst on an adjacent vertical face the corresponding angle is 48°. The angle between the two faces is 110°. What is the true dip or inclination of the strata? [B.E.]

22. Two inclined faces of rock A, B in a quarry intersect in L. The lines of steepest ascent on A make 10° with the vertical and go due North. Those on B make 20° and have a direction 12° North of East. A seam of coal intersects these faces in lines which both go upwards from L, making 70° with L on the face A and 65° with L on the face B. Find the direction, and inclination to the horizontal, of the lines of steepest descent of the seam. [B.E.]

23. The plans of four points O, A, B, and C are given in Fig. 470. O is the centre of a sphere of 2·5 inches radius. A, B, and C lie on the surface of the sphere, A and C being above, and B below the level of O. Draw the plan of the spherical triangle ABC, and an elevation on a ground line parallel to *ox*. Add the plan and elevation of the circumscribed circles of the spherical triangle. Lastly, determine the plan and elevation of the polar triangle.

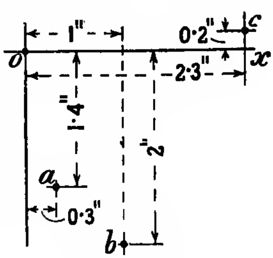


FIG. 470.

24. Solve, by construction, the following trihedral angles or spherical triangles. The results, as obtained by calculation, correct to the nearest tenth of a degree, are added.

(1) Given, $a = 40°$, $b = 50°$, $c = 60°$.
Results. A = 47·9°, B = 62·2°, C = 89·1°.

(2) Given, $a = 70°$, $b = 80°$, $c = 90°$.
Results. A = 69·7°, B = 79·4°, C = 93·7°.

(3) Given, $a = 45°$, $b = 55°$, C = 60°.
Results. A = 58·4°, B = 80·7°, $c = 46°$.

(4) Given, $a = 60°$, $b = 70°$, C = 120°.
Results. A = 50·5°, B = 56·9°, $c = 103·6°$.

(5) Given, $a = 45°$, $b = 55°$, A = 45°.
Results. $c = 75·1°$, B = 55°, C = 104·9°,
or, $c = 15·5°$, B = 125°, C = 15·5°.

(6) Given, $a = 60°$, A = 70°, C = 90°.
Results. $b = 39·1°$, $c = 67·2°$, B = 43·2°,
or, $b = 140·9°$, $c = 112·8°$, B = 136·8°.

(7) Given, A = 120°, B = 45°, C = 35°.
Results. $a = 78·7°$, $b = 53·2°$, $c = 40·5°$.

(8) Given, A = 130°, B = 120°, $c = 90°$.
Results. $a = 126°$, $b = 113·9°$, C = 108·7°.

CHAPTER XVII

SECTIONS OF SOLIDS

207. Sections of Solids.—Many objects of which mechanical drawings have to be made are of such a form that their construction is not completely shown by outside views only. The construction of the interior of a house, for instance, cannot be seen from the outside. In order to exhibit the interior of such an object it is imagined to be cut in pieces by planes and these pieces are then represented separately. But in representing an object of comparatively simple form, the addition of a sectional drawing of it often adds very much to the illustration of it, although such a sectional drawing may not be absolutely necessary for the complete representation of the object.

That surface which is produced when a plane cuts a solid is called a *section*, and if that part of the solid which is below or behind the cutting plane is also shown on the projection of a section the projection is called a *sectional plan* or a *sectional elevation*. But in the application of these terms to architectural and engineering drawings the term section is often used in the same sense as sectional plan or sectional elevation.

The projection of a section is distinguished in various ways, one way, used in this work, is by drawing across it parallel diagonal lines at equal distances apart. These lines are called section lines.

If the true form of a section is required it must be projected on a plane parallel to that of the section.

In this chapter sections of solids having plane faces are considered. Sections of the sphere, cylinder and cone are considered in chapter XVIII.

208. Section of a Prism by a Plane perpendicular to one of the Planes of Projection.—A plane section of a prism will be a rectilineal figure whose angular points are at the points where the plane of section cuts the edges of the prism. Since the plane of section is perpendicular to one of the planes of projection one of the projections of the section will be a straight line coinciding with the trace of the plane of section on the plane of projection to which the plane of section is perpendicular.

The determination of sections of a right square prism is illustrated by Fig. 471. The plan (1) and the elevation (2) represent the prism

when standing with one end on the horizontal plane. PQ is the horizontal trace of a vertical plane of section. This plane of section cuts two of the vertical faces of the prism in vertical lines of which the points *a* and *b* are the plans, and *a'a'* and *b'b'* perpendiculars to XY, the elevations. The same plane intersects the ends of the prism in horizontal straight lines of which *ab* is the plan and *a'b'* and *a'b'* the elevations. The rectangle *a'b'b'a'* is the complete elevation of the section.

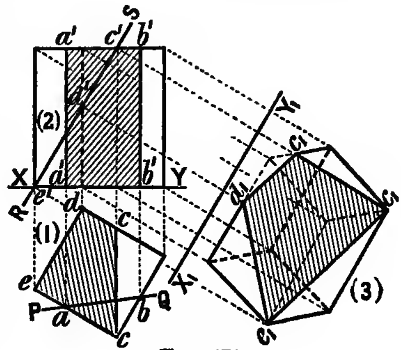


FIG. 471.

RS is the vertical trace of another plane of section. This second plane of section is perpendicular to the vertical plane of projection. On the plan (1) this second section appears as the figure *cdec*. Another plan (3) of this second section is shown on a ground line X_1Y_1 parallel to RS. The plan (3) shows the true form of the section of the prism by the plane RS. The whole solid is also shown in plan (3), the part above the section by the plane RS being represented by thin dotted lines

209. Section of a Pyramid by a Plane perpendicular to one of the Planes of Projection.—The first paragraph of the preceding Article on the section of a prism applies also to the section of a pyramid, and all that need be done here is to give an example which the student should work out.

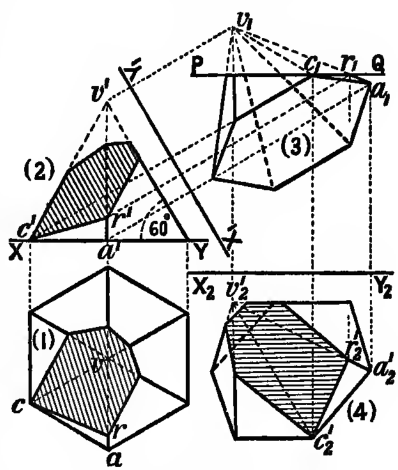


FIG. 472.

In Fig. 472, (1) is the plan and (2) the elevation of a right hexagonal pyramid when standing on the horizontal plane with two sides at right angles to XY. (3) is a plan of the pyramid when one triangular face is horizontal. The plan (3) is projected from the elevation (2) as shown. (4) is an elevation projected from (3) on a ground line X_2Y_2 parallel to XY. The straight line PQ is the horizontal trace of a plane of section which is perpendicular to the plane of the plan (3). PQ is parallel to XY and passes through c_1 the plan (3) of one angular point of the base of the pyramid. The section of the pyramid by the

plane PQ is shown projected on to each of the other views of the pyramid, and in each view the part of the solid between the vertex and the section is shown by thin dotted lines. The side of the hexagonal base may be taken 1·25 inches long.

210. Sections of Mouldings.—A *moulding* is an ornament of uniform cross section which may be formed on a piece of a structure or it may be a separate piece attached to the structure for ornament only, or it may be the main part of the structure as in a picture frame. Mouldings are of frequent occurrence in cabinet making, joinery, and masonry.

The principal problem in the geometry of mouldings is the determination of the cross section of one moulding which will mitre correctly

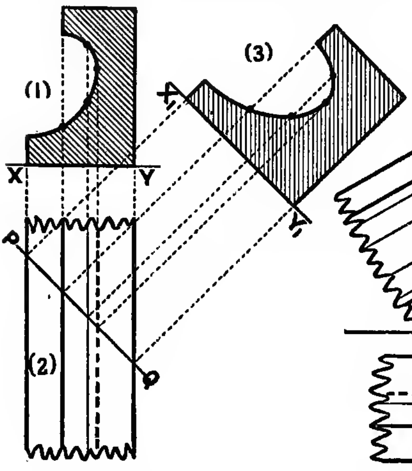


FIG. 473.

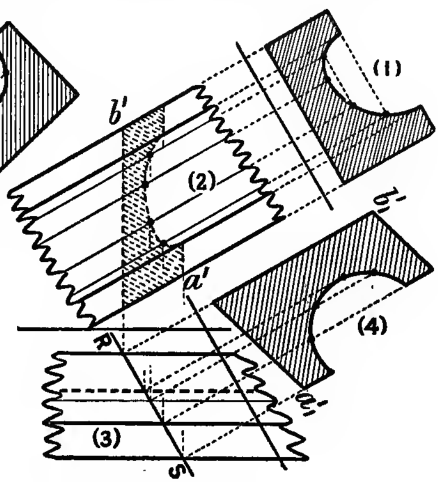


FIG. 474.

with another when faces of the one are in different planes from corresponding faces of the other.

Given the cross section of a moulding the determination of any other section is a simple problem and is worked exactly as for a prism. In Fig. 473, (1) is the true form of the cross section of a straight piece of moulding. (2) is a plan of the moulding, the long edges being horizontal. PQ is the horizontal trace of a vertical plane of section, and (3) is the elevation of the section PQ on a ground line X_1Y_1 parallel to PQ.

The same moulding is shown in Fig. 474. (1) is the cross section, (2) is a side elevation, and (3) is a plan when the long edges are parallel to the vertical plane of projection but inclined to the horizontal plane. RS is the horizontal trace of a vertical plane of section. $a'b'$ is the projection of the section RS on the plane of the elevation (2), and $a_1'b_1'$ is a projection of the same section on a plane parallel to it and which therefore shows the true form of the section.

In Fig. 474 the projections (1), (2), (3), and (4) have been drawn in the order in which they are numbered, but a study of the figure will show that if the section $a_1'b_1'$ by the plane RS be given instead of the cross section (1) the latter may be found by working backwards.

In Fig. 475 the section a' is the cross section and a is the plan of a straight piece of moulding whose long edges are horizontal and which is fixed to the face of a vertical wall whose plan is rs. b is the plan and b' the elevation of another straight piece of moulding whose long edges are inclined to the horizontal plane and which is fixed to the face of a vertical wall whose plan is st. The faces of the two walls are at right angles to one another, and the two mouldings intersect in a vertical plane, whose horizontal trace, or plan, is the straight line ns which bisects the angle between sr and st. A moulding such as bb'

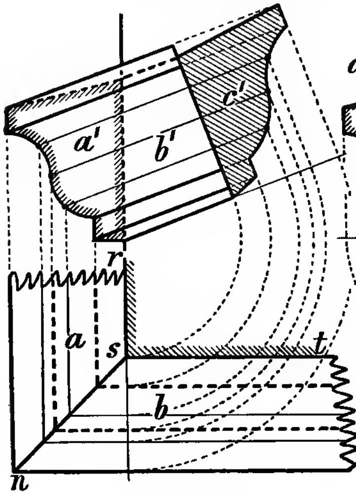


FIG. 475.

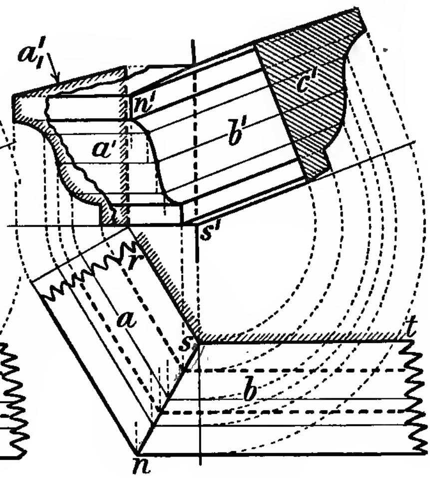


FIG. 476.

whose long edges are inclined to the horizontal is called a *raking moulding.*

The problem is now to find the true form of the cross section of the raking moulding. The condition which the raking moulding must satisfy is that its section by the plane ns of the joint with the other moulding must be the same as the section of the other moulding by that plane. From this condition it follows that the corresponding longitudinal edges or lines on the two mouldings must intersect in the plane of the joint. The projections of the longitudinal lines on the raking moulding may therefore now be drawn and the cross section c' determined as shown.

The case illustrated by Fig. 476 differs from that illustrated by Fig. 475 in that the angle rst between the faces of the walls in Fig. 476 is greater than a right angle. a_1', the cross section of the horizontal

moulding, is shown rabatted into the vertical plane of projection. It will be seen that in order that the elevations of the longitudinal lines on the raking moulding may be drawn the elevation $n's'$ of the joint must first be determined.

Two straight mouldings, of the same cross section, may be mitred correctly together if they are attached to a plane surface, or if they are attached to plane surfaces inclined to one another, provided that the longitudinal lines of the two mouldings are perpendicular to the line of intersection of the plane surfaces to which they are attached.

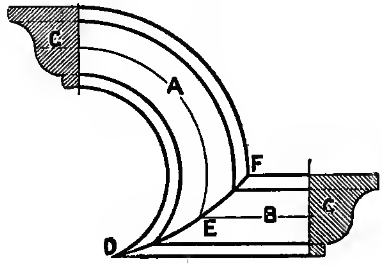


Fig. 477.

Two curved mouldings of the same cross section but of different curvatures and attached to a plane surface may be jointed together but the joint is a curved surface. Fig. 477 shows a curved moulding A jointed to a straight moulding B both having the same cross section C. The two mouldings are supposed to be attached to the plane of the paper. The projection of the joint on the plane of the paper is the curved line DEF determined as shown.

211. Geometry of Rafters.—Practical problems on the sections of prisms occur in timber construction, and illustrations will now be given of these problems as applied to a timber roof frame.

Fig. 478 shows plan and elevation line diagrams of part of what is called a *hipped roof*. The various members of the frame of this roof are mainly of rectangular cross section. vu, $v'u'$ is the ridge piece. vr, $v'r'$ is the *hip* or *hip rafter* which passes from one end of the ridge piece to the top corner of two intersecting walls. cd, $c'd'$ is one of the *jack rafters* which lie between the hip rafter and the *wall plate* on the top of one of the walls. ef, $e'f'$ is one of the *common rafters* which lie between the ridge piece and a wall plate.

The dihedral angle between the two roof surfaces intersecting in the line VR is determined by the construction of Art. 191, p. 224, or by that construction slightly modified as shown here. vv_1' is drawn at right angles to vr and is made equal to the height of v' above r'. Joining $v_1'r$ determines the true length of VR. mn is drawn at right angles to vr to meet the horizontal traces of the roof surfaces at the level of R at m and n. mn is the horizontal trace of a plane at right angles to VR. This plane intersects the roof surfaces whose line of intersection is VR in two straight lines the angle between which is the angle required. os' is drawn at right angles to $v_1'r$ and oS is made equal to os'. Joining S to m and n determines mSn the dihedral angle required.

At h and k are shown two forms of the cross-section of the hip rafter to an enlarged scale. TP and TQ are parallel to Sm and Sn respectively. This determines the angle ϕ which the carpenter requires

in backing off the top face of the rafter to bring it into the planes or plane of the roof surface.

Fig. 479 shows in detail a plan (1), a side elevation (2) and a front elevation (3) of the upper part of a jack rafter where it joins the hip rafter. All the rafters have their side faces vertical, and X_2Y_2 is the plan of the vertical face of the hip rafter to which the jack rafter is

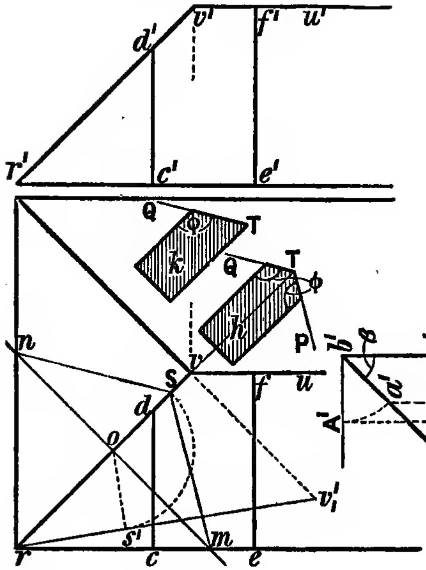


Fig. 478.

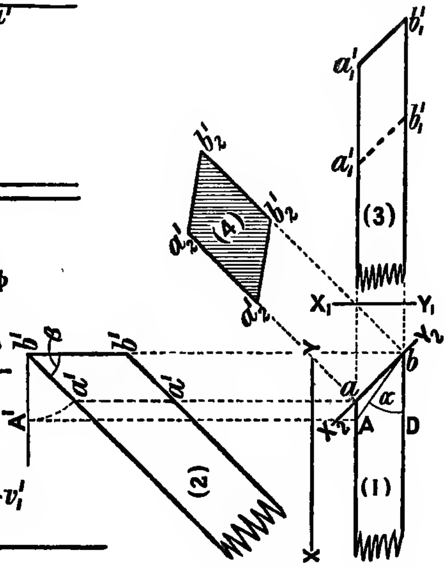


Fig. 479.

attached. This end of the jack rafter requires to be bevelled in two directions in order to fit against the face of the hip rafter. The angles for these bevels are marked α and β. The angle β is shown on the side elevation (2). To determine α the upper surface of the rafter is brought into a horizontal position by turning it about a horizontal axis through B at right angles to XY, as shown in the plan (1) and elevation (2). The true form of the end of the rafter is shown at (4).

Exercises XVII

1. The solid whose projections are given in Fig. 480 is cut in two by a vertical plane whose horizontal trace is HT. Draw an elevation of the part to the left of the plane of section on a ground line parallel to HT.

2. The solid whose projections are given in Fig. 481 is cut in two by a vertical plane whose horizontal trace is PQ. Draw an elevation of the larger portion on a plane parallel to the plane of section.

3. The solids whose projections are given in Fig. 482 are cut by a vertical plane

whose horizontal trace is RS. Draw an elevation of the portions whose plans lie between RS and XY on a ground line parallel to RS.

4. The plan of a right square prism with a rectangular hole through it is given in Fig. 483. A side of the square base is on the horizontal plane. Draw the

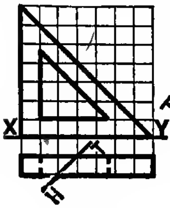


FIG. 480.

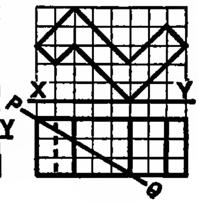


FIG. 481.

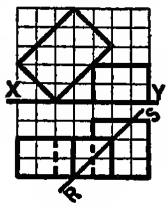


FIG. 482.

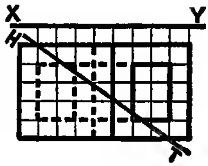


FIG. 483.

In reproducing the above diagrams take the small squares as of 0·3 *inch side.*

elevation on XY and show on this elevation the section of the solid by the vertical plane of which HT is the horizontal trace. Determine also the true form of the section.

5. A right square prism, side of base 1 inch, altitude 3 inches, is cut into two equal parts by a plane. The true form of the section is a rhombus of 1·5 inches side. Draw a plan of one part when it stands with its section face on the horizontal plane.

6. A pyramid has for its base a square of 2 inches side which is on the horizontal plane. The altitude of the pyramid is 2·5 inches and the plan of the vertex is at the middle point of one side of the plan of the base. This pyramid is cut into two portions by a horizontal plane which is 1 inch above the base. Draw the plan of the lower portion.

7. A right hexagonal pyramid, side of base 1 inch, altitude 2 inches is cut by a plane which contains one edge of the base and is perpendicular to the opposite face. Determine the true form of the section.

8. The plan and elevation of one of the rectangular faces ABCD of a right hexagonal prism are given in Fig. 484, AB and CD being on the ends of the prism. Complete the plan and elevation of the prism and show on the elevation the section by the vertical plane whose horizontal trace is PQ.

9. The plan and elevation of one of the triangular faces VAB of a right square pyramid are given in Fig. 485, V being the vertex of the pyramid. Complete the plan and elevation of the pyramid and show on the elevation the section by the vertical plane whose horizontal trace is RS.

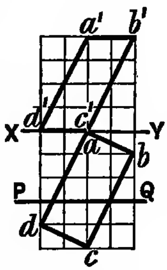


FIG. 484.

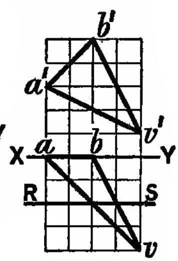


FIG. 485.

Take small squares 0·4 *inch side.*

10. A horizontal moulding has (*e*), Fig. 486, for its cross-section. Determine the true form of a vertical section which is inclined at 45° to the longitudinal lines of the moulding.

11. A moulding whose cross-section is given at (*f*) in Fig. 486 is placed with its back on the vertical plane of projection and its longitudinal lines inclined at 30° to the horizontal plane. Draw the plan and elevation and show the true form of a section by a vertical plane whose horizontal trace is inclined at 60° to XY.

12. Two mouldings of the same cross-section (c), Fig. 486, are fixed to the face

of a vertical wall. One moulding is horizontal and the other rakes at 30° to the horizontal. The two mouldings join correctly together. Determine the true form of the joint section.

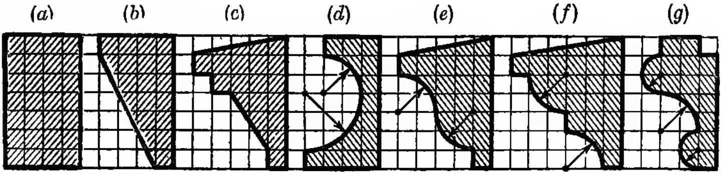


FIG. 486.

In reproducing the above sections take the small squares as of ¼ inch side.

13. A horizontal moulding mitres correctly with a raking moulding. Inclination of raking moulding to the horizontal, 25°. The backs of the mouldings are in vertical planes which are at right angles to one another as shown in plan in Fig. 487. Select from Fig. 486 a cross-section for the horizontal moulding and then determine the cross-section of the raking moulding.

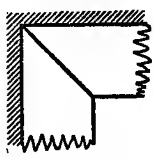


FIG. 487.

14. Same as exercise 13 except that the angle between the vertical planes is 120° instead of 90°.

15. Two raking mouldings on vertical walls which are at right angles to one another as in Fig. 475, p. 244, mitre correctly. Both mouldings are inclined at 20° to the horizontal. Select a cross-section for one of the mouldings from Fig. 486 and then determine the cross-section of the other.

16. Same as exercise 15 except that the angle between the walls is 120° instead of 90°.

17. A, C, and E (Fig. 488) are three pieces forming part of a timber frame, each piece being of rectangular cross-section. The projections a' and c are incomplete. Complete the projections indicated and draw an elevation on a ground line inclined at 45° to XY.

18. A piece of timber having plane faces is shown in plan and elevation in Fig. 489. Draw two other elevations, one on a ground line perpendicular to XY, and the other on a ground line parallel to the longer sides of the plan. Determine the true form of the cross-section of the piece.

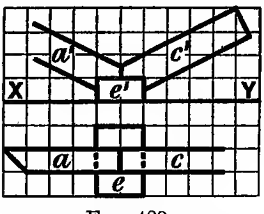


FIG. 488.

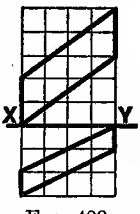


FIG. 489.

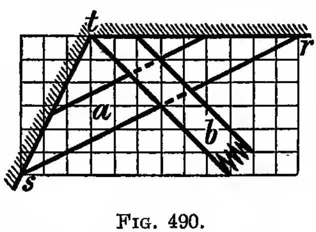


FIG. 490.

In reproducing the above diagrams take the small squares as of half-inch side.

19. Fig. 490 shows the plan of a piece of timber A, of uniform square cross-section, which lies between the faces RT and ST of two vertical walls. The longitudinal edges of A are inclined at 20° to the horizontal, rising from R to S. B is another piece of timber of uniform square cross-section. The longitudinal edges of B are horizontal. The piece B fits into a notch on A, the greatest vertical depth of this notch being equal to half the thickness of A. Draw two elevations of A and B, one on a ground line parallel to rs, and the other on a ground line parallel to rt. Determine all the bevels for the ends and the notch of the piece A.

CHAPTER XVIII

THE SPHERE, CYLINDER, AND CONE

212. Surfaces and Solids.—When a surface encloses a space and that space is filled with solid matter the whole is called a solid, and when the surfaces and solids have certain definite shapes they have certain definite names. But in speaking of certain solids and their surfaces the same name is frequently used to denote the solid and its surface. For example, in speaking of the sphere, the cylinder, and the cone the solids bearing these names may be referred to or it may be that it is their surfaces that are referred to, although the words "solid" or "surface" may not be used. In general the omission of the words "surface of" when a solid is mentioned by name will not lead to any confusion when it is the surface that is referred to. For example, by "the curve of intersection of two cylinders" is obviously meant "the curve of intersection of the *surfaces* of two cylinders."

213. The Sphere.—The *sphere* may be generated by the revolution of a semicircle about its diameter which remains fixed. The surface of a sphere is also the locus of a point in space which is at a constant distance from a fixed point called the *centre of the sphere.* The constant distance of the surface of the sphere from its centre is the *radius of the sphere.* A straight line through the centre of the sphere and terminated by the surface is a *diameter* of the sphere.

The orthographic projection of a sphere is always a circle of the same diameter as the sphere.

214. Plane Sections of a Sphere.—All plane sections of a sphere are circles. When the plane of section contains the centre of the sphere the section is a *great circle* of the sphere. All sections of the sphere by planes which do not contain its centre are called *small circles.*

A portion of a sphere lying between two parallel planes is called a *zone,* and a portion lying between two planes containing the same diameter is called a *lune.* See Fig. 491.

The determination of a plane section of a sphere is illustrated by Figs. 492 and 493. In Fig. 492 the plane of section is perpendicular to the vertical plane of projection and has V.T. for its vertical trace. In Fig. 493 the plane of section is inclined to both planes of projection and has V.T. for its vertical trace and H.T. for its horizontal trace. In both cases points on the projections of the section may be found

as follows. PQ parallel to XY is taken as the vertical trace of a horizontal plane. This plane cuts the sphere in a circle whose plan is a circle concentric with the plan of the sphere and whose diameter is equal to the portion of PQ within the circle which is the elevation of the sphere. This same plane cuts the given plane of section in a straight line. The points *rr* in which the plan of this line intersects the plan of the circle are the plans of points on the required intersection, and *r'r'* the elevations of these points are on PQ. In like manner by taking other positions for PQ any number of points on the projections of the required intersection may be found.

Since the required intersection is a circle and the projections of a circle are ellipses the axes of these ellipses may be found and the ellipses then be drawn by the trammel method (Art. 45, p. 41).

Referring to Fig. 492 *a'a'* the intercept of V.T. on the elevation of

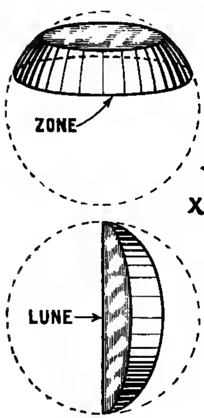


FIG. 491.

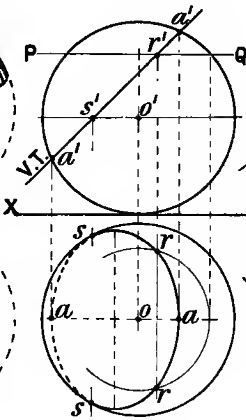


FIG. 492.

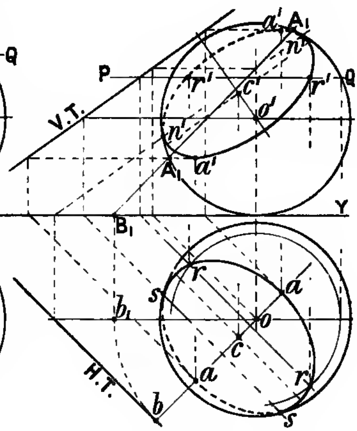


FIG. 493.

the sphere is the elevation of the circle which is the section of the sphere by the given plane. Projectors from *a'* and *a'* to meet a parallel to XY through *o* the plan of the centre of the sphere determines *aa* the minor axis of the ellipse which is the plan of the required section. The major axis bisects the minor axis at right angles and has a length equal to *a'a'*.

Referring to Fig. 493, draw *aab* through *o* at right angles to H.T. *aab* is the horizontal trace of a vertical plane which intersects the sphere in a great circle and the given plane of section in a straight line, and AA the portion of this line within the great circle has for its plan the minor axis of the ellipse which is the plan of the required section. Let this vertical plane with the line AAB and the great circle in it be turned round about a vertical axis through O until it is parallel to the vertical plane of projection. $A_1A_1B_1$ is the elevation of the line AAB in its new position and the elevation of the sphere is the

elevation of the new position of the great circle. This determines the levels of the points A and A on the given plane of section and the plans a and a are then readily found, aa is the minor axis of the ellipse, the major axis of which is equal to A_1A_1. By a similar construction, using a plane perpendicular to the vertical plane of projection with its vertical trace through o' at right angles to V.T., the axes of the ellipse in the elevation may be found. It should be remembered that the major axes of these ellipses are parallel to the horizontal and vertical traces respectively of the given plane of section and the minor axes are perpendicular to these traces.

By taking a horizontal section of the sphere and given plane of section through the centre of the sphere the points s and s where the ellipse in the plan touches the circle which is the plan of the sphere are determined (Figs. 492 and 493). These points determine the limits of the portions of the required section which are on the upper and lower halves of the sphere respectively. In like manner (Fig. 493) a section of the sphere and given plane by a plane parallel to the vertical plane of projection and through the centre of the sphere determines n' and n' where the ellipse in the elevation touches the circle which is the elevation of the sphere. And these points determine the limits of the portions of the required section which are on the front and back halves of the sphere respectively.

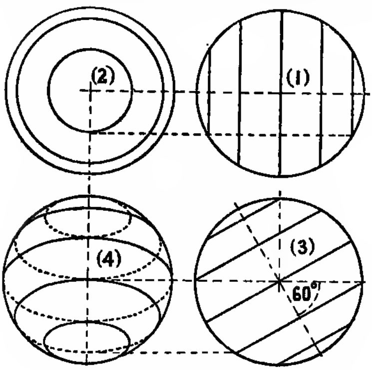


FIG. 494.

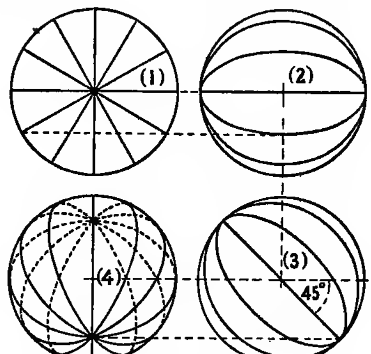


FIG. 495.

Useful exercises on the projection of plane sections of the sphere are illustrated by Figs. 494 and 495. In Fig. 494 the sections, except the middle one, are small circles of the sphere, while in Fig. 495 they are all great circles. The different views in each Fig. should be drawn in the order in which they are numbered. The sphere may be taken, say, 2·5 inches in diameter.

Another useful exercise is illustrated by Fig. 496. This is a hexagonal nut with a spherical chamfer. The various curves, other than those representing the hole in the nut, are plane sections of a sphere whose centre is O and radius R. In this example the various

views are arranged according to the American system (p. 169). In the elevations (2) and (3) the elliptic arcs a' and b' are usually drawn as arcs of circles. The arc c' in the elevation (3) is a true arc of a circle whose centre is o'. In working this example the following dimensions may be taken.—Diameter of screw, 1·75 inches; width of nut across the flats, 2·75 inches; height of nut, 1·75 inches; radius of spherical chamfer, 2·1 inches.

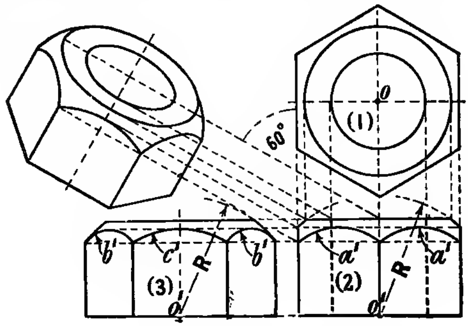


FIG. 496.

215. Projections of Points on the Surface of a Sphere.—Suppose that one projection of a point on the surface of a sphere to be given and that it is required to find the other. Take a plane parallel to one of the planes of projection and containing the point. The section of the sphere by this plane will have for one projection a circle and for the other a straight line, and the given projection of a point will lie on one of these and the other on the other.

216. Intersection of a Straight Line and a Sphere.—Take a plane perpendicular to one of the planes of projection and containing the given line. Draw a projection (A) of the section of the sphere by this plane on a plane (B) parallel to it. Draw also on the plane (B) another projection (C) of the given line. (A) is a circle which intersects (C) at points which are projections of the points of intersection of the line and sphere. From these the projections of the points of intersection on the original planes of projection may be found.

217. The Cylinder.—The *cylinder* may be generated by a straight line which moves in contact with a fixed curve and remains parallel to a fixed straight line. If the fixed curve is a circle whose plane is at right angles to the fixed straight line, the cylinder is a *right circular cylinder*, and a straight line through the centre of the fixed circle at right angles to its plane is the *axis of the cylinder*. The fixed circle referred to above is a *normal section* of the right circular cylinder, and its diameter is the *diameter of the cylinder*.

When a cylinder is limited in length by two planes which intersect all positions of the generating line, the sections of the cylinder by these planes become the *ends of the cylinder*, and when these ends are at right angles to the generating line the cylinder is a *right cylinder*. In speaking of a right circular cylinder, when its ends are considered, the ends are generally understood to be at right angles to the axis. But a right circular cylinder may have its ends inclined to its axis at an angle which is not a right angle.

218. Projections of a Right Circular Cylinder.—Starting with the cylinder in the position in which its axis is vertical, its plan is a circle and its elevation a rectangle as shown at (1) and (2) in Fig. 497.

Taking a second ground line X_1Y_1 inclined to $r's'$ the first elevation of the axis of the cylinder, and projecting from (2), a new plan (3) is obtained. In this new plan (3), rs, the plan of the axis of the cylinder, is parallel to X_1Y_1 and at a distance from it equal to the distance of rs in (1) from XY. Also the plans of the ends of the cylinder are equal ellipses whose major axes are at right angles to rs and equal in length to the diameter of the cylinder. The minor axes bisect the major axes at right angles and their lengths are found by projectors from (2) as shown.

Taking a third ground line X_2Y_2 inclined to rs in the plan (3), $r's'$, a new elevation of the axis of the cylinder is first obtained, the distances of r' and s' in (4) from X_2Y_2 being equal to the distances of

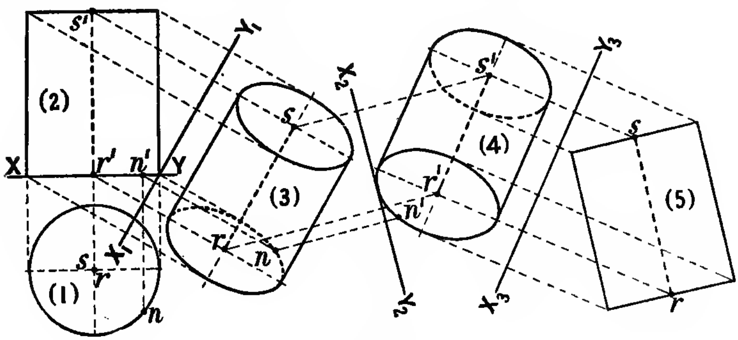


FIG. 497.

r' and s' respectively in (2) from X_1Y_1. Before going further with the elevation (4), take another ground line X_3Y_3 parallel to $r's'$ in (4) and draw rs a new plan of the axis of the cylinder as shown in (5). The distances of r and s in (5) from X_3Y_3 are equal to the distances of r and s respectively in (3) from X_2Y_2. The new plan (5) being on a plane parallel to the axis of the cylinder will be a rectangle as in (2) and may therefore be now drawn. The axes of the ellipses in (4) are next determined by projectors from (5) as shown, just as the axes of the ellipses in (3) were obtained by projectors from (2).

Instead of finding the axes of the various ellipses which are the projections of the ends of the cylinder and then constructing the ellipses in the usual way, a number of points on the ends may be projected as is shown for one point N.

If rs in (3) and $r's'$ in (4) be given and also the diameter of the cylinder, it is obvious that by drawing the projections (2) and (5) the projections (3) and (4) may be completed.

219. Plane Sections of a Right Circular Cylinder.—All

sections of a right circular cylinder by planes at right angles to its axis are circles of the same diameter, and all sections of the curved surface by planes parallel to the axis are parallel straight lines. All sections of the curved surface by planes inclined to the axis are ellipses whose centres are on the axis of the cylinder.

Referring to Fig. 498, PQ is the vertical trace of a horizontal plane cutting the inclined cylinder whose axis SS_1 is parallel to the vertical plane of projection. S and S_1 are the centres of two spheres inscribed in the cylinder and touching the plane of section at F and F_1 respectively. The points F and F_1 are the foci of the ellipse which is the section of the cylinder by the given plane of section. $a'a_1'$ the portion of PQ lying within the elevation of the cylinder is the elevation of the major axis of the ellipse and from this the plan aa_1 is projected. The

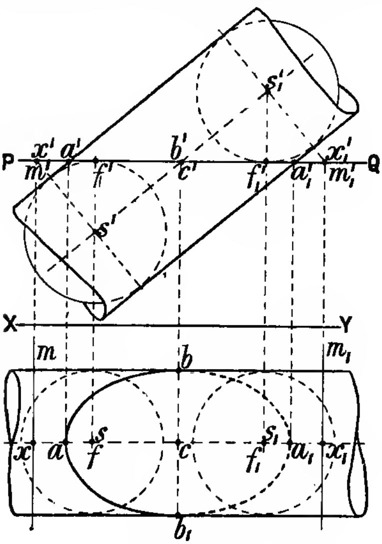


FIG. 498.

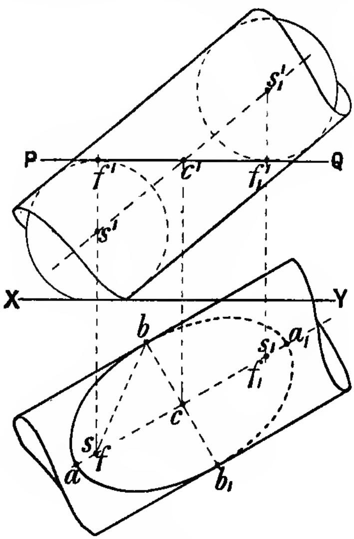


FIG. 499.

minor axis bb_1 of the plan of the ellipse is equal to the diameter of the cylinder.

Planes through S and S_1 at right angles to the axis of the cylinder intersect the given plane of section in straight lines XM and X_1M_1 which are the directrices of the ellipse (Art. 35, p. 32).

Fig. 499 differs from Fig. 498 in that the axis of the cylinder is inclined to both planes of projection instead of being inclined to the horizontal plane only. The intercept of PQ within the elevation of the cylinder does not now give the major axis of the ellipse, but the foci and the minor axis are determined as before, but by making ca and ca_1 each equal to bf, the major axis is determined.

The determination of the section of a cylinder, whose axis is

inclined to both planes of projection, by an oblique plane is illustrated by Fig. 500. AB is the axis of the cylinder and H.T. and V.T. are the traces of the plane of section.

The ellipse uv, which is the horizontal trace of the cylinder, is the section of the cylinder by the horizontal plane of projection and it may be determined by the construction shown in Fig. 499, or by the construction shown in Fig. 503.

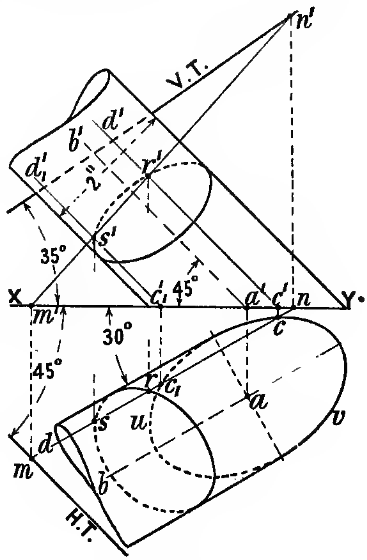


FIG. 500.

Taking mu, parallel to ab, as the horizontal trace of a vertical plane, this plane intersects the given plane of section in a line of which $m'n'$ is the elevation. This same vertical plane intersects the cylinder in two straight lines whose elevations $c'd'$ and $c_1'd_1'$ are parallel to $a'b'$, the points c' and c_1' being projected from c and c_1, the points where mn cuts the ellipse uv. The points r' and s' where $m'n'$ intersects $c'd'$ and $c_1'd_1'$ are the elevations of points on the required section. Projectors from r' and s' to meet mn determine points r and s on the plan of the required section.

By taking other vertical planes parallel to the axis of the cylinder other points on the required section may be found.

It should be noted that all the lines of intersection of the assumed vertical planes with the given plane of section will be parallel.

The student should work out this example to the dimensions marked on the figure.

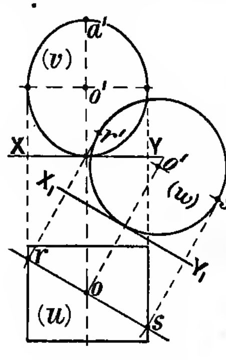


FIG. 501.

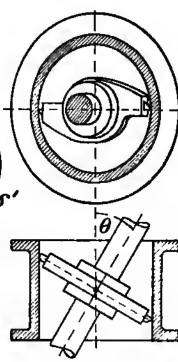


FIG. 502.

220. Circular Sections of a Right Elliptical Cylinder.—A *right elliptical cylinder* is one in which a section by a plane at right angles to the axis of the cylinder is an ellipse. In Fig. 501, (u) is the plan and (v) the elevation of such a cylinder, the axis of the cylinder being perpendicular to the vertical plane of projection, and the major axis of a section at right angles to the axis is vertical.

The elevation (v) shows the true form of the normal section and is an ellipse whose semi-major axis is $o'a'$.

Taking a point o on the plan of the axis as centre and a radius equal to $o'a'$ an arc of a circle is described cutting the outline of the plan which is parallel to the plan of the axis at r and s. The straight line rs is the horizontal trace of a vertical plane which will cut the cylinder in a circle whose diameter is equal to the major axis of the normal section. An elevation (w) of the circular section on a ground line X_1Y_1 parallel to rs is shown. All sections of the curved surface of the cylinder by planes parallel to one circular section are equal circles.

The property of a right elliptical cylinder of having circular sections makes it possible to bore such a cylinder with an ordinary boring bar as shown in Fig. 502. The axis of the boring bar is set at an angle θ to the axis of the cylinder to be bored, θ being such that $\cos\theta = \frac{d}{D}$, where d is equal to the minor axis and D is equal to the major axis of the normal elliptic section of the cylinder. While the boring bar is rotated the cylinder is moved in the direction of its axis, or if the cylinder is stationary the boring bar is moved bodily in the direction of the axis of the cylinder. It is evident that with the arrangement sketched in Fig. 502 the length of cylinder which may be bored is limited, the limit depending on the angle θ and the diameter of the boring bar.

221. Cylinder Enveloping a Sphere.—A cylinder which envelops a sphere will have its axis passing through the centre of the sphere and the curve of contact will be a "great circle" of the sphere whose plane is perpendicular to the axis of the cylinder.

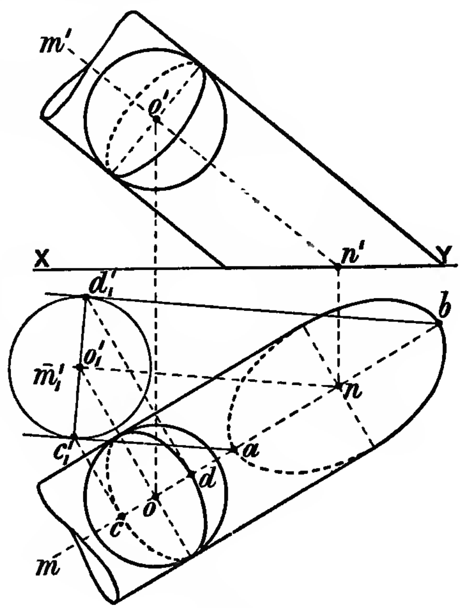


FIG. 503.

Referring to Fig. 503, oo' is the centre of the sphere and mn, $m'n'$ is the axis of the cylinder. The sphere and the direction of the axis of the cylinder are supposed to be given. In problems requiring a cylinder to envelop a sphere the cylinder is generally of indefinite length and the ends are not required to be shown but

usually the trace of the surface on one of the planes of projection is necessary. Neglecting the ends, the projections of the cylinder will consist of the tangents to the projections of the sphere parallel to the projections of the axis of the cylinder. Fig. 503 shows the construction for finding the horizontal trace of the cylinder and the plan of the circle of contact. An elevation of the cylinder and sphere is drawn on mn as a ground line. $m_1'n$ is the new elevation of the axis of the cylinder and o_1' that of the centre of the sphere. The points a and b, where the tangents to the circle which is the new elevation of the sphere, parallel to $m_1'n$, meet mn, are the extremities of the major axis of the ellipse which is the horizontal trace of the cylinder. The minor axis of this ellipse passes through n, the horizontal trace of the axis of the cylinder, and is equal to the diameter of the cylinder or sphere.

The line $c_1'd_1'$, passing through o_1', and perpendicular to $m_1'n$ is the new elevation of the circle of contact. Perpendiculars from c_1' and d_1' to mn determine cd, the minor axis of the ellipse which is the plan of the circle of contact. The major axis of this ellipse is a diameter of the circle which is the plan of the sphere and is at right angles to mn.

The elevation of the circle of contact is obtained by making an auxiliary plan of the cylinder and sphere on a plane parallel to the axis of the cylinder, say on $m'n'$ as a ground line. The construction is similar to that already described for the plan. The vertical trace of the cylinder may also be obtained from this same auxiliary plan.

222. Projections of Points on the Surface of a Right Circular Cylinder.—Suppose that one projection, say the plan r of a point R, on the surface of a right circular cylinder to be given and that it is required to find the other. Take an elevation of the cylinder on a vertical plane parallel to its axis. Take any point oo' (Fig. 504) on the axis of the cylinder as the centre of a sphere inscribed in the cylinder. Through r draw a straight line rs parallel to the plan of the axis of the cylinder and assume this to be the horizontal trace of a vertical plane which is parallel to the axis of the cylinder. This vertical plane cuts the sphere in a circle whose diameter is equal to ac the intercept of rs on the plan of the sphere. The elevation of this circle is a circle with its centre at o'. This same vertical plane cuts the curved surface of the cylinder in two straight lines whose elevations are tangents to the circle just mentioned and are parallel to the elevation

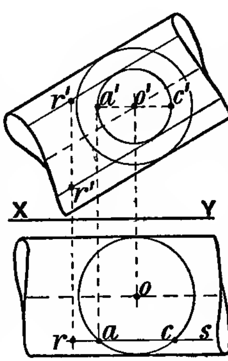


FIG. 504.

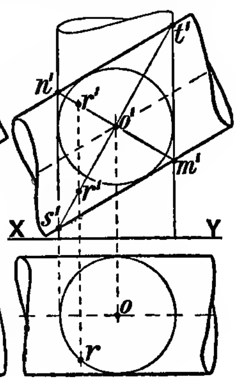


FIG. 505.

of the axis of the cylinder. There are two points on the surface of the cylinder having the point r for their plan. The elevations of these points are on the projectors from r and on the lines which are the elevations of the lines of intersection of the assumed vertical plane and the cylinder.

Another solution of the problem is shown in Fig. 505. A sphere is inscribed in the cylinder as before and a second cylinder, with its axis vertical, is made to envelop this sphere. The two cylinders intersect in two ellipses whose elevations on the vertical plane parallel to the axes of the cylinders are the straight lines $m'n'$ and $s't'$. The points whose plan is r are evidently on the intersection of the two cylinders and their elevations are found as shown.

223. Intersection of a Straight Line and a Right Circular Cylinder.—First obtain from the given projections of the line and cylinder projections of them on planes parallel and perpendicular respectively to the axis of the cylinder. Next take a plane parallel to the axis of the cylinder and containing the given line. This plane will intersect the curved surface of the cylinder in two straight lines which will intersect the given line at the points of intersection required. Working backwards to the original projections the required projections of the points of intersection of the line and cylinder are found.

224. The Cone.—The *cone* may be generated by a straight line which moves in contact with a fixed curve and passes through a fixed point. The fixed point is the *vertex of the cone.* If the fixed curve is a circle and the fixed point is on the straight line which passes through the centre of the circle and is at right angles to the plane of the circle, the cone is a *right circular cone.* The fixed circle is a circular section of the right circular cone. The straight line joining the vertex of a right circular cone to the centre of a circular section is the *axis of the cone.* The right circular cone may also be defined as the surface described by a straight line which intersects a fixed line at a fixed point and moves so that its inclination to the fixed line is constant. The fixed line in this case is the axis of the cone.

If a right circular cone terminates at one end at the vertex and at the other end at a circular section, that circular section is called the *base of the cone.*

225. Projections of a Right Circular Cone.—Starting with the cone in the position in which its axis is vertical, its plan is a circle and its elevation is an isosceles triangle as shown at (1) and (2) in Fig. 506.

Taking a second ground line X_1Y_1 inclined to $v'r'$ the first elevation of the axis of the cone, and projecting from (2), a new plan (3) is obtained. In this new plan (3), vr the plan of the axis of the cone, is parallel to X_1Y_1 and at a distance from it equal to the distance of vr in (1) from XY. Also the plan of the base of the cone is an ellipse whose major axis is at right angles to vr and equal in length to the diameter of the base of the cone. The minor axis bisects the major axis at right angles and its length is found by projectors from (2) as shown.

Taking a third ground line X_2Y_2 inclined to vr in the plan (3), $v'r'$, a new elevation of the axis of the cone is first obtained, the distances of v' and r' in (4) from X_2Y_2 being equal to the distances of v' and r' respectively in (2) from X_1Y_1. Before going further with the elevation (4), take another ground line X_3Y_3 parallel to $v'r'$ in (4) and draw vr a new plan of the axis of the cone as shown in (5). The distances of v and r in (5) from X_3Y_3 are equal to the distances of v and r respectively in (3) from X_2Y_2. The new plan (5) being on a plane parallel to the axis of the cone will be an isosceles triangle as

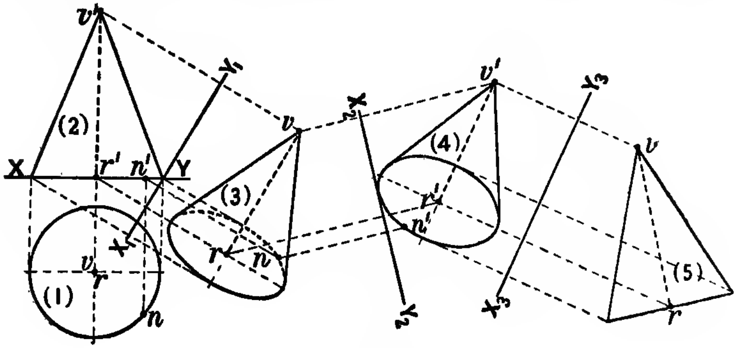


FIG. 506.

in (2) and may therefore now be drawn. The axes of the ellipse in (4) are next determined by projectors from (5) as shown, just as the axes of the ellipse in (3) were obtained by projectors from (2).

Instead of finding the axes of the various ellipses which are the projections of the base of the cone and then constructing the ellipses in the usual way, a number of points on the base may be projected as is shown for one point N.

If vr in (3) and $v'r'$ in (4) be given and also the diameter of the base of the cone it is obvious that by drawing the projections (2) and (5) the projections (3) and (4) may be completed.

226. Plane Sections of a Right Circular Cone.—In what follows only the curved surface of the cone is considered, the base being supposed to be beyond the limits of the section or the part of the section represented. When the term "cone" is used "right circular cone" is to be understood.

In studying the sections of the cone it is necessary to consider that the straight line which generates the surface is of unlimited length and consequently that the vertex of the cone is not at one end of the generating line. The cone then consists of two *sheets* generated by the parts of the generating line which are on opposite sides of the vertex. The two sheets of a cone are exactly alike and the axis of one is the axis of the other produced. The two sheets of a cone are shown in Fig. 510.

The following are the various sections of the cone. (1) Two

straight lines when the plane of section passes through the vertex of the cone. (2) The *circle* when the plane of section is perpendicular to the axis of the cone. (3) The *ellipse* when the plane of section cuts all the generating lines on the same side of the vertex. (4) The *hyperbola* when the plane of section *cuts both sheets of the cone* and does not pass through the vertex. (5) The *parabola* when the plane of section is parallel to a tangent plane to the cone, or, to give a common but less exact definition, when the plane of section is parallel to the *slant side* of the cone.

The circle (2) is a particular case of the ellipse (3), and two straight lines (1) are a particular case of the hyperbola (4). As the plane of

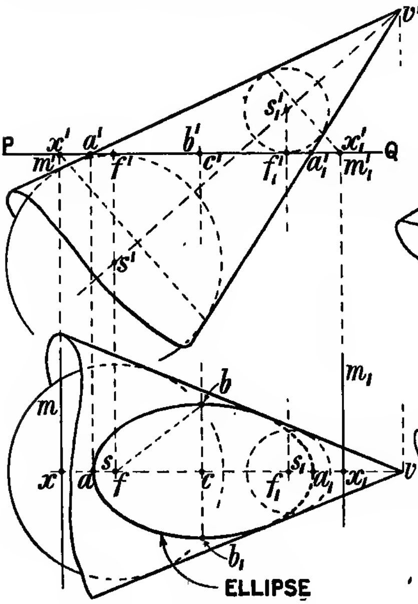


Fig. 507.

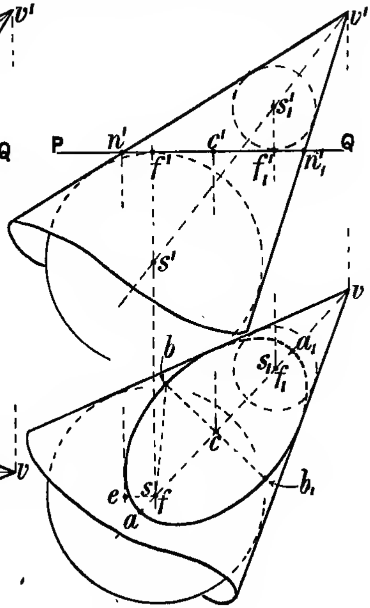


Fig. 508.

section turns round from a position which gives an ellipse to a position which gives an hyperbola it passes through the position which gives a parabola. Sections (1) and (4) are those which lie on both sheets of the cone.

The sections which will now be considered are, the ellipse, the parabola, and the hyperbola.

Taking the elliptic section first and referring to Fig. 507, a cone, whose axis VS is inclined to the horizontal plane but is parallel to the vertical plane of projection, is cut by a horizontal plane of which PQ is the vertical trace. S and S_1 are the centres of two spheres inscribed in the cone and touching the plane of section at F and F_1

respectively. The points F and F_1 are the foci of the ellipse which is the section of the cone by the given plane of section. $a'a_1'$, the portion of PQ lying within the elevation of the cone, is the elevation of the major axis of the ellipse and from this the plan aa_1 is projected. The minor axis bb_1 of the plan of the ellipse bisects aa_1 at right angles at c. The length of the minor axis may be determined by taking f as centre and a radius equal to ac and describing arcs to cut bb_1 at b and b_1.

Planes containing the circles of contact between the cone and the inscribed spheres intersect the given plane of section in straight lines XM and X_1M_1 which are the directrices of the ellipse (Art. 35, p. 32).

Fig. 508 differs from Fig. 507 in that the axis of the cone is inclined to both planes of projection instead of being inclined to the horizontal plane only. $n'n_1'$ the intercept of PQ within the elevation of the cone does not now give the major axis of the ellipse, but the foci are determined exactly as before. A projector from n' to the plan will however be a tangent to the plan of the ellipse. From f draw fe at right angles to this tangent, meeting it at e. With centre c, the middle point of ff_1, and with a radius equal to ce describe arcs to cut ff_1 produced at a and a_1. Then aa_1 is the major axis of the plan of the ellipse. This follows from the fact that the foot of a perpendicular from a focus of an ellipse on to a tangent lies on the auxiliary circle (Art. 42, p. 39). The minor axis bb_1 may now be determined as before.

Consider next a parabolic section, which is illustrated by Fig. 509. The cone is placed so that it is tangential to a horizontal plane and the axis is parallel to the vertical plane of projection. PQ is the vertical trace of a horizontal plane of section. S is the centre of the sphere which is inscribed in the cone and which touches the plane of section. Only one such sphere can be drawn in this case. The point F at which the inscribed sphere touches the plane of section is the focus of the parabola and A is the vertex.

The plane of the circle of contact between the cone and the inscribed sphere intersects the plane of section in the straight line XM which is the directrix of the parabola. It will be found that ax is equal to af. The parabola may now be constructed as described in Art. 34, p. 30, or as follows. Take a point o' on the elevation of the axis of the cone. Draw $o'd'$ at right angles to $v'o'$ to meet the outline of the elevation of the cone at d' and PQ at r'. With o' as centre and $o'd'$ as radius describe an arc to meet a straight line $r'R_1$ parallel to $v'o'$ at R_1. Through r' draw a projector to the plan cutting the plan of the axis of the cone at t. On this projector make tr, above and below t, equal to $r'R_1$, then r and r are two points on the plan of the parabola. The theory of this construction is that $r'o'd'$ is the part elevation of a circular section of the cone and the arc $d'R_1$ is a rabatment of part of this circle into the plane of the elevation. It will be seen that $r'R_1$ is the half width of the cone measured at right angles to the vertical plane of projection through the point R. In like manner any number of points on the parabola may be found. This

construction may also be used to find points on the ellipse of an elliptic section when the cone is placed as in Fig. 507.

An hyperbolic section is illustrated by Fig. 510. The axis of the cone is parallel to the vertical plane of projection but is inclined to the horizontal plane. A horizontal plane whose vertical trace is PQ cuts both sheets of the cone. The foci, the transverse axis, and the directrices of the hyperbola are determined as in the case of the elliptic section Fig. 507.

The hyperbola may be constructed from its foci and transverse axis as described in Art. 43, p. 40, or points such as *r* and *r* may be

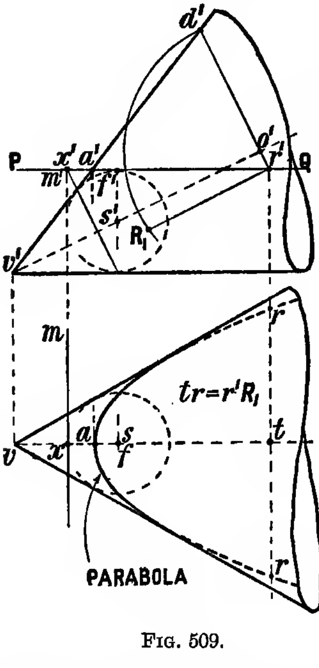


FIG. 509.

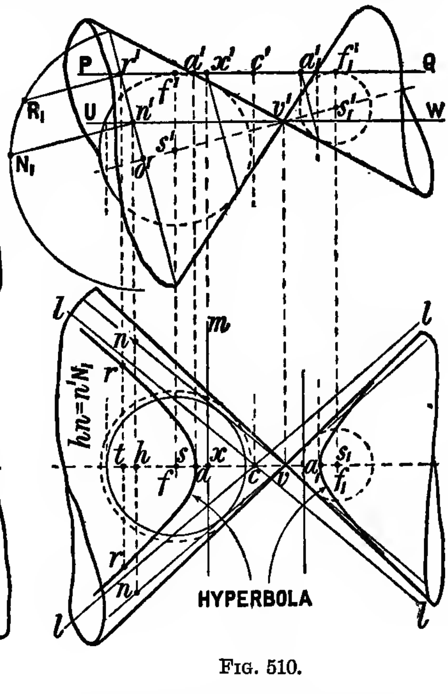


FIG. 510.

found by the construction shown and already explained in connection with the parabolic section.

The asymptotes pass through C the middle point of the transverse axis and are parallel to the straight lines which form the section of the cone by the plane UW through the vertex and parallel to the given plane of section PQ. *vn* and *vn* are the plans of the lines which form the section of the cone by the plane UW, the points *n* and *n* being determined in a manner similar to that for the points *r* and *r* as shown. The lines *vn* and *vn* may however be found by drawing them as tangents through *v* to the circle which is the plan of the section of one of the inscribed spheres by the plane UW as shown.

The asymptotes *lcl* and *lcl* may now be drawn through *c* parallel to *vn* and *vn*.

227. Cone Enveloping a Sphere.—A cone which envelops a sphere will have its axis passing through the centre of the sphere and the curve of contact will be a "small circle" of the sphere whose plane is perpendicular to the axis of the cone.

Referring to Fig. 511, oo' is the centre of the sphere and vv' is the vertex of the cone. The sphere and the vertex of the cone are supposed to be given. The constructions for finding the horizontal trace of the cone and the plan of the circle of contact are shown. An auxiliary

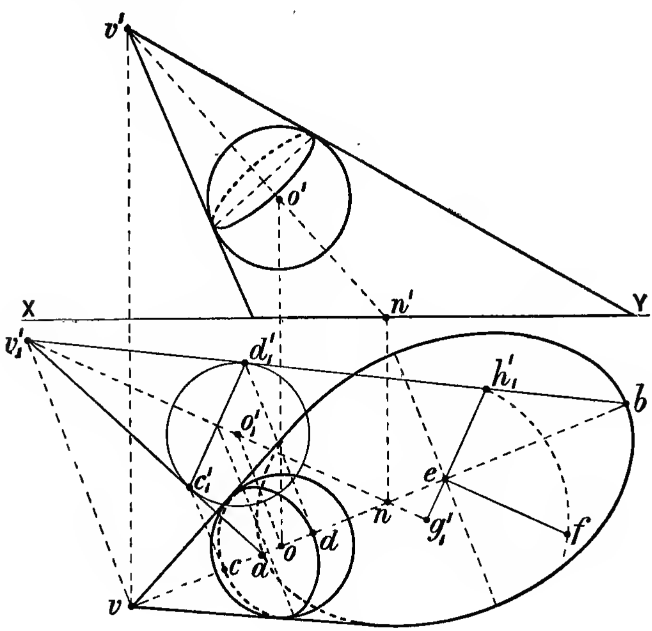


FIG. 511.

elevation of the cone and sphere is drawn on the plan *vn* of the axis of the cone as a ground line. $v_1'n$ is the new elevation of the axis of the cone and o_1' that of the centre of the sphere. The points *a* and *b* where the tangents from v_1' to the circle which is the new elevation of the sphere meet *vn* are the extremities of the major axis of the ellipse which is the horizontal trace of the cone. The minor axis of this ellipse must of course bisect *ab* at right angles, but it should be noticed that the middle point of *ab* is not at *n* the horizontal trace of the axis of the cone. To determine the length of the minor axis of the ellipse which is the horizontal trace of the cone the construction is as follows. Through *e* the middle point of *ab* draw $g_1'eh_1'$ at right angles to $v_1'n$ meeting $v_1'n$ at g_1' and

$v_1'b$ at h_1'. With centre g_1' and radius $g_1'h_1'$ describe the arc $h_1'f$ and draw ef at right angles to eh_1' to meet this arc at f. ef is equal in length to the semi-minor axis. The theory of this construction is that a section of the cone perpendicular to its axis has been taken whose trace on the plane of the auxiliary elevation is $g_1'h_1'$. The true form of this section is a circle a portion of which is shown turned round into the plane of projection. The chord of this circle drawn through e perpendicular to eh_1' gives the greatest width of the cone at the level of the horizontal plane.

The line $c_1'd_1'$ joining the points of contact of the tangents from v_1' to the circle which is the auxiliary elevation of the sphere is the auxiliary elevation of the circle of contact. Perpendiculars from c_1' and d_1' to vn determine c and d the extremities of the minor axis of the ellipse which is the plan of the circle of contact. The major axis of this ellipse is equal to the diameter of the circle of contact and is therefore equal to $c_1'd_1'$.

The axes of the ellipse which is the elevation of the circle of contact are found by making an auxiliary plan of the cone and sphere on a plane parallel to the axis of the cone, say on $v'n'$ as a ground line. The construction is similar to that already described for the plan. The vertical trace of the cone may also be found from this same auxiliary plan.

228. Projections of Points on the Surface of a Right Circular Cone.—Suppose that one projection, say the plan r, of a point R on the surface of a given right circular cone to be given and that it is required to find the other. Take an elevation of the axis of the cone on a ground line parallel to vr (Fig. 512). Take a point oo' on

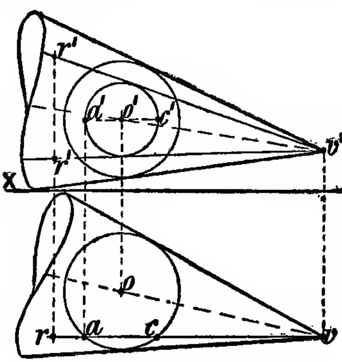


FIG. 512.

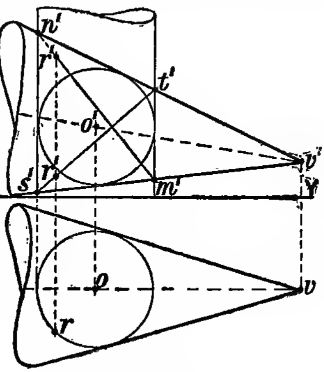


FIG. 513.

the axis of the cone as the centre of a sphere inscribed in the cone. Assume vr to be the horizontal trace of a vertical plane. This vertical plane cuts the sphere in a circle whose diameter is equal to ac the intercept of vr on the plan of the sphere. The elevation of this circle is a circle with its centre at o'. This same vertical plane cuts the

curved surface of the cone in two straight lines whose elevations pass through v' and are tangents to the circle just mentioned. There are two points on the surface of the cone having the point r for their plan. The elevations of these points are on the projector from r and on the elevations of the lines of intersection of the assumed vertical plane and the cone.

Another solution of the problem is shown in Fig. 513. A sphere is inscribed in the cone so that the plan of the sphere passes through r, and a cylinder with its axis vertical is made to envelop this sphere. The cone and cylinder intersect in two ellipses whose elevations on the vertical plane parallel to the axes of the cone and cylinder are the straight lines $m'n'$ and $s't'$. The points whose plan is r are evidently on the intersection of the cone and cylinder and their elevations are therefore found as shown.

229. Intersection of a Straight Line and a Right Circular Cone.—First obtain from the given projections of the line and cone projections of them on planes parallel and perpendicular respectively to the axis of the cone. Next take a plane to contain the vertex of the cone and the given line. This plane will intersect the curved surface of the cone in two straight lines which will intersect the given line at the points of intersection required. Working backwards to the original projections the required projections of the points of intersection of the line and cone are found.

230. The Oblique Cylinder.—The *oblique cylinder* may be generated by a straight line which moves in contact with a fixed circle and remains parallel to a fixed straight line which is not perpendicular to the plane of the circle. The line parallel to the fixed line and passing through the centre of the fixed circle is the *axis of the cylinder.* A section of the oblique cylinder by a plane perpendicular to the plane of the fixed circle and containing the axis of the cylinder is called the *principal section.*

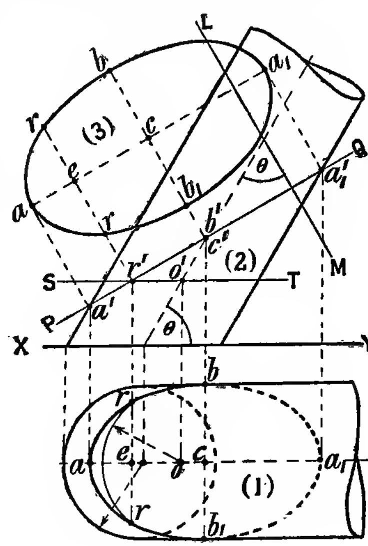


FIG. 514.

An oblique cylinder is shown in Fig. 514 by a plan (1) and elevation (2). The fixed circle is on the horizontal plane of projection and may be called the base of the cylinder. The axis of the cylinder is parallel to the vertical plane of projection. The principal section is therefore also parallel to the vertical plane of projection.

PQ is the vertical trace of a plane of section which is perpendicular

to the vertical plane of projection. The section by this plane PQ is an ellipse whose major axis is equal to $a'a_1'$ and whose minor axis is equal to the diameter of the base circle. Two plans of this section, which are ellipses, are shown, the lower one (1) on XY as a ground line, and the upper one (3) on PQ as a ground line. The latter plan shows the true form of the section.

To find points on the lower plan (1) of the section, take a horizontal plane whose vertical trace is ST and which cuts the axis of the cylinder at oo'. The section of the cylinder by this plane is a circle equal to the base circle. The plan of this circle is a circle whose centre is o. Draw this circle, and from r', the point of intersection of PQ and ST, draw the projector $r'er$ cutting the circle at r and r which are points on the plan of the section. In like manner other points may be found. The plan of the section may however be constructed from its axes aa_1 and bb_1 by the trammel method.

Referring next to the upper plan (3) which shows the true form of the section, aa_1, the major axis, is parallel and equal to $a'a_1'$. The semi-minor axis cb is equal to the radius of the base circle and er in (3) is equal to er in (1).

A section by a plane LM which is perpendicular to the principal section and makes with the axis an angle θ equal to the angle between the axis and the base is called a *sub-contrary section* and is a circle equal to the base circle.

Planes of section parallel to the base or to a sub-contrary section are called *cyclic planes* because the sections by these planes are circles.

Since a section of an oblique cylinder by a plane at right angles to its axis is an ellipse the oblique cylinder is evidently also a *right elliptical cylinder* (Art. 220).

231. The Oblique Cone.—The *oblique cone* may be generated by a straight line which moves in contact with a fixed circle and passes through a fixed point which is not on the straight line through the centre of the circle at right angles to its plane. The fixed point is the *vertex of the cone* and the straight line joining the vertex to the centre of the fixed circle is the *axis of the cone.* A section of the oblique cone by a plane perpendicular to the plane of the fixed circle and containing the axis of the cone is called the *principal section.*

An oblique cone is shown in Fig. 515 by a plan (1) and elevation (2). The fixed circle is on the horizontal plane of projection and may be called the base of the cone. The axis of the cone is parallel to the vertical plane of projection. The principal section is therefore also parallel to the vertical plane of projection.

PQ is the vertical trace of a plane of section which is perpendicular to the vertical plane of projection. The section by this plane PQ is an ellipse whose major axis is equal to $a'a_1'$. Two plans of this section, which are ellipses, are shown, the lower one (1) on XY as a ground line, and the upper ono (3) on PQ as a ground line. The latter plan shows the true form of the section.

To find points on the lower plan (1) of the section, take a horizontal

plane whose vertical trace is ST and which cuts the axis of the cone at oo'. The section of the cone by this plane is a circle whose diameter is the intercept of ST on the elevation of the cone. The plan of this circle is a circle whose centre is o. Draw this circle and from r', the point of intersection of PQ and ST, draw the projector $r'er$ cutting the circle at r and r, which are points on the plan of the section. In like manner other points may be found. The above construction when applied to the point c' which is the middle point of $a'a_1'$ gives the minor axis bb_1 of the ellipse which is the plan of the section.

Referring next to the plan (3) which shows the true form of the section, aa_1, the major axis, is parallel and equal to $a'a_1'$. The semi-minor axis cb in (3) is equal to cb in (1), also er in (3) is equal to er in (1).

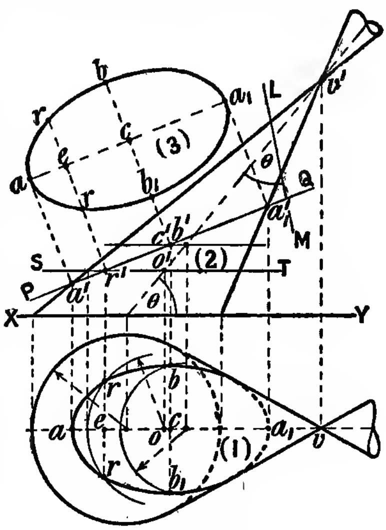


FIG. 515.

A section by a plane LM which is perpendicular to the principal section and makes with the axis an angle θ equal to the angle between the axis and the base is called a *sub-contrary section* and is a circle.

Planes of section parallel to the base or to a sub-contrary section are called *cyclic planes* because sections by these planes are circles.

As in the case of a right circular cone, a section of an oblique cone by a plane which is parallel to a tangent plane to the cone is a parabola. Also a section of an oblique cone by a plane which cuts both sheets of the cone is an hyperbola.

Exercises XVIII

1. The circle (Fig. 516) is the plan of a sphere resting on the horizontal plane. H_1T_1 and H_2T_2, parallel to XY, are the horizontal traces of two parallel planes. The plane of which H_1T_1 is the horizontal trace contains the centre of the sphere. Draw the plan and elevation of the zone of the sphere which lies between these two planes.

2. The parallel lines H_1T_1 and H_2T_2 (Fig. 517) are the horizontal traces of two planes which pass through the centre of a sphere resting on the horizontal plane, the given circle being the plan of the sphere. Draw the plan and elevation of the lower lune of the sphere lying between the given planes.

3. The circle (Fig. 518) is the plan of a sphere resting on the horizontal plane. H_1T_1 inclined at 45° to XY and H_2T_2 perpendicular to XY are the horizontal traces of two planes which contain the centre of the sphere. Draw the plan and elevation of the lower portion of the sphere which lies between these two planes.

4. The given circle (Fig. 519) is the plan of a sphere, centre C. The points a, a, b, are the plans of points A, A, B on its upper surface. N and S are the upper and lower ends of a vertical diameter.

Draw the plan of a figure on the surface made up of three arcs of great circles joining NA, NA, AA, and the arc of a small circle ABA.

Draw the stereographic projection of this figure on the horizontal plane

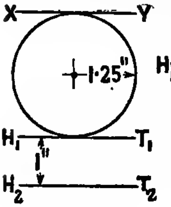


FIG. 516.

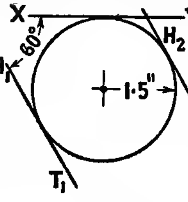


FIG. 517.

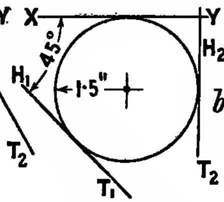


FIG. 518.

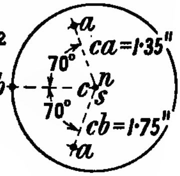


FIG. 519.

through C. That is, find the plane section of a cone, vertex S, of which the figure is a spherical section. [B.E.]

5. The horizontal and vertical traces of a plane make angles of 30° and 45° respectively with XY. A hemisphere of 1·25 inches radius has its base on this plane and touching the planes of projection. Draw the plan and elevation of the hemisphere.

6. A square of 1·5 inches side and a circle of 1·25 inches radius, the centre of the circle being at the centre of the square, form the plan of a sphere with a square hole through it. From this plan project two elevations, one on a ground line XY parallel to a side of the square and the other on a ground line X_1Y_1 parallel to a diagonal of the square. Also from the second elevation project a plan on a ground line X_2Y_2 making an angle of 45° with X_1Y_1.

7. A cylinder 2 inches in diameter and 3 inches long has its axis horizontal and inclined at 45° to the vertical plane of projection. The cylinder is cut in halves by a vertical plane which is inclined at 60° to the axis of the cylinder and 15° to the vertical plane of projection. Draw the elevation of one of the halves of the cylinder.

8. *rs*, *r′s′* (Fig. 520) is the axis of a hollow cylinder, whose external diameter is 2·1 inches and whose internal diameter is 1·4 inches. Draw the plan and elevation.

9. A plan and an elevation of a hollow half cylinder, with one end closed, are

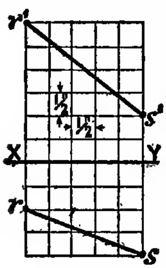


FIG. 520.

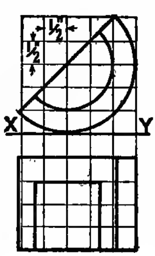


FIG. 521.

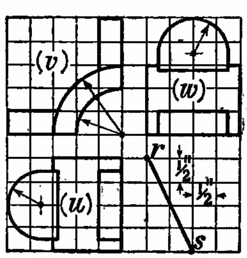


FIG. 522.

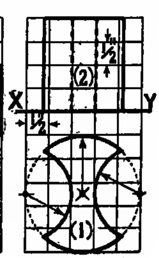


FIG. 523.

given in Fig. 521. Draw these projections and from the plan project an elevation on a ground line inclined at 60° to XY.

10. A plan (*u*) and elevations (*v*) and (*w*) of a solid are given in Fig. 522. Draw these projections and from the plan project an elevation on a ground line parallel to *rs*. Also from the elevation (*v*) project a plan on a ground line perpendicular to *rs*.

11. A plan (1) and an elevation (2) of a right circular cylinder with two longitudinal grooves in it are given in Fig. 523. Draw these projections and add others

corresponding to (3), (4), and (5) in Fig. 497, p. 253. The angle between X_1Y_1 and XY to be 60°, and the angle between X_2Y_2 and X_1Y_1 to be 45°.

12. A right circular cylinder of indefinite length has its axis vertical. A plane whose horizontal and vertical traces make angles of 30° and 45° with the ground line cuts the cylinder. Draw the elevation of the section and determine its true form.

13. A split wrought-iron collar is shown in Fig. 524. SS is a vertical section plane. Draw a sectional elevation on a ground line parallel to SS, the portion A in front of the section plane being supposed removed. [B.E.]

Note. Fig. 524 is to be reproduced double size.

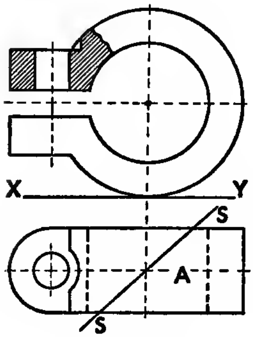


FIG. 524.

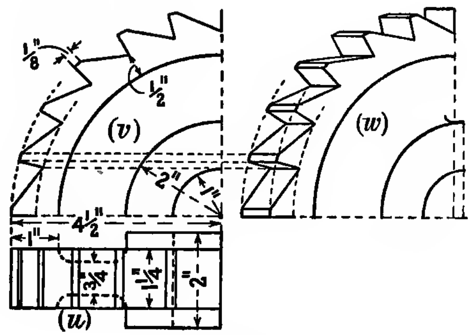


FIG. 525.

14. Referring to Fig. 525, (*u*) is a plan and (*v*) a side elevation of a quarter of a ratchet wheel having 24 teeth of the form shown, (*w*) is an elevation on a vertical plane inclined at 30° to the side of the wheel. Draw these projections for the whole wheel to the dimensions given.

15. A sphere 2·2 inches in diameter rests on the horizontal plane and touches the vertical plane of projection. The plan of the axis of a cylinder which envelops the sphere makes 45° with XY and the elevation of the axis makes 60° with XY. Draw the horizontal and vertical traces of the surface of the cylinder,

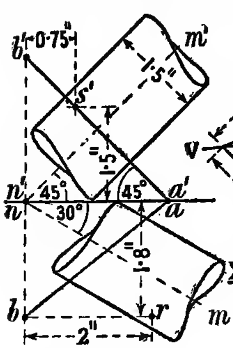


FIG. 526.

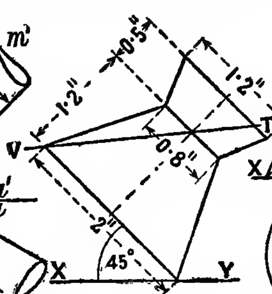


FIG. 527.

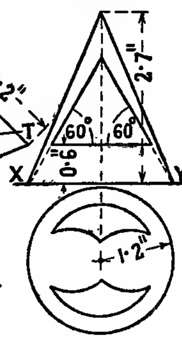


FIG. 528.

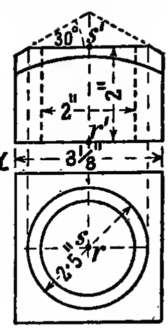


FIG. 529.

also the plan and elevation of the circle of contact between the sphere and cylinder.

16. *mn*, *m'n'* (Fig. 526) is the axis of a right circular cylinder 1·5 inches in

diameter, r is the plan of a point on the upper surface of the cylinder and s' is the elevation of a point on the front surface. Find r' and s. Find also the plans and elevations of the points of intersection of the line ab, $a'b'$ with the surface of the cylinder.

17. The elevation of a solid made up of two truncated right circular cones is given in Fig. 527. Draw the plan of the solid. Show also the plan of the section of the solid by a plane which is perpendicular to the vertical plane of projection and which has VT for its vertical trace.

18. A right circular cone, base 2·5 inches in diameter and altitude 2 inches, lies with its slant side on the ground. Draw the plan of the cone and show the plan of a straight line which lies on the surface of the cone and is inclined at 45° to the ground.

19. The plan and an elevation of a right circular cone, whose axis is vertical and which has a triangular hole through it, are given in Fig. 528. Draw these projections and add an elevation on a ground line inclined at 45° to XY.

20. A plan and an elevation of a square nut with a conical chamfer are given in Fig. 529. Draw these projections and add an elevation on a ground line parallel to one diagonal of the plan. Also, project from the given elevation a new plan on a ground line inclined at 60° to $r's'$.

21. An equilateral triangle $v'a'b'$, of 2 inches side, is the elevation of a right circular cone, $a'b'$ the elevation of the base being parallel to XY. A circle inscribed in the triangle is the elevation of a plane section of the cone. Draw the plan of the section.

22. A right circular cone, base 2 inches diameter, altitude 2·5 inches, lies with its slant side on the horizontal plane. The cone is divided by a vertical plane which passes through the centre of the base and whose horizontal trace is inclined at 30° to the plan of the axis. Draw the elevation of the larger part of the cone on a plane parallel to the plane of section.

23. vr, $v'r'$ (Fig. 530) is the axis of a right circular cone, vv' being the vertex and rr' the centre of the base. The cone rests with its slant side on the hori-

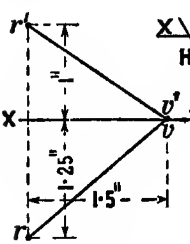


FIG. 530.

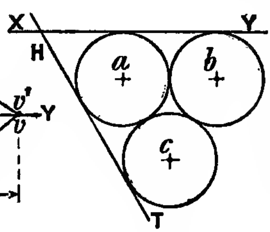


FIG. 531.

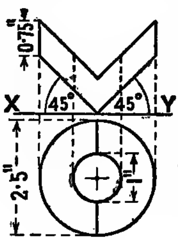


FIG. 532.

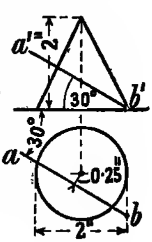


FIG. 533.

zontal plane. Draw the plan and elevation of the cone. Show the projections of a point on the surface of the cone which is 0·75 inch above the horizontal plane and 1·75 inches distant from the vertex of the cone.

24. The given circles a, b, and c (Fig. 531) are each 2 inches in diameter. These circles touch one another. a and b touch XY and the straight line HT is tangential to a and c. a is the plan of a cone, altitude 2·5 inches; b is the plan of a cylinder, length 4 inches, and c is the plan of a sphere. All these solids stand on the horizontal plane. HT is the horizontal trace of a plane which cuts the solids and is inclined at 45° to the horizontal plane. Show in plan and elevation the sections of the given solids by the given plane.

25. A solid cut from a hollow cylinder is given in Fig. 532 by a plan and an elevation. Draw an elevation on a ground line which is perpendicular to XY, also an elevation on a ground line which is inclined at 45° to XY.

26. Show the points of intersection of the given line ab, $a'b'$ (Fig. 533) with the surface of the given cone.

27. A sphere 1·5 inches in diameter rests on the horizontal plane and its centre is 1·5 inches in front of the vertical plane of projection. A point on the horizontal plane, 3·75 inches in front of the vertical plane of projection and 2·25 inches from the plan of the centre of the sphere is the vertex of a cone which envelops the sphere. Draw the vertical trace of the cone, also the plan and elevation of the circle of contact between the cone and sphere.

28. A circular section of an oblique cylinder is 2 inches in diameter and the axis of the cylinder is inclined at 40° to this section. Draw the true form of the section of the cylinder by a plane perpendicular to its axis.

29. An oblique cylinder is given in Fig. 534. Draw the vertical trace of the cylinder, that is, determine the section of the cylinder by the vertical plane of projection. Draw also the plan and elevation of a sub-contrary section.

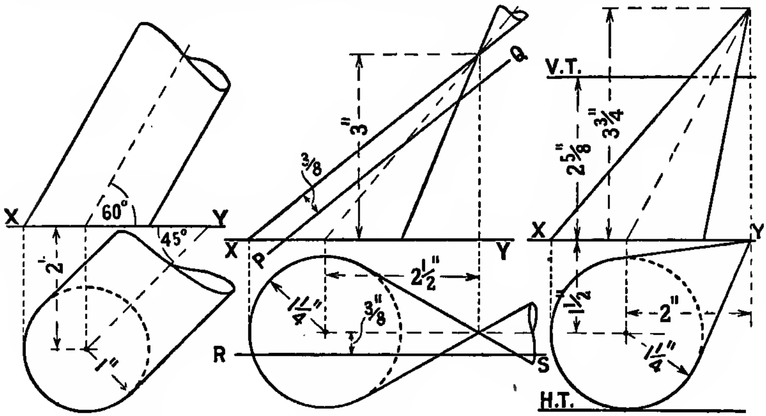


FIG. 534. FIG. 535. FIG. 536.

30. An oblique cone is given in Fig. 535. Draw the plan and also the true form of the parabolic section of this cone by a plane whose vertical trace is PQ and which is perpendicular to the vertical plane of projection. Draw also the elevation of the hyperbolic section by the vertical plane whose horizontal trace is RS.

31. Draw the plan and elevation of the section of the oblique cone (Fig. 536) by the plane whose traces are H.T. and V.T.

CHAPTER XIX

SPECIAL PROJECTIONS OF PLANE FIGURES AND SOLIDS

232. Projections of a Figure whose Plane and a Line in it have Given Inclinations.—First determine the traces of a plane containing the figure. The working of the problem is much simplified by first assuming this plane perpendicular to the vertical plane of projection. In this position the elevation of the figure will be a straight line coinciding with the vertical trace of the plane. Afterwards any other elevation may be obtained by Art. 157, p. 191.

Draw L′M (Fig. 537) making with XY an angle θ equal to the given inclination of the plane of the figure. Draw MN the horizontal trace at right angles to XY. In the plane L′MN place a line RS having the inclination a of the line of the figure given (Art. 174, p. 214).

Now imagine the plane L′MN, with the line RS upon it, to rotate about its horizontal trace MN until it comes into the horizontal plane. The point S being in MN will remain stationary while the point R will describe an arc of a circle in the vertical plane of projection with M as centre. Hence, if with M as centre and Mr' as radius the arc $r'R_1$ be described, meeting XY at R_1, R_1s will be the position of the line RS when that line is brought into the horizontal plane.

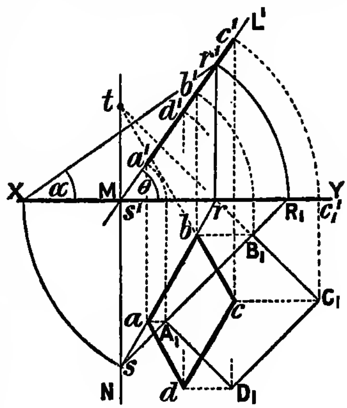


FIG. 537.

Mark off on R_1s a length A_1B_1 equal to the length of that line of the figure whose inclination a is given, and on A_1B_1 construct the given figure. Note that the line AB of the figure is not necessarily one of its sides, but may be any line whatever in the plane of the figure and occupying a definite position in relation to the figure. Next imagine the figure thus drawn on the horizontal plane to rotate about MN until its inclination is θ. All points in the figure will describe arcs of circles whose plans will be straight lines perpendicular to MN and whose elevations will be arcs of circles having their centres at M and having radii equal to the distances of the points

from MN. Thus the point C_1 will describe an arc of a circle whose plan is C_1c perpendicular to MN and whose elevation is the arc $c_1'c'$ of a circle whose centre is M. c is of course in a straight line through c' at right angles to XY.

As a verification of the construction it should be noticed that if any line, say C_1B_1, of the figure constructed on the horizontal plane be produced, if necessary, to meet MN it will do so at the point t where the plan bc of the same line in its inclined position meets it.

It must be borne in mind that no line of the figure can have a greater inclination than that of its plane, that is, α must not be greater than θ.

233. Projections of a Plane Figure, the Inclinations of two Intersecting Lines in it being given.—Determine by Art. 187, p. 219, the traces of the plane containing the lines whose inclinations are given, taking the horizontal trace at right angles to the ground line. Rotate this plane, with the lines upon it, about its horizontal trace, as in the preceding Article, until it comes into the horizontal plane. On the lines thus brought into the horizontal plane construct the given figure and proceed to determine the plan and elevation of the figure in its inclined position exactly as in the preceding Article.

234. Projections of a Plane Figure having given the Heights of three Points in it.—Note that the difference between the heights of any two points must not exceed the true distance between the points.

Let A, B, and C be the points whose heights are given.

Construct on the horizontal plane a triangle $A_1B_1C_1$ (Fig. 538) equal to the triangle ABC. With centre A_1 and radius equal to the height of the point A describe the circle EFH. With centre B_1 and radius equal to the height of the point B describe the circle KLM. With centre C_1 and radius equal to the height of the point C describe the circle NPQ. Draw HL to touch the circles EFH and KLM, and produce it to meet A_1B_1 produced at R. Also draw EP to touch the circles EFH and NPQ, and produce it to meet A_1C_1 produced at S. RS is the horizontal trace of the plane which will contain the points A, B, and C.

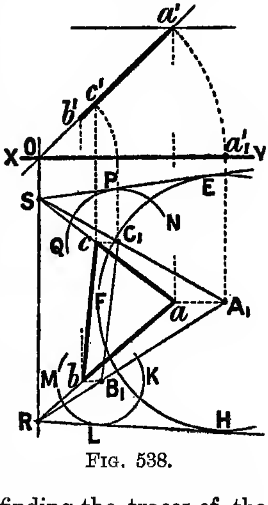


FIG. 538.

Draw XY perpendicular to RS meeting it at O. Draw A_1a_1' perpendicular to XY to meet it at a_1'. With centre O and radius Oa_1' describe an arc of a circle to meet at a' a parallel to XY which is at a distance from XY equal to the height of the point A. Oa' is the vertical trace of the plane which will contain the points A, B, and C.

The theory of the above construction for finding the traces of the

plane containing the points A, B, and C is similar to that of the construction given in Art. 187, p. 219, which should be again referred to. It is evident that the angles A_1RH and A_1SE are the inclinations of the straight lines AB and AC respectively.

The plan *abc* and the elevation $a'b'c'$ of the triangle ABC are next determined as shown, the construction being the same as in Art. 232.

If ABC is not the complete figure given, a figure equal to it must be built up on the triangle $A_1B_1C_1$. Then the projections of the remainder of the figure are obtained in the same way as the projections of the part ABC.

235. Projections of a Plane Figure when it has been turned about a Horizontal Line in it till the Plan of an Opposite Angle is equal to a given Angle.—Let ABCD denote the given figure, AC the horizontal line about which it is turned, and B the angle whose plan is to be equal to a given angle.

Construct the figure aB_1cD_1 (Fig. 539) equal to the given figure ABCD. On *ac* describe a segment of a circle (Art. 14, p. 18) to contain an angle equal to that into which the angle B is to be projected.

As the figure revolves about AC the point B will describe an arc of a circle whose plan is a straight line through B_1 perpendicular to *ac*. Let this perpendicular meet the arc of the segment of a circle which has been described on *ac* at the point *b*. Join *ab* and *bc*. Then *abc* is the plan of the angle B when the figure has been turned as required.

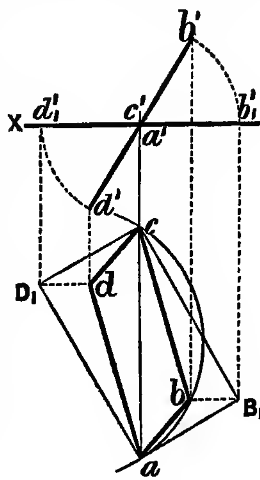


FIG. 539.

Draw XY at right angles to *ac*, meeting *ac* produced at a'. Draw B_1b_1' at right angles to XY to meet it at b_1'. With centre a' and radius $a'b_1'$ describe the arc $b_1'b'$, and through *b* draw a perpendicular to XY to meet this arc at b'. $a'b'$ is the vertical trace and aa' is the horizontal trace of the plane containing the figure ABCD when it occupies the position required.

From D_1 draw D_1d_1' perpendicular to XY to meet it at d_1'. With centre a' and radius $a'd_1'$ describe the arc $d_1'd'$ to meet $b'a'$ produced at d'. Draw $d'd$ perpendicular to XY to meet a line through D_1 parallel to XY at *d*. Join *ad* and *cd*. *abcd* is the plan and $d'b$ is an elevation of the figure ABCD as required.

236. Projections of a Solid when a Plane Figure on it and a Line in that Figure have given Inclinations.—The plane figure may be a face of the solid or it may be a section of it. Considering the most general case, the first step is to determine the projections of the plane figure as in Art. 232, p. 272. On the rabatment of the plane figure on the horizontal plane the feet of the

perpendiculars from the angular points of the solid on the plane of the figure must next be located. The projections of the feet of these perpendiculars are next found, and the projections of the perpendiculars can then be drawn. The projections of the angular points of the solid are therefore determined.

Two examples are illustrated by Figs. 540 and 541.

In the first example (Fig. 540) a right square prism, side of base 1·5 inches, altitude 1·8 inches, is shown when its base is inclined at 50° and one side of that base is inclined at 20°. The plan *abcd* and elevation *a′b′c′d′* of the square base are first determined as in Art. 232, p. 272. In this case the feet of the perpendiculars from the angular points of the solid on the plane of the base are at the angular points of that base. Hence it is now only necessary to draw from *a′*, *b′*, *c′*, and *d′* perpendiculars to the vertical trace of the plane of the base and make them 1·8 inches long in order to obtain the elevations *e′*, *f′*, *g′*, and *h′* of the other angular points of the solid. The plans of these perpendiculars are at right angles to the horizontal trace of the plane of the base and are therefore parallel to XY.

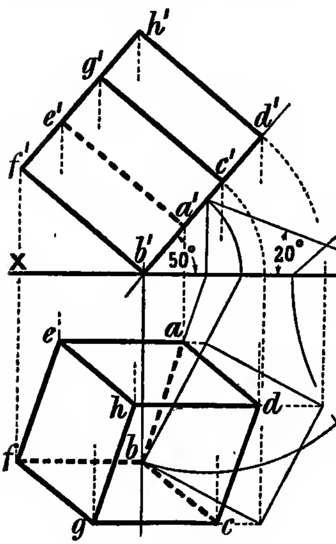


FIG. 540.

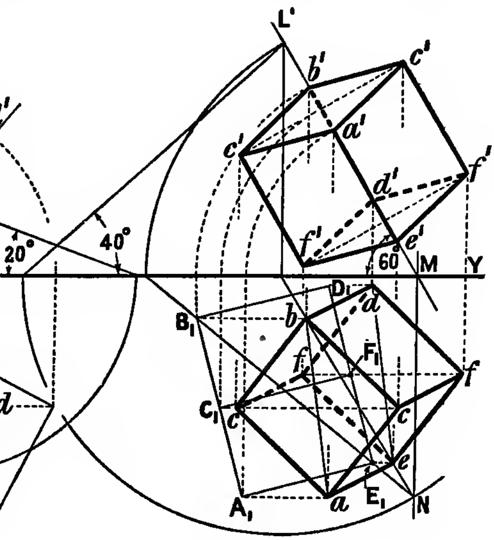


FIG. 541.

In the second example (Fig. 541) a cube of 1·5 inches edge is shown when the plane of two diagonals of the solid is inclined at 60° and one of these diagonals is inclined at 40°. The plane of the two diagonals divides the cube into two right triangular prisms, the triangular ends of which are each equal to one half of a face of the cube. The common face of these two triangular prisms is a rectangle of which two opposite sides are edges of the cube while the other two

sides are diagonals of opposite faces of the cube and the diagonals are diagonals of the solid.

The plan *abde* and elevation *a'b'd'e'* of the rectangle which is the section of the cube by the plane of two of its diagonals are determined by Art. 232, p. 272. In this case the feet of the perpendiculars from the other angular points of the cube on to the plane of two diagonals are at the middle points of the longer sides of the rectangle ABDE and the lengths of these perpendiculars are each equal to the half of a diagonal of a face of the cube. Hence, perpendiculars to the vertical trace of the plane of the above mentioned rectangle from the middle points of *a'b'* and *d'e'* and equal to half of a diagonal of a face of the cube determines the elevations *c'*, *c'*, *f'*, and *f'* of the other corners of the cube. The plans of these perpendiculars are parallel to XY.

237. Projections of a Solid when two Intersecting Lines connected with it have given Inclinations.—Determine by Art. 187, the plane containing the intersecting lines whose inclinations are given. Rabat this plane with the lines upon it into the horizontal plane by turning it about its horizontal trace. On these lines thus brought into the horizontal plane complete the plan of the solid and then proceed exactly as in Art. 236.

238. Projections of a Solid when the Heights of three Points in it are given.—Determine by Art. 234, the plane containing the three points whose heights are given. Rabat this plane with the points upon it into the horizontal plane by turning it about its horizontal trace. About these points thus brought into the horizontal plane complete the plan of the solid and then proceed exactly as in Art. 236.

239. Projections of a Solid when two Faces A and B, which are at Right Angles to one another, have given Inclinations.—If one of the two faces referred to is a base denote it by A. First determine L'MN, the plane of the face A, the horizontal trace MN being perpendicular to XY. Next determine by Art. 194, p. 226, a plane L'PN perpendicular to the plane L'MN and inclined at an angle equal to the given inclination of the face B.

Find LN the line of intersection of these two planes. Now rabat the plane L'MN with the line LN upon it into the horizontal plane by turning it about MN. On this line thus brought into the horizontal plane construct the face A, that side of the face A which is adjacent to the face B being made to coincide with the line, and then proceed exactly as in Art. 236.

240. Projections of a Solid when two Faces A and B, which are not at Right Angles to one another, have given Inclinations.—If one of the two faces is a base denote it by A. First determine L'MN, the plane of the face A, the horizontal trace MN being perpendicular to XY. Next determine by Art. 194, p. 226, a plane L'PN having an inclination equal to that of the face B and making with the plane L'MN an angle equal to the angle between the faces A and B. The procedure is now the same as in the second paragraph of the preceding Article.

Exercises XIX

1. Draw the plan of an equilateral triangle of 2·5 inches side when its plane is inclined at 45°, one side inclined at 30°, and one angular point on the horizontal plane. From this plan project an elevation on a ground line parallel to the plan of that side of the triangle which is inclined at 30°.

2. Draw the plan of a square of 2·5 inches side when its plane is inclined at 60° and one diagonal is inclined at 45°. What is the inclination of the other diagonal?

3. A regular hexagon ABCDEF of 1·25 inches side has the side AB horizontal and the plan of the diagonal BD is 2 inches long. Draw the plan and from it project an elevation on a ground line parallel to *ab*. What is the inclination of the plane of the hexagon?

4. The plane of a square of 2 inches side is inclined at 60° and the plan of one diagonal is 2 inches long. Draw the plan of the square.

5. The horizontal trace of a plane makes an angle of 45° and the vertical trace makes an angle of 50° with XY. A regular pentagon of 1·5 inches side lies on this plane with one side inclined at 30° to the horizontal plane. Draw the plan and elevation of the pentagon.

6. A square of 2 inches side has one side inclined at 45° and an adjacent side inclined at 30°. Draw the plan of the square and an elevation on a ground line parallel to the plan of the diagonal which is inclined at 45°.

7. ABC is a triangle. AB = 2 inches, BC = 2·75 inches, and CA = 3 inches. Draw a plan of this triangle when the sides AB and BC are inclined at 40° to the horizontal plane.

8. A regular pentagon of 2 inches side has one side inclined at 30° and a diagonal through one end of that side inclined at 40°. Draw the plan.

9. ABC is an equilateral triangle of 2·5 inches side. A is on the horizontal plane, B is 1 inch and C is 1·5 inches above the horizontal plane. Draw the plan and an elevation on a ground line parallel to *ab*.

10. Draw the plan of a square of 2 inches side when the heights of its centre and two angular points above the horizontal plane are 2 inches, 1·5 inches, and 0·75 inch respectively.

11. Draw the plan of a regular pentagon of 1·5 inches side when one side is on the horizontal plane and the plan of the opposite angle is 120°.

12. A 60° set-square revolves about its shortest side, which is horizontal, until the plan of the opposite angle is 60°. Find what is then the inclination of the plane of the set-square.

13. ABC is a triangle, AB = 1·5 inches, BC = 3 inches, and CA = 2 inches. Draw the plan of this triangle when the side AB is horizontal and the plan of the angle C is 20°.

14. Draw a triangle *oab*, *oa* = 1 inch, *ob* = 1·25 inches, and *ab* = 1·5 inches. *o* is the plan of the centre and *ab* is the plan of one side of a regular hexagon. Complete the plan of the hexagon.

15. Draw a triangle *abc*. *ab* = 1 inch, *bc* = 1·3 inches, and *ca* = 2 inches. *ab* and *bc* are the plans of adjacent sides of a regular octagon. Complete the plan of the octagon.

16. Draw *ab* 2 inches long. From the middle point *c* of *ab* draw *cd* 3 inches long and making the angle *acd* = 60°. These are the plans of two intersecting straight lines which are at right angles to one another. The points A and D are each 0·5 inch above the horizontal plane. Find the height of the point B.

17. *a* is a point on a straight line HT. *ab* is a straight line 2 inches long making an angle of 30° with HT. *ab* is the plan of one edge of a regular tetrahedron of 2·5 inches edge. HT is the horizontal trace of the plane of a face of the tetrahedron containing the edge AB. Complete the plan of the tetrahedron and draw an elevation on a ground line parallel to HT.

18. Draw the plan of the solid given in Fig. 373, p. 196, when the base is inclined at 45° and one side of that base is inclined at 30°. From this plan project an elevation on a ground line parallel to the horizontal trace of the plane of the base.

19. Draw the plan of the solid given in Fig. 372, p. 196, when the base is inclined at 50° and one diagonal of the base is inclined at 40°. From this plan project an elevation on a ground line parallel to the plan of the diagonal which is inclined at 30°.

20. A right pyramid has for base a regular pentagon of which the diagonals measure 2·5 inches. The vertex is 2 inches above the base. Draw the plan and elevation of the pyramid, with its base in a plane inclined at 55° to the vertical plane and at 60° to the horizontal plane; one diagonal inclined at 30°, and one end of that diagonal in the vertical plane. [B.E.]

21. Two diagonals of a cube of 2 inches edge are inclined at 35° to the horizontal plane. Draw the plan of the cube.

22. A right prism 3 inches long has for its ends regular hexagons of 1·25 inches side, AB is an edge of one end and BH is one of the long edges. Draw the plan of this prism when the heights of the points A, B, and H above the horizontal plane are 0·2 inch, 1 inch, and 2·5 inches respectively.

23. ABC is the base of a pyramid and V its vertex. AB = 2·5 inches, AC = 2 inches, BC = 2·75 inches, AV = CV = 3 inches, and BV = 2·5 inches. Draw the plan of the pyramid when A is 1·5 inches, B 2·5 inches and C 1 inch above the horizontal plane. [B.E.]

24. Draw the plan of a cube of 2 inches edge when one face is inclined at 55° and another face is inclined at 75°.

25. One face of a regular tetrahedron of 2·5 inches edge is inclined at 45° and another face is inclined at 70°. Draw the plan of the tetrahedron.

CHAPTER XX

HORIZONTAL PROJECTION

241. Figured Plans.—If the plan of a point is given, and also its distance from the horizontal plane, the position of the point is fixed without showing an elevation of it. The distance of the point from the horizontal plane is shown by placing a number or index adjacent to its plan. For example, if the point is 6 units above the horizontal plane the figure 6 is placed adjacent to its plan, and if the plan is also lettered the figure is placed to the right of the letter and at a slightly lower level thus, a_6. If the negative or minus sign be placed in front of the index this denotes that the given distance is *below* the horizontal plane. Thus, a point whose plan is marked -6 or a_{-6} is 6 units below the horizontal plane. A plan such as has been described is called a *figured plan* or *indexed plan.*

If the plan of a straight line is given and also the figured plans of two points in it, it is obvious that the line is completely fixed.

In *horizontal projection* points and lines are shown by their figured plans. The form and position of a curved line can only be shown in horizontal projection by giving the figured plans of a sufficient number of points in it.

242. Scales of Slope.—It has already been shown that planes may be represented by their traces on the co-ordinate planes. In horizontal projection, planes are represented by their *scales of slope.* The scale of slope of a plane is really the figured plan of a straight line lying in the plane and perpendicular to the horizontal trace of the plane. If a straight line be given and it is understood that a plane whose horizontal trace is perpendicular to the line contains the line then it is obvious that the plane is completely fixed. To show that the given line represents a plane and not simply a line a second and thicker line is ruled near to and parallel to it. It is usual to place the thicker of the two lines to the left of the other when looked at by a person ascending the plane.

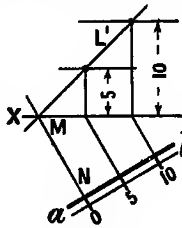


FIG. 542.

Fig. 542 shows the connection between the scale of slope ab and the traces L'M, MN of a plane. The numbers at the different points of the scale denote the distances of these points from the horizontal plane.

243. Applications of Horizontal Projection.—Most of the problems on points, lines and planes can be worked as conveniently by horizontal projection as by ordinary plan and elevation, but it may be observed that although the solution of a problem may be given by a figured plan only the working may involve constructions which are equivalent to the drawing of one or more elevations.

As the principles involved in the solution of problems on points, lines, and planes are generally the same whether they are solved by plan and elevation or by horizontal projection, a selection of a few problems only will be taken in this chapter to illustrate the method of horizontal projection.

244. Simple Problems on the Straight Line.—Let a_4b_{13} (Fig. 543) be the figured plan of a straight line AB. From a and b draw perpendiculars to ab, and make them respectively equal to the indices of a and b. The line A_1B_1 joining the tops of these perpendiculars will have a length equal to the *true length* of AB. The angle between A_1B_1 and ab will measure the *inclination* of AB, and the point where A_1B_1 meets ab will be the *horizontal trace* of AB. It is evident that the index of the horizontal trace of a line is O. If one of the indices is positive and the other negative the perpendiculars aA_1 and bB_1 must be drawn on opposite sides of ab.

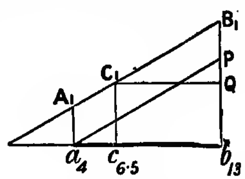


FIG. 543.

In the above construction A_1B_1 may be looked upon as an elevation of AB on ab as a ground line, or it may be taken as the line AB rebatted on to the horizontal plane about ab as an axis.

The true length and inclination of AB may also be found by drawing bP perpendicular to ab and equal to the difference between the indices of a and b. The line aP will then be the true length of AB, and the angle Pab will be the inclination of AB.

To determine a point c in ab which shall have a given index, say 6·5, make bQ equal to 6·5, draw QC_1 parallel to ab to meet A_1B_1 at C_1. A perpendicular C_1c to ab determines the point required.

To determine the index of a given point c in ab, draw cC_1 perpendicular to ab to meet A_1B_1 at C_1. The length of cC_1 is the index required.

To draw a line through a given point c_5 (Fig. 544) parallel to a given line a_2b_9, draw through c a line cd parallel to ab, and make $cd = ab$. The index of d will be 7 greater than 5 the index of c because the index of b is 7 greater than the index of a. If the line cd be produced in the opposite direction, and ce be made equal to ba, then the index of e will be 7 less than 5 the index of c because the index of a is 7 less than the index of b. The index of e will therefore be -2, or the point e is 2 units below the horizontal plane.

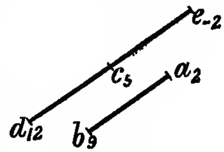


FIG. 544.

245. Inclination of a Given Plane.—Let ab (Fig. 545) be

the scale of slope of the plane. Since the long lines of the scale of slope are at right angles to the horizontal trace of the plane, the inclination of these lines must be the same as the inclination of the plane. Hence if $b\mathrm{B}_1$ be drawn perpendicular to ab, and made equal to the difference between the indices of b and a, the angle B_1ab will measure the inclination of the plane.

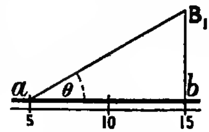


FIG. 545.

246. Two Problems on Parallel Planes.—(1) To determine a plane to contain a given point p_7 (Fig. 546), and be parallel to a given plane ab. Since the horizontal traces of parallel planes are parallel it is clear that their scales of slope which are at right angles to these traces, must also be parallel. Draw, therefore, the long lines cd of the required scale of slope parallel to ab, and in any convenient position. Through p draw a line pq, at right angles to cd. pq will be the plan of a horizontal line lying in the required plane, and q will therefore have the same index as p. The scale of slope cd must be graduated in the same way as ab; that is to say the difference between the indices of a given length on cd must be equal to the difference between the indices on an equal length of ab.

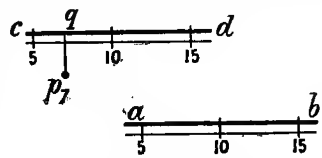


FIG. 546.

(2) To determine the distance between two parallel planes ab and cd (Fig. 547). If the traces of these planes on a vertical plane perpendicular to their horizontal traces, and therefore parallel to their scales of slope, be drawn, the distance between these traces will be the distance between the planes.

ab is taken as the ground line, and ac' and bb' are drawn perpendicular to ab. bb' is made equal to the index of b, and ac' equal to the index of c. As in the example shown in the figure, the index of a is 0, ab' is the vertical trace of the plane whose scale of slope is ab, and as the other plane is parallel to this one, a line $c'd'$ parallel to ab' will be its vertical trace on the assumed vertical plane. The distance between ab' and $c'd'$ is the distance between the given planes.

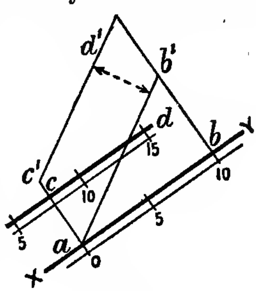


FIG. 547.

247. The Plane containing three given Points.—Let a_{17}, b_{13}, and c_{15} (Fig. 548) be the given points. Find by Art. 244 a point d in ab having the same index as c. Join cd. cd is the plan of a horizontal line lying in the plane containing the three given points; therefore the scale of slope of the plane containing these three points must be perpendicular to cd.

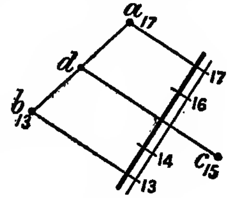


FIG. 548.

Where cd cuts the scale of slope determines a point on it having the same index as c or d. If through a a line be drawn parallel to cd to meet the scale of slope, a point is determined on the latter having the same index as a. These two points on the scale of slope having been found, the scale may be graduated if required.

248. In a Given Plane to place a Line having a Given Inclination so that it shall contain a Given Point in the Plane.—Let ab (Fig. 549) be the given plane and p_{20} the given point. Through r, any point in the scale of slope not having the same index as p, draw rq perpendicular to ab. Draw pQ_1, in any convenient direction, say parallel to rq, and a line pP_1 at right angles to pQ_1. Make pQ_1 equal to the difference between the indices of p and r, and draw P_1Q_1 making the angle P_1Q_1p equal to the given inclination. With centre p and radius pQ_1 describe an arc cutting rq at q. $p_{20}q_{15}$ is the line required. Except when the given inclination is the same as the inclination of the given plane there are obviously two straight lines which will satisfy the given conditions.

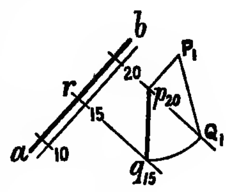


FIG. 549.

249. Intersection of Two Given Planes.—Let ab and cd (Fig. 550) be the given planes. Through a and b, any two points in the scale of slope ab, draw the lines ap and bq at right angles to ab. Through points c and d in cd, having the same indices respectively as a and b, draw lines perpendicular to cd. The line ap is the plan of a horizontal line lying in the plane ab. cp is the plan of a horizontal line lying in the plane cd. Since these lines have the same indices, they are in the same horizontal plane, therefore they will intersect at a point of which p is the plan. Therefore p_{20} is a point in the intersection of the two planes. In like manner, q_{10} is a point in the intersection, therefore the line $p_{20}q_{10}$ is the intersection of the given planes.

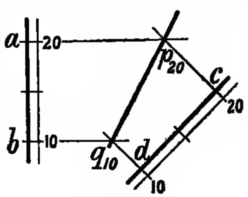


FIG. 550.

If the given scales of slope are parallel, the above construction will evidently fail, because the horizontals of both planes will be parallel, and therefore never meet. In this case a third plane may be taken, not parallel to either of the given planes, and its intersection with each of them found by the method just given. This determines two lines whose intersection with one another will be a point in the intersection required. It is evident that the intersection of the given planes in this case will be a horizontal line, therefore a line perpendicular to the given scales of slope through the point found determines the intersection of the given planes.

If the third plane mentioned above be taken perpendicular to each of the given planes, it will be a vertical plane, and the plans of its intersection with the given planes will coincide, and an elevation of

them must be drawn to determine the point where they meet. This elevation is best taken on the third plane itself.

If the given scales of slope are nearly parallel, the horizontals of the planes will meet at a very acute angle, and it is more accurate to find two points in the required intersection by the method explained for planes whose scales of slope are parallel, that is, by cutting each of the given planes by two other planes.

250. Intersection of a Line and Plane.—Let $a_{28}b_{13}$ (Fig. 551) be a given line and *cd* a given plane. Through any two points *a* and *b* in the given line, draw the parallel lines *ar* and *bq* in any convenient direction. Through *c* and *d*, points on the scale of slope having the same indices as *a* and *b* respectively, draw the horizontals *cr* and *dq*, meeting the parallels *ar* and *bq* respectively at *r* and *q*. Join *rq*. The point P where the line *rq* or *rq* produced meets *ab* or *ab* produced is the plan of the point where the given line meets the given plane.

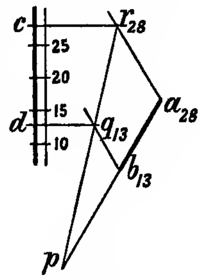


Fig. 551.

The theory of the above construction is as follows—$a_{28}r_{28}$ and $b_{13}q_{13}$ are horizontals of a plane containing the given line. The line $r_{28}q_{13}$ is evidently the intersection of this plane with the given plane. The point P is therefore a point in both planes and also in AB, therefore it is the intersection required.

The intersection may also be found by taking an elevation of the line and plane on a ground line parallel to the scale of slope.

251. The Normal to a given Plane through a given Point.—The plan of a line which is perpendicular to a plane is at right angles to the horizontal trace of the plane, and will therefore be parallel to its scale of slope.

To figure the plan of the line (which of course passes through the figured plan of the point), determine the trace of the plane and the elevation of the point on a vertical plane parallel to the scale of slope. Through the elevation of the point draw a perpendicular to the trace of the plane, this perpendicular will be the elevation of the normal and from it the plan may be figured.

252. The Plane through a given Point perpendicular to a given Line.—Taking the figured plan of the line as a ground line, determine the elevations of the point and line. Through the elevation of the point draw a line perpendicular to the elevation of the given line. This perpendicular will be the vertical trace of the required plane. The scale of slope of the plane will be parallel to the given figured plan of the line, and it may be graduated from the vertical trace found as above.

253. Contour Lines.—The plan of a portion of the earth's surface is made to show the form of that surface very clearly, by drawing on it the sections of the surface by a series of horizontal planes, at equal distances from one another. These sections are called *contours* or *contour lines.* It is evident that the relative closeness of the

contour lines shows the relative steepness of the different parts of the surface, the surface being steepest where the contour lines are closest together. It is usual to affix to the contour lines their heights above some fixed horizontal plane.

254. Intersection of a Plane and a Contoured Surface.—Fig. 552 shows the scale of slope of a plane and the contoured plan of

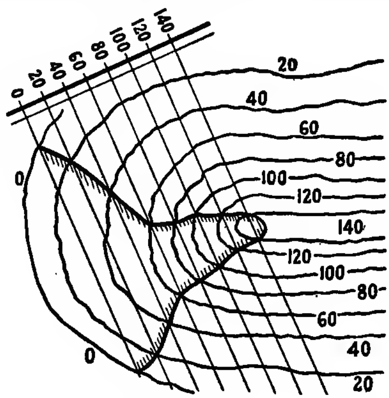


FIG. 552.

a surface. To determine the intersection of the plane and the surface, draw the plans of a number of horizontal lines lying in the plane, and having the same indices as the contour lines. The points where these lines meet the contour lines having the same indices, are points in the intersections required. The complete intersection is obtained by drawing a fair curve through the points thus found.

255. Contouring a Surface from its Equation.—A formula or equation which contains three variables is represented graphically by a surface and in representing this surface on paper a system of contour lines may be necessary. As an example take the formula, $H = \frac{(62800 - 3v^2)d^2v}{230000}$ which gives the relation between the horse-power H transmitted by a cotton rope and the diameter d of the rope in inches and the velocity v of the rope in feet per second, allowing for the stress in the rope due to centrifugal force.

For each diameter of rope the relation between H and v is shown by a plane curve, and if the curves for ropes of different diameters be constructed these curves will be contours of the surface which represents the original formula. The contours of the surface corresponding to diameters of $\frac{3}{4}$ inch, 1 inch, $1\frac{1}{4}$ inches, $1\frac{1}{2}$ inches, $1\frac{3}{4}$ inches, and 2 inches are shown at (a) Fig. 553 between the limits $v = 0$, and $v = 120$. To construct these curves it is first necessary to construct, by calculation, the table given on the opposite page.

Horse-power H, for different values of v and d.

v	Diameter d.					
	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2
10	1·5	2·7	4·2	6·1	8·3	10·9
20	3·0	5·4	8·4	12·1	16·4	21·4
30	4·4	7·8	12·2	17·6	24·0	31·4
40	5·7	10·1	15·8	22·7	30·9	40·3
50	6·8	12·0	18·8	27·0	36·8	48·1
60	7·6	13·6	21·2	30·5	41·5	54·3
70	8·2	14·6	22·9	32·9	44·8	58·6
80	8·5	15·2	23·7	34·1	46·4	60·7
83·5	8·6	15·2	23·8	34·2	46·6	60·8
90	8·5	15·1	23·5	33·9	46·1	60·3
100	8·0	14·3	22·3	32·1	43·7	57·0
110	7·1	12·7	19·8	28·5	38·8	50·7
120	5·8	10·2	16·0	23·0	31·3	40·9

In addition to the velocities at intervals of 10 feet per second given in the first column of the table it will be noticed that the velocity 83·5 feet per second is given. This velocity 83·5 is the velocity at which the horse-power for any given rope is a maximum. The velocity 83·5 is obtained by calculation as follows. The horse-power for any rope will evidently be a maximum when $62800\,v - 3v^3$ is a maximum. Let $y = 62800v - 3v^3$. Differentiating this, $\frac{dy}{dv} = 62800 - 9v^2$ and y is a maximum when $62800 - 9v^2 = o$, that is when $3v = \sqrt{62800}$, or $v = 83·5$.

Assuming that the curves at (a) Fig. 553 are in vertical planes, horizontal sections of the surface at levels whose intervals are 10 horse-power are shown at (b) Fig. 553 between the limits $d = \frac{3}{4}$ and $d = 2$.

The student should work out this example to the following scales. H, 1 inch to 10 horse-power. d, 1 inch to $\frac{1}{2}$ inch diameter. v, 1 inch to 20 feet per second. In addition to the contours shown at (a) Fig. 553 the student should add the curves for ropes of $\frac{7}{8}$ inch, $1\frac{1}{8}$ inches, $1\frac{3}{8}$ inches, $1\frac{5}{8}$ inches, and $1\frac{7}{8}$ inches diameter. Also in addition to the contours shown at (b) he should add the curves for 5, 15, 25, 35, 45, and 55 horse-power. Lastly, on a base line parallel to MN he should construct the contours of the surface showing the relation between H and d for velocities at intervals of 10 feet per second. These contours are vertical sections of the surface parallel to MN just as the curves at (a) are vertical sections by planes parallel to LM.

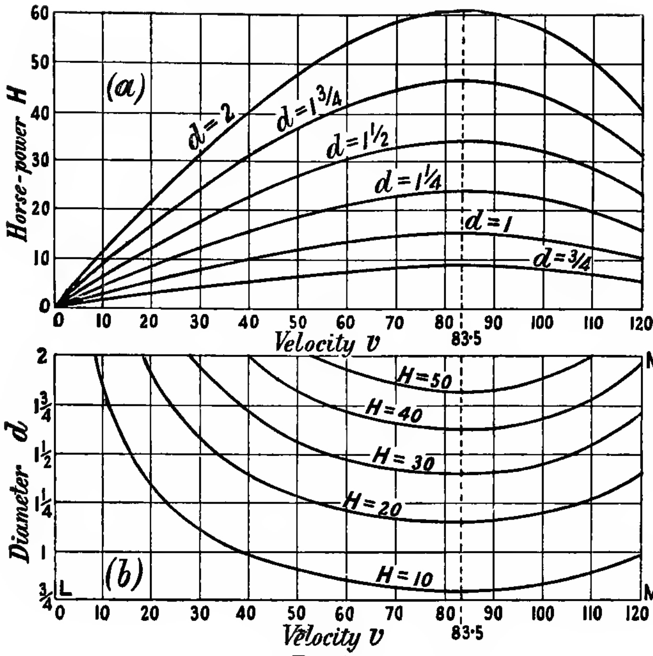


FIG. 553.

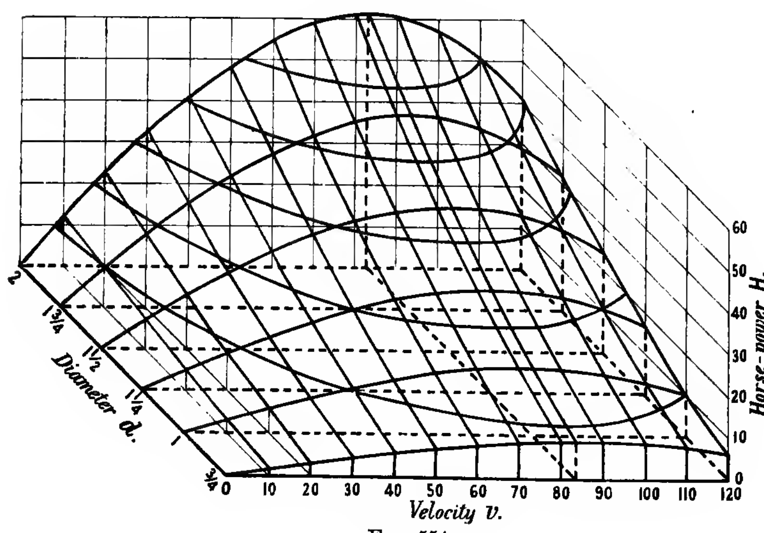


FIG. 554.

An oblique parallel projection of the surface which has just been considered is shown in Fig. 554. The student should have no difficulty in making such a projection after he has studied Chapter XXI. This projection may be very readily drawn on squared paper, taking the axis for d at 45° to the axis for v.

When the student draws the oblique parallel projection shown in Fig. 554 he should put in the additional contours suggested with reference to Fig. 553.

Exercises XX

Note. *Unless otherwise stated the unit for the indices is $\frac{1}{10}$ or 0·1 of an inch, so that an index 25 denotes a height of 2·5 inches above the horizontal plane.*

1. Two points a and b in the plan of a straight line are 2 inches apart. The index of a is 3 and the index of b is 11. Determine, (1) the index of a point c in ab which is $\frac{1}{2}$ inch from a; (2) a point $d_{7.5}$ in the plan of the line; (3) a point e_{14} in the plan of the line; (4) a point f_{-2} in the plan of the line; (5) the true length of EF.

2. Draw a triangle $a_2\, b_{-4}\, c_{12}$ (ab = 2·7 inches, bc = 2·2 inches, ca = 1·7 inches). Find a point d in bc whose index is 8, and show the figured plans of two straight lines passing through B and C, and parallel to AD.

3. A straight line making an angle of 30° with the ground line is both horizontal and vertical trace of a plane. Show the scale of slope of this plane.

4. Show the scales of slope of two parallel planes inclined at 50°, the distance between the planes being 0·7 inch.

5. Determine the scale of slope of the plane of the triangle given in exercise 2, also the true shape of the triangle.

6. An equilateral triangle of 2·5 inches side has its angular points indexed 3, 12, and 15. Show the figured plans of the bisectors of the angles of the triangle of which the given triangle is the plan. Draw also the ellipse which is the plan of the circumscribing circle of the triangle.

7. Draw the scale of slope of a plane whose inclination is 50° and show the figured plan of a triangle ABC which lies in this plane. The inclinations of AB and BC are 30° and 45° respectively, the indices of a, b and c are 3, 24, and 8 respectively.

8. A plane is inclined at 35° to the horizontal, the lines of steepest upward slope going due East. A second plane has an inclination of 52°, the direction of the lines of upward slope being due North. Represent the planes by scales of slope, unit for heights 0·1 inch. Show the figured plan of the intersection of the planes. Find and measure the angle between the planes. [B.E.]

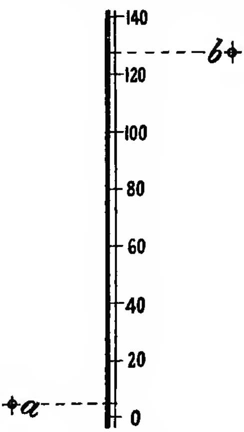


FIG. 555.

9. Fig. 555 is a plan, drawn to a scale of 1 inch to 200 feet, showing at a, b two places A, B on a hill-side, the surface of the ground being an inclined plane represented by a scale of slope, heights being indexed in feet. Draw Fig. 555 to a scale of 1 inch to 100 feet and add the plan of a zigzag path connecting A and B, made up of three straight lengths of a constant inclination equal to half that of the hill, and deviating equally on each side of a straight line through A and B. Ascertain and measure in feet the total length of this path. [B.E.]

10. Two lines AB and CD are given by their figured plans in Fig. 556. Draw the scale of slope of a plane which will contain the line CD and make an angle of 15° with the line AB.

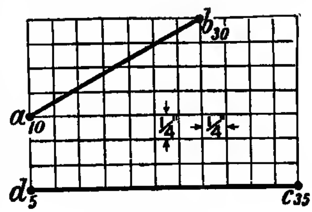


FIG. 556.

11. The plans of six points a_0, b_5, c_{10}, d_{15}, e_{20}, and f_{25}, taken in order, are situated at the angular points of a regular hexagon of $1\frac{1}{2}$ inches side. Find the intersection of the plane containing A, C, and F, with the plane containing B, D, and E, and state its inclination.

12. Draw a square $a_{20}\,b_{15}\,c_{10}\,d_0$ of 2 inches side. Determine the plan of the sphere on whose surface the points A, B, C, and D are situated.

13. Draw the figured plan of the common perpendicular to the straight lines AC and BD of the preceding exercise.

14. $a_{50}\ b_{60}\ c_{60}\ d_{50}$ (Fig. 557) is the figured plan of a straight roadway which is to be made, partly by cutting, and partly by embankment, on the ground

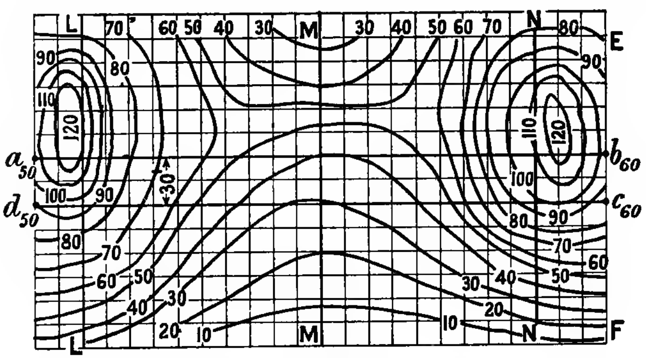


FIG. 557.

whose surface is given by its contoured plan. The sloping faces of the cutting and embankment are to be inclined at 40°. Draw the plan of the intersections of the faces of the cutting and embankment with the surface of the ground.

Show also, on EF as ground line, vertical sections at LL, MM, and NN.

N.B.—Fig. 557 is to be enlarged four times. The unit in this exercise is 1 foot.

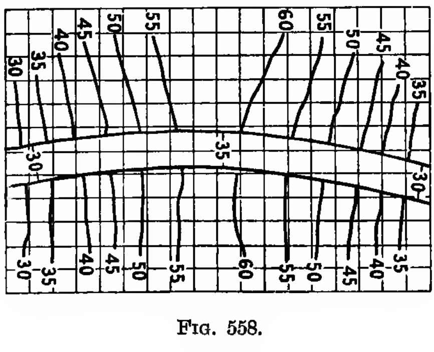


FIG. 558.

15. The surface of a piece of ground is given by contours at vertical intervals of 5 feet (Fig. 558), the linear scale of the plan being 1 inch to 100 feet. A road is to be cut at the given heights, the face of the cutting on each side having a slope of 38° to the horizontal. Draw Fig. 558 to a scale of 1 inch to 50 feet and complete the plan of the finished earthwork. [B.E.]

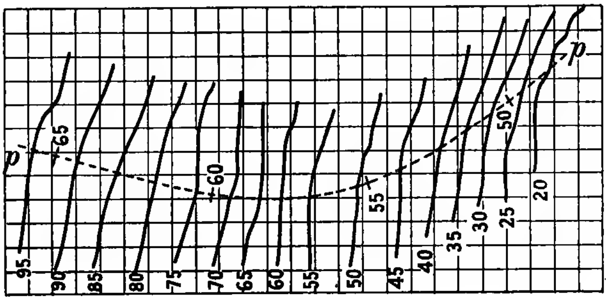


FIG. 559.

16. You are given, in Fig. 559, the plan of the surface of a piece of ground, contoured in feet, the linear scale being 1 inch to 100 feet. The curve pp is the centre line of a road, 20 feet wide, which is to be made partly by cutting and partly by embankment, the former having a slope of 45° and the latter one of 38° to the horizontal. Draw Fig. 559 to a scale of 1 inch to 50 feet and complete the plan of the finished earthwork within the limits of the data. [B.E.]

17. Represent by contour lines, as in the example worked out in Art. 255, the surface whose equation is $M = \frac{1 \cdot 8\sqrt{v}}{t - 60}$. The axes to be arranged as shown at (a), (b), and (c), Fig. 560. v denotes velocity in feet per minute, and t denotes temperature in degrees Fahrenheit. Scales.—For v, 1 inch to 50 feet per minute. For t, 1 inch to 20°. For M, 2 inches to 1. The limits of v and t are shown at (a) and (b).

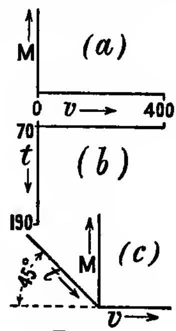


FIG. 560.

CHAPTER XXI

PICTORIAL PROJECTIONS

256. Pictorial Projections.—Since the orthographic projection of a line only shows its true length when the line is parallel to the plane of projection, it is usual, in making *working drawings* of an object, to arrange it so that as many of its lines as possible are parallel to at least one of the co-ordinate planes. The object is then in a simple position. Fig. 561 shows a plan and two elevations of a rectangular solid when the solid is in the simplest possible position in relation to the planes of projection. These projections are very easily drawn, but, although they represent the solid completely, they are not at all "pictorial," and a special training is necessary before the observer can form from them a correct mental picture of the object.

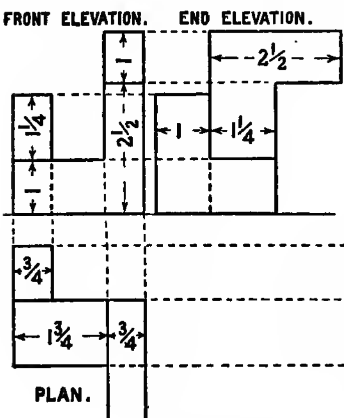


Fig. 561.

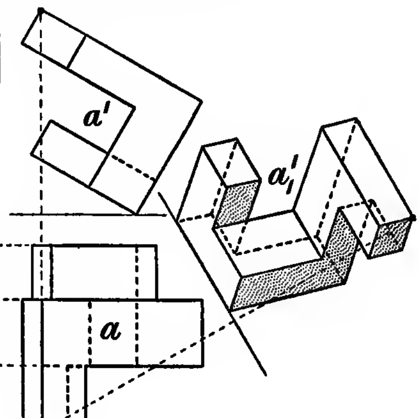


Fig. 562.

Now let the solid be tilted over as shown by the plan a and elevation a', Fig. 562, and let a new elevation a_1' be drawn in the manner explained in Chap. XIV. It will now be observed that the elevation a_1' by itself gives a much better idea of the form of the object than any of the other projections shown in Figs. 561 and 562, but unless a much simpler method of drawing such a pictorial projection than that shown in Fig. 562 can be devised, and unless some simple method of

measuring the various dimensions of the object direct from the drawing can be given, such a pictorial projection would be of little value as a substitute for ordinary working drawings.

In the subsequent articles of this chapter, methods are described which enable the draughtsman to make pictorial projections resembling that shown at a_1' in Fig. 562 with almost as little trouble as is required for the simple projections of the kind shown in Fig. 561, and it will be found that the dimensions of the object represented may be determined from these pictorial projections as easily as from the ordinary projections.

Fig. 563 shows an *isometric projection* of the solid represented in Figs. 561 and 562, and Fig. 564 shows an *oblique parallel projection* of

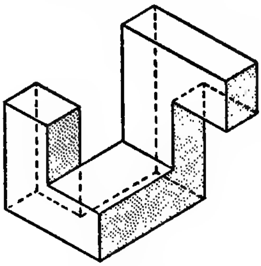


FIG. 563.

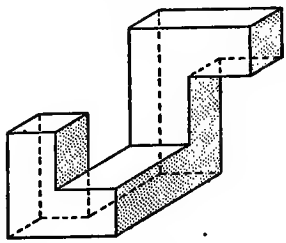


FIG. 564.

the same solid, and a glance at either of these pictorial projections conveys to the mind at once a clear picture of the object. Since no lines other than those shown require to be drawn, and since the dimensions of length, breadth and thickness may be measured directly on these projections, it will be seen that such drawings may often be most useful.

The theorems upon which the methods of this chapter depend are :—

(1) *Parallel lines have parallel projections.*

(2) *Parallel lines have equal inclinations to the same plane.*

(3) *Parallel lines and lines which are equally inclined to the plane of projection have the lengths of their projections to the same scale.*

In stating the above theorems it is assumed that the projectors are parallel to one another.

257. Isometric Projection.—When a solid is rectangular each edge is parallel to one or other of three lines or axes which are mutually perpendicular, and when the solid is placed so that these axes are equally inclined to the plane of projection their projections make angles of 120° with one another, and the projections of the edges of the solid may be measured with the same scale since the edges are parallel to lines which are equally inclined to the plane of projection. Hence follows the simple construction shown in Fig. 565 for making an isometric projection of a rectangular solid. The edges of the solid meeting at one angular point are taken as axes and their

plans *oa*, *ob*, and *oc* are drawn by the aid of the T-square and 30° set square as shown. The lengths of the plans of the edges are marked off with the *isometric scale*, the construction of which is explained in the next article.

The axes referred to in this article are called *isometric axes* and a plane containing any two of them or a plane parallel to this plane is called an *isometric plane*. The word *isometric* means equal measure. It should be noted that only those lines which are parallel to one or other of the isometric axes are *isometric lines*.

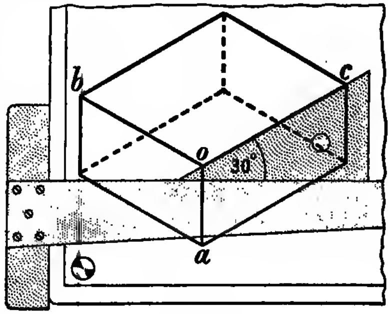


FIG. 565.

258. The Isometric Scale.—Let *oa*, *ob*, and *oc* (Fig. 566) be the projections of three lines OA, OB, and OC which meet at O and are mutually perpendicular and which are equally inclined to the plane of projection; it is required to find the ratio of say *ob* to OB.

Draw *bc* at right angles to *oa*. On *bc* as diameter describe a semicircle cutting *oa* at O_1. O_1b is the true length of the line OB of which *ob* is the projection. For the proof of this construction see Art. 199, p. 229, on the projection of a solid right angle.

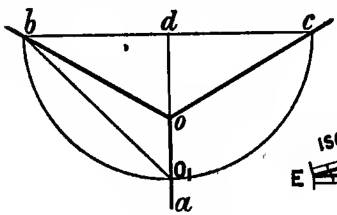


FIG. 566.

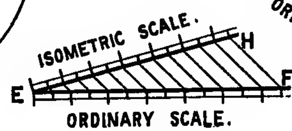


FIG. 567.

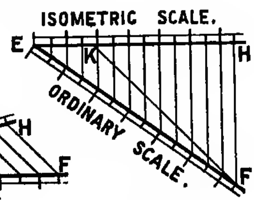


FIG. 568.

If *ao* be produced to cut *bc* at *d*, then it is easy to show that, angle dbO_1 = angle $dO_1b = 45°$; also that angle $dbo = 30°$, and angle $dob = 60°$; hence angle $obO_1 = 15°$. Then it follows that $ob : O_1b :: \sqrt{2} : \sqrt{3}$, that is, the length of the isometric projection of a line is to the true length of the line as $\sqrt{2}$ is to $\sqrt{3}$. Hence to draw an isometric scale corresponding to a given ordinary scale it is necessary to get two intersecting lines whose lengths are to one another as $\sqrt{2}$ to $\sqrt{3}$.

Fig. 567 shows one construction for the isometric scale. EF is the ordinary scale. Angle FEH = 15°. Angle EFH = 45°. EH is the isometric scale, the sub-divisions of which are obtained by drawing lines through the sub-divisions on EF parallel to FH as shown. It will be

seen that the triangle EFH in Fig. 567 is similar to the triangle bO_1o in Fig. 566.

Another construction for the isometric scale is shown in Fig. 568. Angle FHK = 90°. FH = HK. HE = FK. The ordinary scale is made on EF and the isometric scale on EH as shown. If FH = 1, HK = 1, and FK = $\sqrt{2}$ = EH. Hence EF = $\sqrt{3}$ and EH : EF :: $\sqrt{2} : \sqrt{3}$.

In actual practice in making drawings in isometric projection, the isometric scale is seldom used. The lengths of the isometric lines are marked off directly from the ordinary scale. The result of this is that the isometric drawing is larger than it would be if an isometric scale were used in the ratio of $\sqrt{3}$ to $\sqrt{2}$.

259. Projection of any Given Figure lying in an Isometric Plane.—Let ABCDE (Fig. 569) be the given figure, the plane of the paper being the plane of the figure, and let OX and OZ be two lines at right angles to one another in that plane. Let the plane XOZ be tilted up until the axes OX and OZ and the axis OY perpendicular to OX and OZ are equally inclined to the plane of projection. The projections *ox*, *oz*, and *oy* (Fig. 570) will be isometric axes. It is required to add to the projections of the axes the projection of the figure ABCDE.

From the extremities of the straight sides of ABCDE draw parallels to OX to meet OZ at A_1, B_1, C_1 and D_1. Also, from a sufficient number of points on the curved side AED, draw parallels to OX to meet OZ as shown in Fig. 569.

The isometric projections aa_1, bb_1, cc_1, etc. of the lines AA_1, BB_1, CC_1, etc. can now be determined and these determine the projections a, b, c, etc. of the points A, B, C, etc. Hence the required projection of the given figure can be completed.

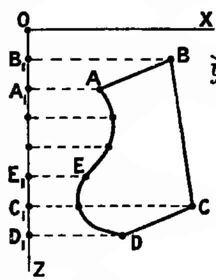


FIG. 569.

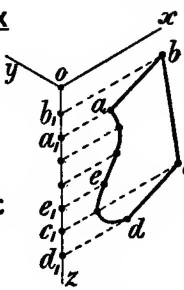


FIG. 570.

260. Projection of a Circle lying in an Isometric Plane.—The method of the preceding article may be applied to find the projections of a sufficient number of points on a circle lying in an isometric plane, and a fair curve drawn through the points thus determined is the projection required. This is shown in Fig. 571, where two diameters of the circle at right angles to one another are taken as two of the three axes. Since, however, the orthographic projection of a circle is an ellipse, a better construction is to find the axes of the ellipse and then determine a sufficient number of points on it by means of a trammel as explained in Art. 45, p. 41.

Referring to Fig. 572, take two diameters of the circle at right angles to one another as the axes OX and OZ. Draw the inscribed square ABCD having two sides parallel to OX and two sides parallel

to OZ. In the isometric projection, *aoc*, the projection of the diameter AOC will evidently be perpendicular to *oy* the third isometric axis.

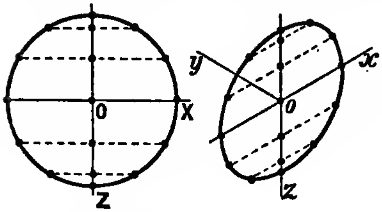


FIG. 571.

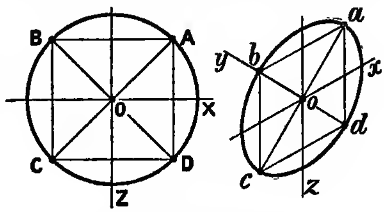


FIG. 572.

aoc is therefore the projection of the horizontal diameter of the circle and will therefore have a length equal to the true diameter. Consequently *aoc* must be the major axis of the ellipse which is the projection of the circle. Also *bod* must be the minor axis of the ellipse. It is easy to show that the angle *bao* is 30°.

Hence, having found *o* the isometric projection of the centre of the circle, the major and minor axes of the ellipse which is the isometric projection of the circle are found as follows.—Draw *aoc* perpendicular to the third isometric axis, and make *oa* equal to *oc* equal to the true radius of the circle. Draw *ab* inclined at 30° to *oa*, or draw *ab* parallel to the isometric axis *ox*. Draw *bod* perpendicular to *aoc* to meet *ab* at *b*. Make *od* equal to *ob*. *aoc* and *bod* are the required axes of the ellipse.

If the ellipse is the projection of a circle connected with an object which is to be projected isometrically without using an isometric scale, then *oa* must be made greater than the true radius of the circle in the ratio of the ordinary scale to the isometric scale, that is, in the ratio of $\sqrt{3} : \sqrt{2}$.

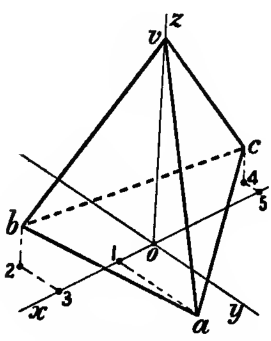


FIG. 573.

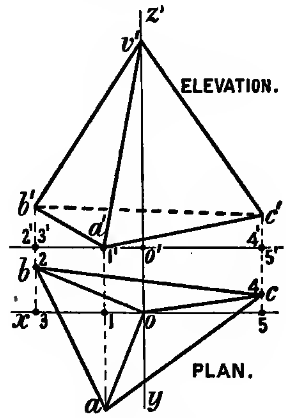


FIG. 574.

261. Isometric Projection of an Object which is not Rectangular.—If three axes, mutually perpendicular, be taken in

relation to any object, the object is said to be projected isometrically when the three selected axes are projected isometrically; but only those lines on the object which are parallel to the selected axes will be projected truly isometrically.

The general construction is to locate a sufficient number of points on the outline of the object by perpendiculars from them on to the planes containing the axes referred to above, and then find the isometric projections of these perpendiculars. For example, take the case of the pyramid shown in Fig. 574 by ordinary plan and elevation. Take a vertical axis OZ through the vertex V and two axes OX and OY at right angles to one another and in the horizontal plane of projection. Fig. 573 shows these three axes projected isometrically and the projection of the pyramid built up on these axes.

262. Axometric Projection.—In *axometric projection* the three principal axes of a rectangular object are not all equally inclined to the plane of projection, and although all those lines which are parallel to one axis have their projections drawn to the same scale, those which are parallel to another axis will, in general, have their projections drawn to a different scale. Thus in the axometric projection of a rectangular box, if no two axes have the same inclination to the plane of projection, one scale is required for measuring its length, another for its breadth, and a third for its depth. These three scales are determined in a manner similar to that for the isometric scale shown in Figs. 566 and 567. The angles at E and F, Fig. 567, must of course be found from the figure corresponding to Fig. 566. The angle at E (Fig. 567) must be made equal to the angle O_1bo, Fig. 566, and the angle at F equal to the angle bO_1o. The scale thus found will be the scale for measuring lines parallel to *ob*.

It will be observed that isometric projection is a particular case of axometric projection.

263. Oblique Parallel Projection.—In *parallel projection* the projectors from the different points of an object to the plane of projection are parallel to one another. In *orthographic projection* the projectors are perpendicular to the plane of projection and are therefore parallel to one another; hence, orthographic projection is a particular case of parallel projection. It is usual, however, to restrict the term *parallel projection* to the case where the parallel projectors are inclined or oblique to the plane of projection. By *parallel projection* is therefore meant *oblique parallel projection.*

A special and useful case of parallel projection is that in which the plane of projection is parallel to one of the principal faces of the object to be projected, and the projectors are inclined at 45° to the plane of projection. It follows that all faces of the solid which are parallel to the plane of projection will have for their projections figures which are of the same size and shape as the faces themselves, also all lines on the object which are perpendicular to the faces just mentioned will have their projections of the same lengths as the lines themselves.

In Fig. 575, *ab* is the ordinary or orthographic plan, and *a′b′* the

corresponding elevation of a rectangular solid placed with two parallel faces parallel to the vertical plane of projection and with two other parallel faces horizontal. The solid being in this simple position in relation to the planes of projection, another projection of it, A′B′, is made on the vertical plane of projection, the projectors being parallel to one another but inclined at 45° to that plane of projection. The elevations of these projectors are shown inclined to XY at 30°, but they may be inclined at any other angle. Having fixed the inclinations of the elevations of the projectors to XY the direction of their plans is found from the further condition that the true inclination of the projectors to the plane of projection is 45°.

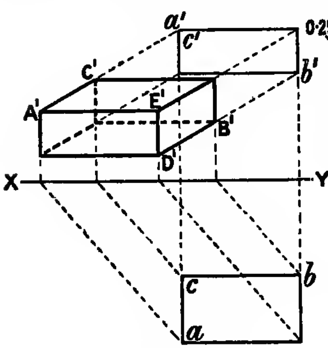


FIG. 575.

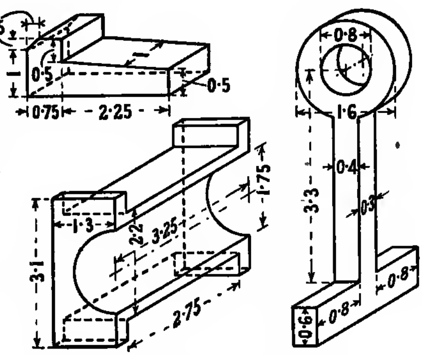


FIG. 576.

It will now be seen that the projection A′B′ may be drawn directly without any preliminary plan and elevation. The rectangle A′D′ is first drawn, being of the same size and shape as the front face of the solid. A′C′ is then drawn at 30° (or any other convenient angle) to A′E′, and its length is made equal to the true length of the edge of which it is the projection. The projection may now be completed by drawing parallel lines as shown.

Fig. 576 shows parallel projections of solids drawn as described above, and the student should draw these, full size, to the dimensions, in inches, marked on them.

If desired, the projections of those lines on the object which are perpendicular to the front face, which is parallel to the plane of projection, may be drawn to a scale which is either smaller or larger than that used for the projection of the front face, and the resulting projection will be a correct parallel projection of the object. This simply means that the inclination of the parallel projectors to the plane of projection is taken either greater or less than 45°. For example, referring to Fig. 577, if the face A′D′ is drawn full size the length A′C′ may be drawn half full size and the complete parallel projection of the solid

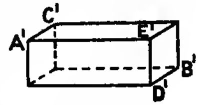


FIG. 577.

will then be as shown. When different scales are used the scales should be stated on the drawing.

When a face of the object perpendicular to the front face contains a figure made up of curved lines, or lines which are not parallel to the edges of that face, the projection of such a figure is determined by using co-ordinates in the manner described in Art. 259 for isometric projection. Also a parallel projection of an object of any form not rectangular may be made by using co-ordinates as described in Art. 261 for isometric projection.

Exercises XXI

Note. *The pictorial projections asked for in the following exercises are to be drawn, as far as possible, directly without first drawing the object in ordinary plan and elevation.*

1–12. Ordinary orthographic projections of various objects are given in Figs. 578 to 589. Represent these objects in isometric projection.

For the dimensions of the objects shown in Figs. 578 *to* 589 *take the sides of the small squares as half an inch.*

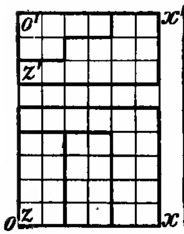


FIG. 578.

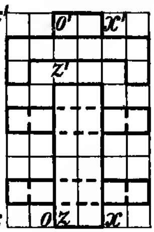


FIG. 579.

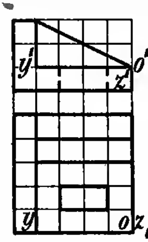


FIG. 580.

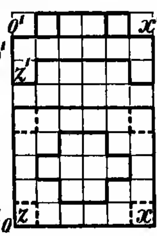


FIG. 581.

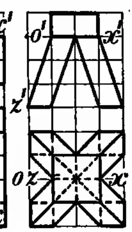


FIG. 582.

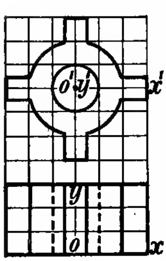


FIG. 583.

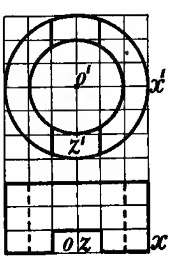


FIG. 584.

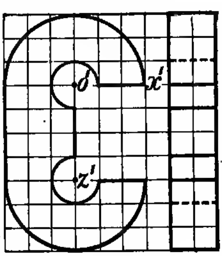


FIG. 585.

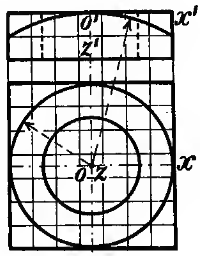


FIG. 586.

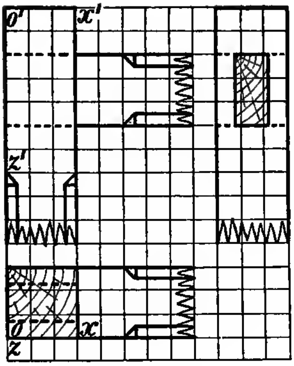


FIG. 587.

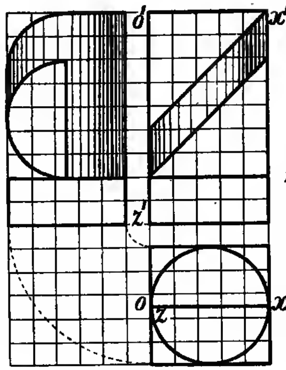


FIG. 588.

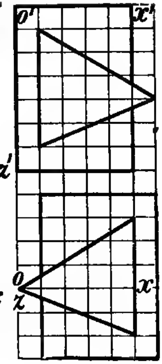


FIG. 589.

13–24. Represent the objects shown in Figs. 578 to 589 in axometric projection, the axes to be arranged as shown in Fig. 590.

25–36. Represent the objects shown in Figs. 578 to 589 in oblique parallel projection, the axes to be arranged as shown in Fig. 591. The scale is to be the same for all the axes of projection.

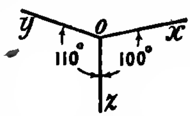


FIG. 590.

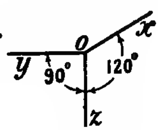


FIG. 591.

37. The axes of two equal cylinders (2 inches in diameter) and their common perpendicular are in isometric projection. Show the intersection of the cylinders without using an auxiliary plan or elevation, (*a*) when their axes intersect, (*b*) when their axes are $\frac{1}{8}$ inch apart where they are nearest together.

38. The same as exercise 37 except that the projection of the axis of one cylinder is parallel to *ox* and the projection of the axis of the other is parallel to *oy*, Fig. 590.

39. Represent the cylinders referred to in exercise 37 in oblique parallel projection. The projection of the axis of one cylinder to be parallel to *ox* and the projection of the axis of the other to be parallel to *oy*, Fig. 591. Show the intersection of the cylinders. The scale is to be the same for all the axes of projection.

40. A solid made up of a hemisphere and part of a circular cylinder is shown in Fig. 592. The centre of the base of the hemisphere is on the axis of the cylinder. Draw an isometric projection of this solid.

41. Fig. 593 shows a locomotive crank axle with the fillets at the junctions of the various cylindrical parts with one another and with the crank arms left out. Draw an isometric projection of this axle to a scale of $1\frac{1}{2}$ inches to a foot.

42. Draw an oblique parallel projection of the crank axle shown in Fig. 593 to a scale of $1\frac{1}{2}$ inches to a foot. The axes of projection to be arranged as shown in Fig. 591, *ox* being the projection of the axis of the axle.

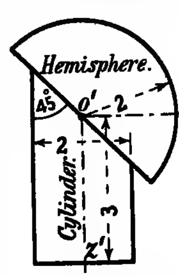


FIG. 592.

43. The relation between the pressure, P (in lb. per square inch), the volume, V (in cubic feet), and the absolute temperature, T (in degrees Fahrenheit) of 1 lb. of air is given by the equation $PV = 0{\cdot}37\ T$. Fig. 594 shows, in oblique parallel projection, contours of this surface, between pressures of 10 and 100 lb. per square inch, at intervals of 500° from 500° to 2500°, the planes of these contours being parallel to the front pressure-

volume plane. Draw this Fig. to the following scales.—Pressure, 1 inch to 20 lb. per square inch. Volume, 1 inch to 4 cubic feet. Temperature, 1 inch to 500°.

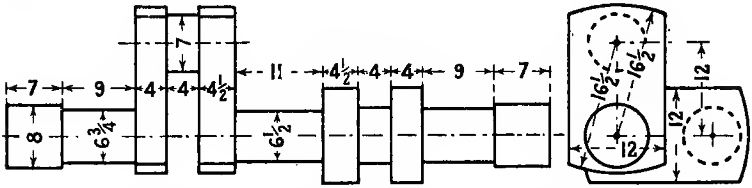


FIG. 593.

Add the contours whose planes are parallel to the bottom volume-temperature plane at intervals of 20 lb. per square inch. Also, add the contours whose planes

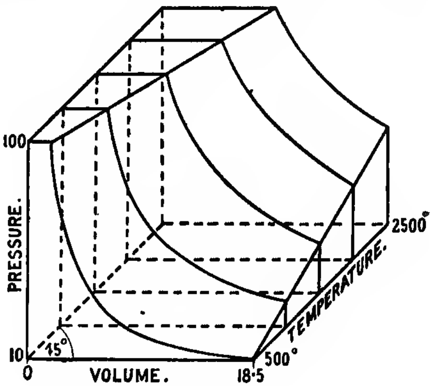


FIG. 594.

are parallel to the left hand pressure-temperature plane at intervals of 4 cubic feet.

Note that the curves in Fig. 594 are rectangular hyperbolas points on which may be found by the construction given in Art. 50, p. 47.

CHAPTER XXII

PERSPECTIVE PROJECTION

264. Difference between Perspective and Orthographic Projection.—In *perspective*, *conical*, or *radial* projection all the projectors converge to a point, while in *orthographic* projection they are perpendicular to the plane of projection. The perspective projection of an object represents it as it would actually appear to the eye of an observer placed at the point to which all the projectors converge.

In perspective projection, the plane of projection, called the *picture plane*, may be situated anywhere but it should meet all the projectors, otherwise the projection or *picture* as it is called would be incomplete. Generally the picture plane is assumed to be placed in a vertical position between the eye of the observer and the object, and perpendicular to the direction in which the observer is looking.

The point to which all the projectors converge is called the *station point* or *point of sight*.

265. Direct Method of drawing a Perspective Projection.—The following method of drawing a perspective projection of an object follows directly from the definition of perspective projection, and it will be seen that it is simply the determination of the plane section of a pyramid or a combination of pyramids which have a common vertex.

First draw the orthographic projections of the object, station point, and picture plane so that the latter is perpendicular to the ground line XY, as shown in Fig. 595, which illustrates the method applied to a square prism standing on the ground. oc, $o'c'$ is the picture plane and ss' is the station point.

Consider the edge ab, $a'b'$ of the prism. Draw the plans sa, sb and the elevations $s'a'$, $s'b'$ of the conical or radial projectors of the extremities of the edge under consideration. These radial projectors meet the picture plane at points of which α and β are the plans and α' and β' the elevations. Through a point o in oc draw on parallel to XY. With centre o and radii $o\alpha$ and $o\beta$ draw arcs of circles to cut on at α_1 and β_1 respectively. From α_1 and β_1 draw perpendiculars to XY to meet lines through α' and β' parallel to XY at A′ and B′ respectively. A straight line A′B′ will be the perspective projection of the edge ab, $a'b'$ of the prism. In like manner the perspectives of the other edges may be obtained as shown.

It is evident that by the first part of the foregoing construction

edge views of the required picture are obtained, one being a plan and the other an elevation, and by the second part of the construction the true shape of this picture is determined.

The points c and c' are the feet of the perpendiculars from s and s' respectively to the edge view of the picture plane. If with centre o and radius oc an arc of a circle be drawn to cut on at c_1, and if c_1C be drawn perpendicular to XY to meet the horizontal through s' at C, then C will be the position, in relation to the picture A′B′, of the foot of the perpendicular from the station point to the picture plane.

The foot of the perpendicular from the station point to the picture plane is called the *centre of vision*. It will be observed that the centre of vision is the orthographic projection of the station point on the picture plane.

The horizontal line drawn on the picture plane through the centre of vision is called the *horizontal line*.

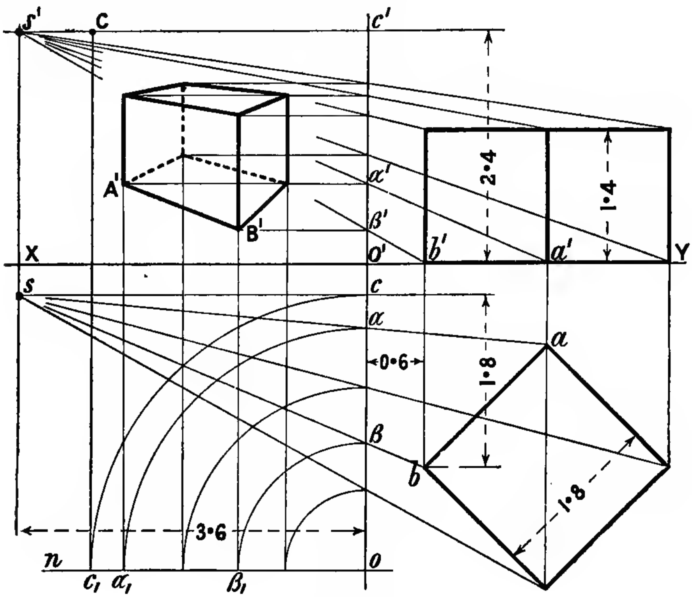


FIG. 595.

The intersection of the picture plane with the ground plane is called the *ground line* or *base line*.

The plane perpendicular to the ground, and containing the station point and centre of vision is called the *vertical plane*.

The line in which the vertical plane intersects the picture plane is called the vertical line.

266. The "School of Art" Method of drawing a Perspective Projection.—The direct method of drawing a perspective

projection, described in the preceding Article, is exceedingly simple in principle, and may be quickly learned, and is easily understood by a student possessing a little knowledge of elementary solid geometry. The direct method may be applied to draw the perspective of any solid in any position, and in some cases it is the best method to use, but if the solid has many parallel lines on it the construction can be much simplified and greater accuracy obtained by making use of a number of principles which are explained in succeeding articles. These principles are applied in what may be called the "School of Art" method of making a perspective drawing. Afterwards a combination of the two methods very commonly used in practice will be described.

267. To determine the Perspective of a Given Point.—The point is given by stating its distances from the ground and from the vertical and picture planes. The first and second of these distances determine the position of the orthographic projection of the point on the picture plane.

In Fig. 596 the plane of the paper is the picture plane, C is the centre of vision, CH is the horizontal line, CL is the vertical line, and p' is the orthographic projection of the given point on the picture plane. On CH make CD equal to the distance of the station point from the picture plane. Draw p'K parallel to CH, and make p'K equal to the distance of the given point P from the picture plane. Join DK and p'C. The point P′ where these lines intersect is the perspective of the point P. It is evident that CP′ : P′p' : : CD : p'K.

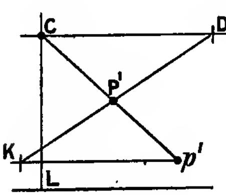


Fig. 596.

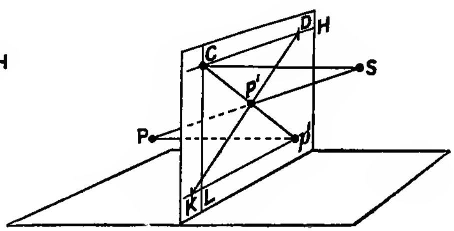


Fig. 597.

Referring now to Fig. 597 which is a distorted perspective projection of the lines of Fig. 596 together with the point P and the station point S. Since SC and Pp' are perpendicular to the picture plane they are therefore parallel to one another, and the triangles SCP′ and Pp'P′ are similar. Hence CP′ : P′p' : : SC : Pp' ; but CD is equal to SC and p'K is equal to Pp', therefore CP′ : Pp' : : CD : p'K. But P′ is the point where SP cuts the picture plane and is therefore the perspective of the point P.

The point D, Fig. 596, is called the *point of distance.*

268. The Perspectives of Parallel Lines converge to a Point.—Let A′B′ and C′D′ (Fig. 598) be the perspectives of the parallel lines AB and CD respectively. Let SV be a line parallel to

AB and CD, and if it meet the picture plane at all, let it meet it at V. SV will be the line of intersection of the planes ABS and CDS.

Since each of the points A′, B′, and V are in the picture plane, and also in the plane ABSV they must all lie on the intersection of these two planes; therefore A′, B′, and V are in the same straight line. Similarly C′, D′, and V are in a straight line; therefore A′B′ and C′D′, the perspectives of the parallel lines AB and CD, converge to the point V.

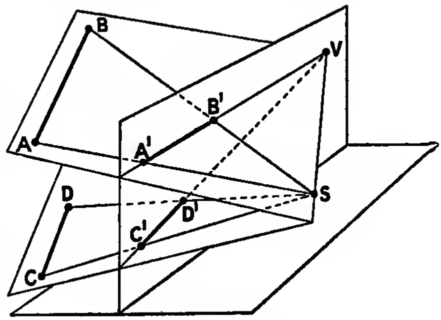


FIG. 598.

The point towards which the perspectives of parallel lines converge is called the *vanishing point* of these lines.

The following particular cases are worthy of notice:—

(*a*) When the lines are perpendicular to the picture plane, SV will also be perpendicular to that plane, and V will coincide with the centre of vision. Hence, *the perspectives of lines which are perpendicular to the picture plane, converge towards the centre of vision.*

(*b*) When the parallel lines are parallel to the picture plane, SV will also be parallel to that plane; therefore V will be at an infinite distance from S, and A′B′ and C′D′ will be parallel to one another. Hence, *the perspectives of parallel lines which are also parallel to the picture plane, are parallel to one another.*

(*c*) When the parallel lines are horizontal, SV will also be horizontal; therefore V must be on the horizontal line. Hence, *the perspectives of parallel horizontal lines converge towards a point on the horizontal line.*

(*d*) When the parallel lines are parallel to the vertical plane, SV will be in the vertical plane; therefore V will be on the vertical line. Hence, *the perspectives of parallel lines which are also parallel to the vertical plane, converge towards a point on the vertical line.*

(*e*) *The perspectives of vertical lines are parallel to the vertical line.*

(*f*) *The perspectives of horizontal lines which are parallel to the picture plane are parallel to the horizontal line.*

(*g*) *The perspectives of parallel horizontal lines which are inclined at* 45° *to the picture plane converge towards the point of distance.*

269. To Determine the Vanishing Point of a Line.—Take the picture plane and the horizontal plane containing the station point for co-ordinate planes, and on these draw a plan *ab* and elevation *a′b′* of the line whose vanishing point is required (Fig. 599). HC, the horizontal line, will be the ground line for this plan and elevation.

Draw C*s* perpendicular to HC and equal to the distance of the

station point from the picture plane, s will be the plan of the station point and C its elevation. Through s draw sv parallel to ab, meeting HC at v. Through C draw CV parallel to $a'b'$ to meet the perpendicular from v to HC at V. The point V is the required vanishing point, because V is the point in which a line through the station point parallel to the original line AB meets the picture plane.

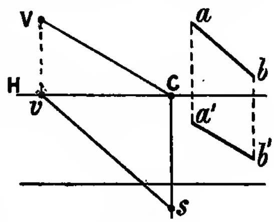


FIG. 599.

If the original line is horizontal CV will coincide with HC, the horizontal line; therefore V is on the horizontal line, and the plan sv (or sV) makes an angle with the horizontal line equal to the inclination of the original line to the picture plane.

270. To mark off on the Perspective of a Line a Part whose True Length is given.—Let TV (Fig. 600) be the perspective of a line whose trace on the picture plane is T, and whose vanishing point is V. As in Fig. 599, s is the plan of the station point, and sv the plan of the line which passes through the station point, and is parallel to the line of which TV is the perspective.

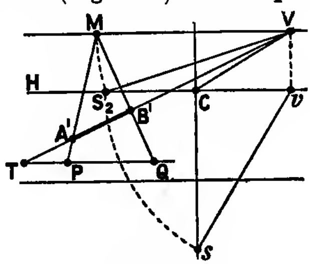


FIG. 600.

With centre v and radius vs describe the arc sS_2 meeting the horizontal line HC at S_2. VS_2 is the real distance of the vanishing point V from the station point. Through V draw VM parallel to HC, and with centre V and radius VS_2 describe the arc s_2M.

Draw TQ parallel to HC. Let A′ be a given point in TV. Draw MA′ and produce it to meet TQ at P. Make PQ equal to the given length. Draw QM meeting TV at B′. The true length of the part whose perspective is A′B′ will be equal to the given length.

The construction is much simplified when the original line is horizontal, for then V is on the horizontal line, and sv is equal to the true distance of V from the station point The construction for this case is shown in Fig. 601.

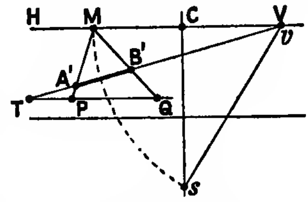


FIG. 601.

The theory of the construction shown in Figs. 600 and 601 will be understood by reference to Fig. 602, which is a perspective view of the ground and picture planes, and the various lines in their natural positions.

In Figs. 600, 601, and 602, the same letters denote the same points. S is the station point, TAB the line whose perspective is TV and V is its vanishing point. T is the trace

of AB on the picture plane. SV is parallel to TAB, and the perspectives of all lines which are parallel to TAB or SV will pass through V as already explained in Art. 268, and illustrated by Fig. 598. Similarly, the perspectives of all lines which are parallel to SM will pass through M, and conversely, lines whose perspectives pass through M, and which are in the same plane, will be parallel to SM. Now AP and BQ are lines whose perspectives A′P and B′Q pass through M, and they are in the same plane, therefore AP and BQ are parallel to SM.

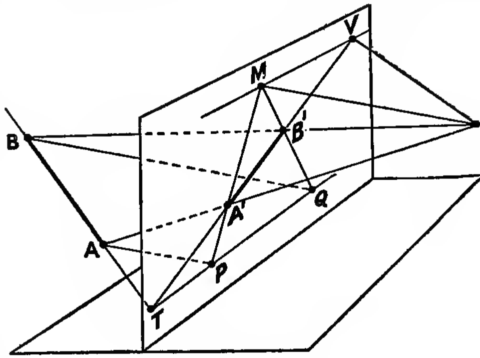


FIG. 602.

Comparing the triangles SVM and BTQ, TB is parallel to VS, TQ is parallel to VM, and BQ is parallel to SM, therefore the triangles are similar. But VM is equal to VS, therefore TB is equal to TQ, and it follows that TA is equal to TP since AP is parallel to BQ, therefore AB is equal to PQ. But PQ is a line in the picture plane, therefore if PQ is made equal to the given length, the construction given will determine the perspective of the part required.

The point M in Figs. 600, 601, and 602 is called a *measuring point* for the vanishing point V. It is evident that each vanishing point can have two measuring points, one to the right and another an equal distance to the left of the vanishing point.

When the original line AB is parallel to the picture plane the foregoing construction fails because the points T and V are "at infinity." A simple and convenient construction in this case is shown in Fig. 603 where A′E′ is the direction of the perspective, and A′ is the point in it from which the part is to be marked off. Take any point M on the horizontal line. Join MA′ and produce it to meet at P a horizontal line PR on the picture plane which is at a height above the ground line equal to the height of A above the ground. Draw PQ parallel to A′E′, and make PQ equal to the given length. Join MQ meeting A′E′ at B′. A′B′ is the part required. The theory of this construction is as follows. PM and QM are the perspectives of parallel horizontal lines passing through the points A and B and meeting the picture plane at P and Q. The figure ABQP in space is a parallelogram and AB is equal to PQ.

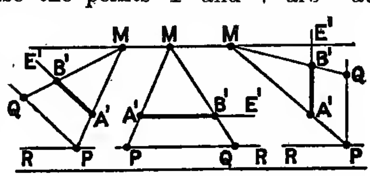


FIG. 603.

But PQ is in the picture plane and may therefore be measured directly.

271. Examples.—(1) *A square prism,* 2 *inches* × 2 *inches* × 4 *inches, rests with one rectangular face on the ground. Nearest corner* ½ *inch to left of vertical plane and* 1 *inch behind picture plane. Long edges inclined at* 50° *to picture plane and vanish to left. Station point* 3½ *inches above ground and* 6¼ *inches from picture plane.*

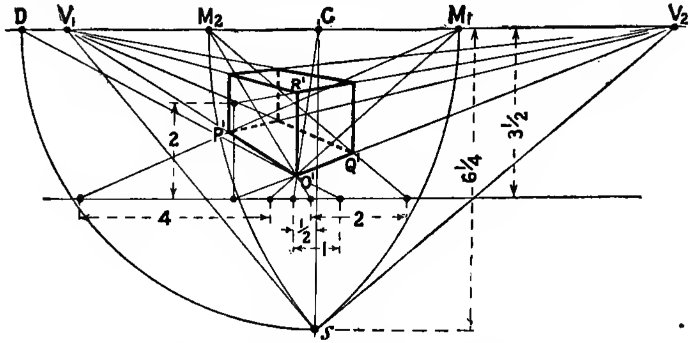


FIG. 604.

Fig. 604 shows this example worked out by the "School of Art" method. V_1 and V_2 are the two vanishing points for the horizontal edges, determined by the construction explained in Art. 269, and M_1 and M_2 are the corresponding measuring points (see Art. 270). D is the point of distance (Art. 267).

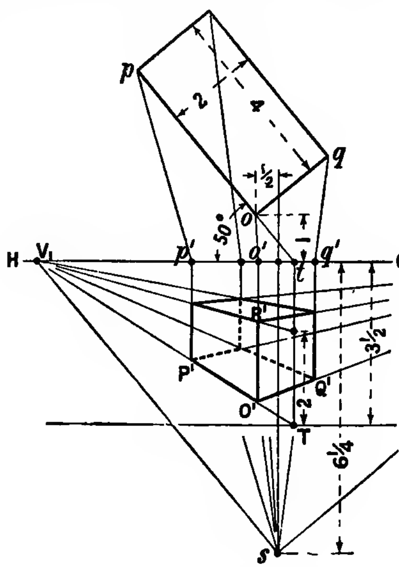


FIG. 605.

The point O′, the perspective of the nearest corner on the ground, is first determined (Art. 267). Joining O′ to V_1 and V_2 determines the directions of the perspectives of the horizontal edges meeting at O. The lengths O′P′ and O′Q′ of the perspectives of these edges are then determined (Art. 270). A vertical line through O′ will be the direction of the perspective of the vertical edge which has its lower extremity at O. The length O′R′ is marked off by the construction explained in the latter part of Art. 270. The remainder of the

construction is obvious. All the construction lines are shown in the figure.

Fig. 605 shows the same example worked out by a combination of the "School of Art" method and the direct method explained in Art. 265. HC, the horizontal line, is taken as a plan of the picture plane, and a plan *pq* of the solid is drawn, the solid being placed in its proper position in relation to the picture plane and vertical plane. *s* is the plan of the station point. Lines from the angular points of the plan to *s* intersect HC at points which are the plans of the angular points of the picture on the picture plane. Vertical lines through these points on HC will contain the angular points of the picture which is to be drawn below HC.

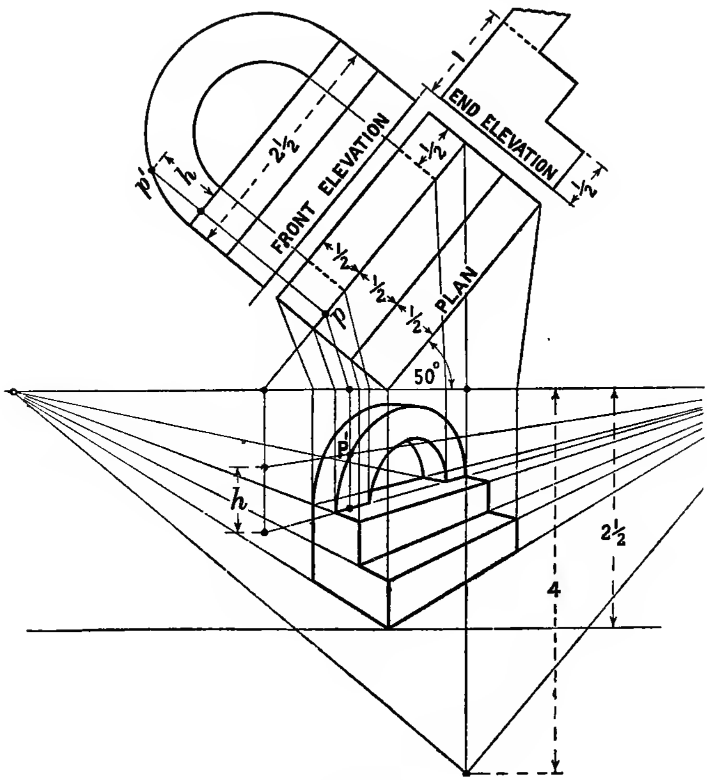


FIG. 606.

The vanishing points V_1 and V_2 (V_2 is outside the figure) are determined as before. Produce *po* to meet HC at *t* and through *t* draw the vertical *t*T to meet the ground line of the picture plane at T. T is obviously the perspective of the point where PO produced

meets the picture plane, and is also the point itself. The line TV_1 will therefore contain the perspective of OP. The points O′ and P′ are determined by vertical lines from o' and p' to meet TV_1 as shown. Joining O′ to V_2 determines the line containing the perspective of OQ, and the point Q′ is found by dropping a vertical line from q' to meet $O'N_2$.

The perspectives of the vertical edges are of course vertical lines. The length O′R′ is determined as in Fig. 604 and the remainder of the construction is obvious.

(2) A second example is shown worked out in Fig. 606 by a combination of the "School of Art" and direct methods. The perspectives of the circular arcs are of course determined by first finding the perspectives of a number of points in them and then joining by fair curves. The construction lines for a point P′ on one of the curves are fully shown, others are omitted for the sake of giving clearness to the figure.

272. Shadows in Perspective.—When a solid which is drawn in perspective casts a shadow, the perspective of the shadow may be drawn in the ordinary way from the shadow as determined from the orthographic projections of the solid as shown in Chap. XXVII., and in cases where the shadow involves the intersection of curved surfaces this is generally the simplest method to adopt. In many cases however the perspective of the shadow can easily be determined from the perspective of the solid, by applying the constructions explained in the remainder of this article.

It will only be necessary to consider the perspective of the shadow of a point. At first all the light will be assumed to come from a point, the modifications of the constructions for parallel rays of light being afterwards described.

Let P′ be the perspective of the point from which all the rays of light diverge, and let A′ be the perspective of a point whose shadow is to be determined. The positions of the points P and A in space are supposed to be known.

(*a*) *To determine the perspective of the shadow cast by the point A on the ground.* Let M′ and N′ (Fig. 607) be the perspectives of the feet of the perpendiculars from A and P on the ground. Join M′N′ and produce it to meet P′A′ produced at Q′. Q′ is the perspective of the trace of PA on the ground, and is therefore the perspective of the shadow of A on the ground.

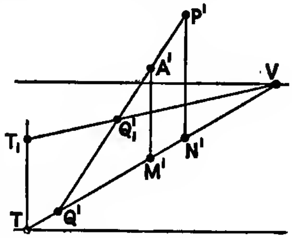


Fig. 607.

(*b*) *To determine the perspective of the shadow cast by A on a horizontal plane which is at a given height above the ground.* Produce M′N′ (Fig. 607) to meet the ground line at T and the horizontal line at V. Draw the vertical line TT_1 and make TT_1 equal to the given height of the horizontal plane above the ground. Draw T_1V to cut P′A′ produced at Q_1'. Q_1' is the perspective

of the shadow required. TV and T_1V are the perspectives of parallel horizontal lines which are in a vertical plane containing PA, therefore Q_1' is the perspective of the trace of PA on a horizontal plane at a height above the ground equal to TT_1.

(*c*) *To determine the perspective of the shadow cast by A on a given vertical plane.* Let EF (Fig. 608) be the perspective of the trace of the given vertical plane on the ground, and let EF cut M′N′ produced at R′. Draw the vertical line R′Q′ to meet P′A′ produced at Q′. Q′ is the perspective of the shadow required. R′Q′ is evidently the perspective of the intersection of the given vertical plane with the vertical plane containing PA, therefore Q′ is the perspective of the trace of PA on the given vertical plane.

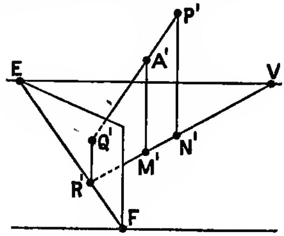


FIG. 608.

(*d*) *To determine the perspective of the shadow cast by A on a given inclined plane.* Let EF (Fig. 609) be the perspective of the trace of the given plane on the ground and let FF_1 be its trace on the picture plane. Produce M′N′ to meet the ground line at T and the horizontal line at V. Draw the vertical line TT_1. At any convenient height draw the horizontal line T_1F_1 to cut TT_1 at T_1 and FF_1 at F_1. Join T_1V and F_1E and let these lines meet at U′. Let TV meet EF at R′. Join R′U′. Produce P′A′ to meet R′U′ at Q′. Q′ is the perspective of the shadow required.

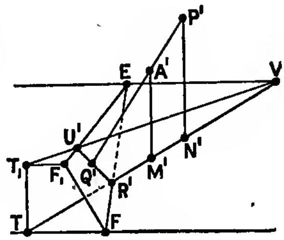


FIG. 609.

T_1V and F_1E are the perspectives of the lines in which a horizontal plane, whose trace on the picture plane is T_1F_1, intersects the vertical plane PM, and the given inclined plane respectively. U′ is therefore the perspective of a point in each of these planes; but R′ is the perspective of another point in each of these planes, therefore R′U′ is the perspective of the line of intersection of these planes. Hence Q′ is the perspective of the trace of PA on the given inclined plane.

When the rays of light are parallel their perspectives will converge to their vanishing point which is determined by the construction explained in Art. 269. This vanishing point may then be taken as the perspective of a luminous point from which the perspectives of all the rays of light diverge, and the perspectives of the shadows are determined as already explained. It should be observed that in this case the perspective of the foot of the perpendicular on the ground from the luminous point (now at infinity) is on the horizontal line.

Exercises XXII.

1. Represent in perspective a point A on the ground plane, 1 foot to the right of the spectator and 2 feet from the ground line; also a point B 4 feet to the left of the spectator, 2 feet from the picture plane, and 3 feet above the ground plane. Join AB. AB is the nearest side of a rectangle of which two other sides are horizontal and 4 feet long. Complete the perspective representation of the rectangle. Position of eye, 11 feet from picture plane and 5 feet from ground. Scale $\frac{1}{2}$ inch to 1 foot. [B.E.]

2. A rectangular stone slab, 10 feet long, 8 feet wide, and 6 inches thick, lies with one face on the ground. The longest edges recede from the picture at an angle of 60° towards the right, and the nearest corner is 1 foot to the left of the spectator, and 2 feet from the ground line. Through the centre of the slab is cut a circular hole, 6 feet in diameter. Represent the slab in perspective. Position of eye and scale as in exercise 1. [B.E.]

3. Two rectangular blocks are shown at Ex. 3, Fig. 612. The lower block lies on the ground. Draw these blocks in perspective projection. The corner aa' is to be $\frac{1}{2}$ inch to the left of the spectator, and $\frac{1}{2}$ inch from the ground line. The edge ab, $a'b'$ is to recede towards the right at an angle of 60° to the picture plane. Position of eye, $5\frac{1}{2}$ inches from the picture plane and $2\frac{1}{2}$ inches above the ground.

4. A solid letter R is shown at Ex. 4, Fig. 612. Draw this in perspective. The corner aa' is to be on the ground 1 inch to the right of the spectator and 3 inches from the ground line. The face of the letter is to be in a vertical plane which is to recede from the picture plane at an angle of 50° towards the left. Position of eye, $5\frac{1}{2}$ inches from the picture plane and $2\frac{1}{2}$ inches above the ground.

5. Put into perspective a point on the ground plane 5 feet to the right of the spectator, and 3 feet within the picture, and, receding from the point, draw a line 10 feet long at right angles with the picture plane. This line is to be the lowest edge of a square slab 1 foot thick, having its square faces inclined to the ground at 60° towards the right. On the upper face of the slab inscribe a circle, and, centrally with it, draw another circle 4 feet in diameter, which is to be the base of a right cone having an altitude of 2 feet. The eye is to be 14 feet from the picture plane, and 8 feet from the ground. Scale, $\frac{1}{2}$ inch to a foot. [B.E.]

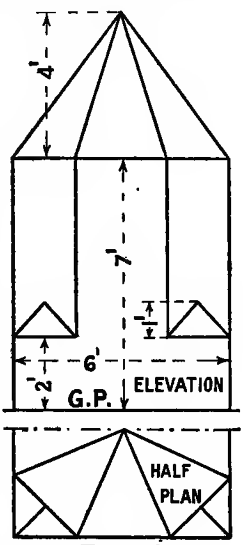


Fig. 610.

6. A right prism $3\frac{1}{2}$ inches high has its base, which is a regular octagon of 1 inch side, on the ground. A cylindrical slab 4 inches in diameter and 1 inch thick rests on the prism, the axes of the two solids being in line. Represent the two solids in perspective. One corner of the base of the prism is to be $\frac{1}{2}$ inch to the right of the spectator and 2 inches from the ground line, and one of the edges of the base adjacent to that corner is to vanish towards the right at an angle of 50° with the ground line. Height of eye above the ground, $2\frac{1}{2}$ inches. Distance of eye from picture plane, 6 inches.

7. Fig. 610 gives the elevation and half plan of an object standing on the ground plane. Put the whole object into perspective, with the sides of its base inclined at angles of 45° to the picture plane, and its nearest vertical edge 2 feet to the right of the spectator, and 2 feet within the picture. The eye is to be 5 feet from the ground, and 12 feet from the picture. Scale $\frac{1}{2}$ inch to a foot. [B.E.]

8. A vertical section of a square ceiling is shown in Fig. 611, the upper part being of the form of a shallow square pyramid. Put the ceiling into perspective with its horizontal edges inclined to the picture plane at angles of 30° to the right and 60° to the left, its nearest corner being 12 feet vertically over a point on the ground plane 3 feet to the left of the spectator, and 1 foot within the picture. The eye is to be 12 feet from the picture, and 5 feet from the ground. Scale $\frac{1}{2}$ inch to a foot. [B.E.]

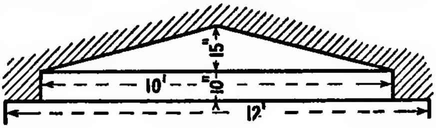


FIG. 611.

9. Draw, by the method of Art. 265, the perspective projection of the solid shown at Ex. 9, Fig. 612. The edge AB is to be on the ground and parallel to the picture plane. The axis of the solid is to be inclined at 50° to the ground, and the nearest point C is to be in the picture plane, and $1\frac{1}{2}$ inches to the right of the spectator. Station point, 4 inches from the picture plane, and 2 inches above the ground.

10. Draw the perspective projection of the solid shown at Ex. 10, Fig. 612. The solid is to stand on the ground with the edge AB in the picture plane, and $1\frac{1}{2}$ inches to the right of the spectator. The face ABC is to be inclined at 40°

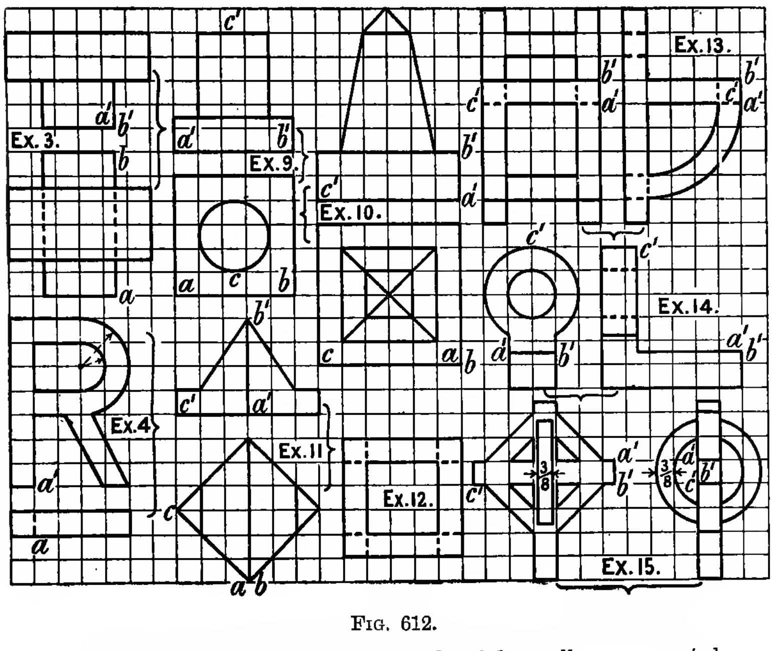


FIG. 612.

In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

to the picture plane. Height of station point 2 inches. The position of the centre of vision, and the distance of the station point from the picture plane is to be determined from the further condition that the distance between the vanishing points for the horizontal edges is to be $7\frac{1}{2}$ inches.

11. Draw the perspective of the solid shown at Ex. 11, Fig. 612. The base of the solid is to be on the ground, and the nearest edge AB ¾ inch behind the picture plane and 1½ inches to the right. The face ABC is to be inclined at 30° to the picture plane. Station point 3 inches from picture plane and 2½ inches from ground.

12. A skeleton cube is shown at Ex. 12, Fig. 612. Draw this object in perspective when the nearest vertical edge is in the picture plane, and 1 inch to the left, and a face containing that edge is inclined at 35° to the picture plane. The station point to be 3½ inches from the picture plane and 1 inch above the top face of the object.

13. Draw the perspective of the object shown at Ex. 13, Fig. 612. The edge AB to be vertical, and in the picture plane, and ½ inch to the left. The face ABC to be inclined at 30° to the picture plane. Station point to be 4 inches from the picture plane and 1½ inches above the point B.

14. Draw the perspective of the object shown at Ex. 14, Fig. 612. The edge AB to be vertical and in the picture plane, and directly opposite to the station point. The point C also to be in the picture plane. Station point, 1½ inches above the level of point C, and 3½ inches from the picture plane.

15. A gable cross is shown at Ex. 15, Fig. 612; draw this in perspective. The edge AB to be vertical and in the picture plane and ½ inch to the left. The face ABC to be inclined at 45° to the picture plane. Station point, 1½ inches below the point A, and 4 inches from the picture plane.

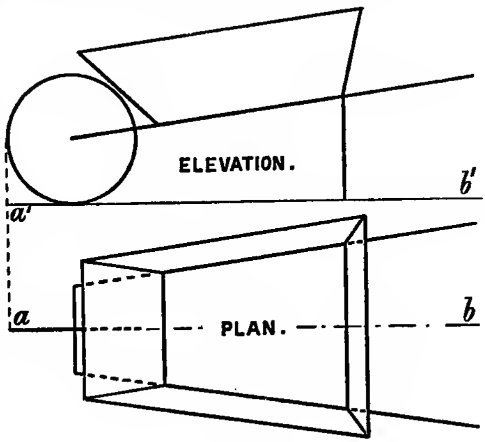


Fig. 613.

16. The plan and elevation of a wheelbarrow standing on the ground are shown in Fig. 613 to the scale, ½ inch to a foot. Represent this object in perspective using the scale, 1 inch to a foot. The line AB is the horizontal trace of the central vertical plane of the wheelbarrow. The point A is to be 1 foot to the left of the spectator and 1 foot from the ground line, and the line AB is to vanish towards the right at 40° to the picture plane. The eye is to be 6 feet, by scale, from the picture plane and 2½ feet above the ground plane. [B.E.]

17. Referring to exercise 14, let the station point be moved 1½ inches nearer to the object, and let the picture plane be moved parallel to itself a distance of 5 inches, so that the station point comes between the solid and the picture plane.

18. Work the example shown in Fig. 606, p. 307, to the dimensions given. Then take a point 3 inches to the left, 5 inches above the ground, and 3 inches behind the picture plane. Consider this as a luminous point and determine the perspective of the shadow cast by the solid on itself and on the ground; all the rays of light to come from the luminous point.

CHAPTER XXIII

CURVED SURFACES AND TANGENT PLANES

273. Generation of Surfaces.—Surfaces may be considered as generated by a line, straight or curved, moving in a definite manner. Thus, a plane may be generated by a straight line moving parallel to one fixed straight line and in contact with another fixed straight line. Again, a sphere may be generated by the revolution of a semicircle about its diameter which remains stationary.

The moving line which generates a surface is called the *generating line* or *generatrix* of the surface, and a line which serves to constrain or direct the motion of the generatrix is called a *directrix.*

The same surface may be generated in numerous ways, but generally there are only a few simple ways in which a surface may be generated. Take the case of the surface of a right circular cylinder. There are two simple ways in which this surface may be generated: (1) by a straight line moving in contact with a fixed circle to the plane of which it remains perpendicular, as shown by the oblique projection in Fig. 614, where the moving line is shown in twelve different positions: (2) by a circle moving so that its centre remains on a fixed straight line to which the plane of the circle is always perpendicular as shown by the oblique projection in Fig. 615, where XX is the fixed straight line and 1, 2, 3, 4, and 5 are positions of the moving circle. This surface may however be generated by an ellipse which moves so that all points on it travel along parallel lines, but the ellipse must be such

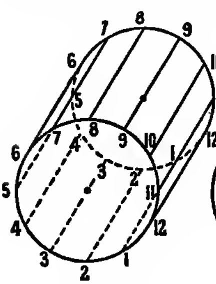


FIG. 614.

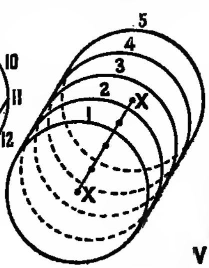


FIG. 615.

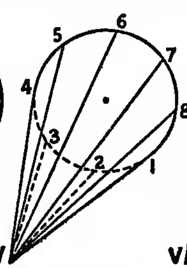


FIG. 616.

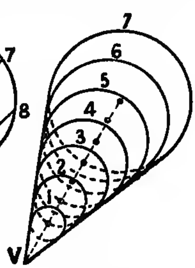


FIG. 617.

that its projection on a plane at right angles to the direction of its motion is a circle.

The two simple ways in which the surface of a right circular cone may be generated are: (1) by a straight line which passes through a fixed point (the vertex of the cone) and moves in contact with the circle which is the base of the cone as shown in Fig. 616: (2) by a circle of changing radius which moves with its centre on the axis and its plane perpendicular to the axis, the radius of the circle being proportional to its distance from the vertex of the cone as shown in Fig. 617.

A surface which may be generated by a line, straight or curved, revolving about a fixed straight line is called a *surface of revolution.* The fixed straight line about which the generating line revolves is called the *axis* of the surface. Sections of a surface of revolution by planes at right angles to its axis are circles. Sections by planes containing the axis are called *meridian sections.* All meridian sections are exactly alike.

A surface which may be generated by the motion of a straight line is called a *ruled surface.* Ruled surfaces may be divided into two classes—*developable surfaces* and *twisted surfaces.* A developable surface may be folded back on one plane without tearing or creasing at any point. The generating line of a developable surface moves in such a manner that any two of its consecutive positions are in the same plane. All ruled surfaces which are not developable are twisted surfaces.

274. Plane Sections of Curved Surfaces.—The way in which a curved surface is generated being known the projections of the generating line in any number of positions can be drawn. The intersections of a given plane with the generating line in each of these positions can then be determined. This will give a number of points on the intersection of the plane with the surface, and a fair curve through them will be the complete intersection required.

When the curved surface can be generated in a number of simple ways, that mode of generation should be made use of which has the projections of its generating line the simplest possible.

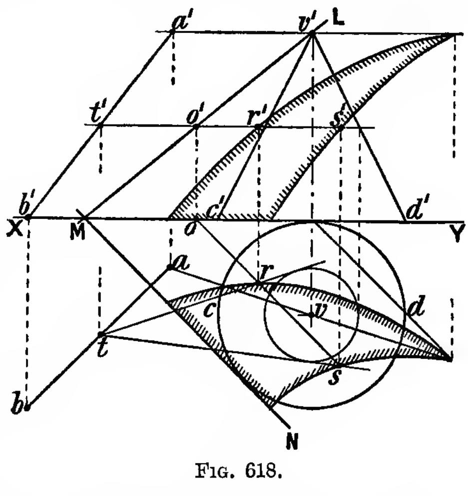


FIG. 618.

EXAMPLE 1. A surface is generated by a horizontal line which moves in contact with the line AB (Fig. 618) and the surface of the cone VCD. To find the section of this surface by the plane LMN.

Take a section of the given cone and plane by a horizontal plane cutting ab, $a'b'$ at tt'. The plan of the section of the cone is a circle and tangents to this circle from t are the plans of two positions of the generating line. The points r and s in which these tangents cut os the plan of the intersection of the assumed plane of section with the plane LMN are the plans of points on the section required. Projectors from r and s to meet the horizontal through t' determine r' and s'. In a similar manner any number of points on the required section may be found.

EXAMPLE 2. Referring to Fig. 619, ab, $a'b'$ is a horizontal circle 3 inches in diameter. cd, $c'd'$ is another horizontal circle $1\frac{1}{2}$ inches

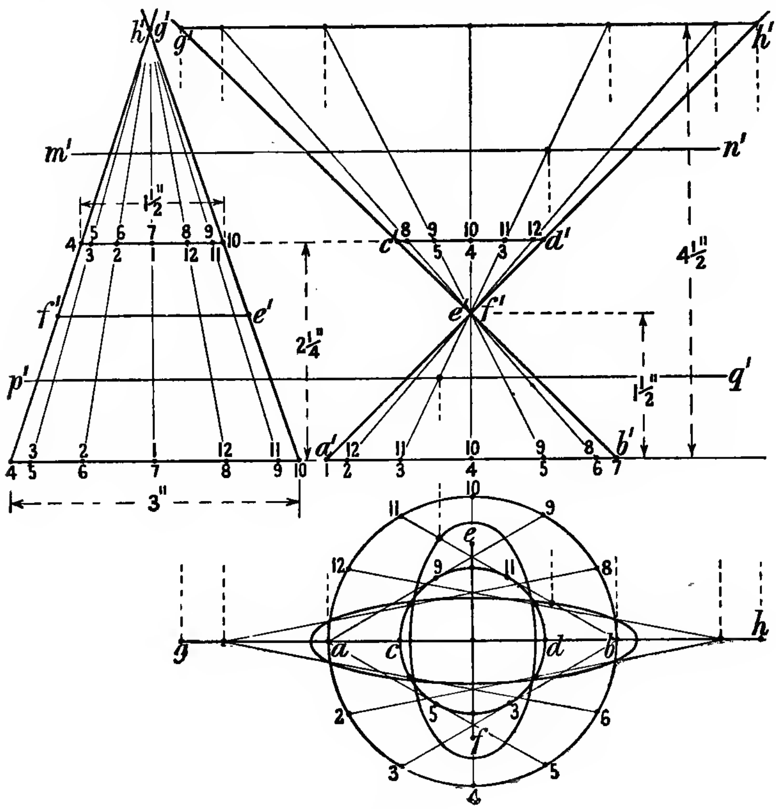


FIG. 619.

in diameter. The line joining the centres of these circles is vertical and $2\frac{1}{4}$ inches long. Two points move, one on each circle, with equal angular velocities in opposite directions. A surface is generated by a straight line which contains the above mentioned moving points.

Twelve positions of the generating line are shown in plan and in two elevations.

It will be found that horizontal sections of the surface at levels $1\frac{1}{2}$ inches and $4\frac{1}{2}$ inches above the plane of the larger circle are straight lines ef, $e'f'$ and gh, $g'h'$ respectively. These straight lines are of definite lengths and are at right angles to one another.

It will also be found that horizontal sections at levels other than that of the circles or the straight lines are ellipses. Two of these elliptic sections are shown, one at the level $m'n'$ and the other at the level $p'q'$. The student should work out this example, full size.

The surface described in this example is one that occurs in the science of optics. It is obvious that the surface may also be generated by a straight line moving in contact with the lines ab, $a'b'$, cd, $c'd'$ and one of the circles or one of the ellipses.

EXAMPLE 3. A surface is generated by a circle whose plane is horizontal and whose centre moves along the straight line AD (Fig. 620). The radius of the circle varies so that the circle

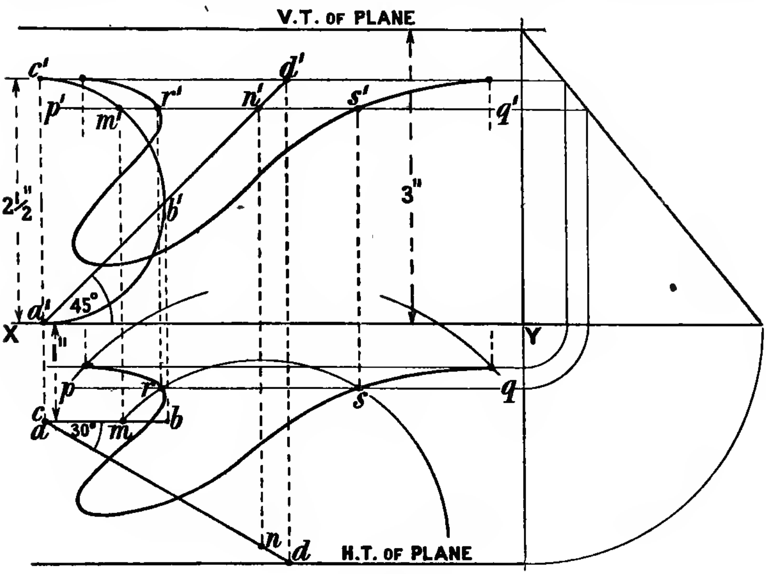


FIG. 620.

intersects the semicircle ABC. It is required to find the section of this surface by the given plane which is parallel to XY.

Take a horizontal plane intersecting the arc ABC at M, the straight line AD at N and the given plane in the straight line PQ. With n as centre and radius nm describe a circle to cut pq at r and s. Perpendiculars to XY from r and s to meet $p'q'$ determine the points

r' and s'. rr' and ss' are points on the section required, and in like manner any number of points may be found.

It is evident that the circle with n as centre and nm as radius is the plan of the generating circle when it is at the level $p'q'$.

275. Intersection of Straight Line and Curved Surface.—Assume a plane to contain the straight line and determine the intersection of this plane with the curved surface by the method described in the preceding article. The intersection of the straight line with the intersection of the assumed plane and the curved surface will be the intersection required. In selecting a plane to contain the straight line choose one whose intersection with the curved surface will have the simplest possible projections or projections which are easiest to determine.

276. Tangent Planes.—If through a given point on a curved surface any two lines be drawn on that surface, the plane containing the tangents to these lines through the given point is the *tangent plane* to the surface at that point. If a straight line can be drawn on the curved surface through the given point, as can be done on all ruled surfaces, the tangent plane at the given point will contain this line.

The *normal to a surface* at a point on it is the perpendicular to the tangent plane at that point.

277. Tangent Plane to a Cone at a Given Point on its Surface.—V (Fig. 621) is the vertex of the cone and P the given

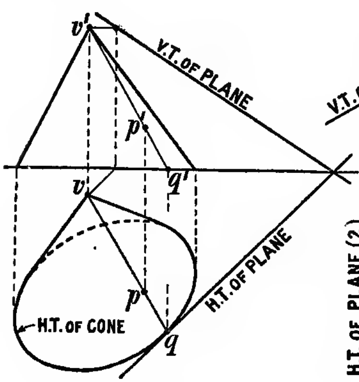


FIG. 621.

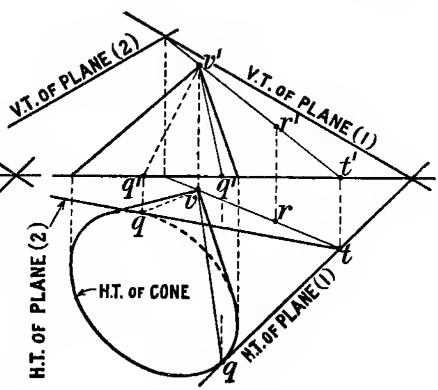


FIG. 622.

point on its surface. Join VP and produce it, if necessary, to meet the horizontal trace of the cone at Q. Through Q draw a tangent to the horizontal trace of the cone. This tangent will be the horizontal trace of the required plane. The vertical trace of the plane can be determined from the condition that the plane contains the vertex of the cone. The straight line VQ is obviously the line of contact between the cone and plane.

278. Tangent Plane to a Cone through a Given External Point.—V (Fig. 622) is the vertex of the cone and R the given point. Join VR. The required plane will contain the straight line VR, and the horizontal trace of the plane will pass through T the horizontal trace of VR. The tangent TQ to the horizontal trace of the cone is the horizontal trace of the required tangent plane and its vertical trace may be found from the condition that the plane contains the straight line VR.

If Q is the point of contact of the tangent from T to the horizontal trace of the cone, then the straight line VQ is the line of contact between the cone and plane.

When T the horizontal trace of VR falls outside the horizontal trace of the cone there will be two tangent planes to the cone passing through the given point R. When T falls on the horizontal trace of the cone there will be one tangent plane only containing the given point. If T falls inside the horizontal trace of the cone the problem is impossible but in that case the point R would be inside the cone.

[*Note.* In the preceding articles on tangent planes to the cone, use has been made of the horizontal trace of the cone; but in practice it may often be more convenient to take a vertical trace of the cone, and first determine the vertical trace of the tangent plane, which will be a tangent to the vertical trace of the cone. For example, if the base of the cone is circular and in a vertical plane, take this vertical plane, or a plane parallel to it, as one of the planes of projection and use the trace of the cone on this plane in determining the tangent plane. These remarks also apply to problems which follow on tangent planes to cones and cylinders.]

279. Tangent Plane to a Cone Parallel to a Given Straight Line.—Through the vertex of the cone draw a line parallel to the given line. By the construction described in the preceding article, determine the planes to contain the former line and touch the cone. These planes are the planes required. In the particular case where the given line is parallel to a generating line of the cone, there will be only one tangent plane to the cone parallel to the given line. If the line through the vertex of the cone parallel to the given line falls within the cone, the problem is impossible.

280. Tangent Planes to Right Circular Cones.—The methods described in the preceding articles on tangent planes to the cone are applicable to any form of cone, it being assumed that the trace of the surface of the cone on one of the planes of projection is given. If a trace of the surface of the cone is not given there would necessarily be sufficient data given to enable a trace on one of the planes of projection to be found.

When the cone is a right circular cone and its axis is inclined to one of the planes of projection its trace on that plane will be one of the conic sections which may be drawn and the methods of the preceding articles may then be applied. But the drawing of the conic section which is the trace of the cone may in general be avoided by the adoption of special constructions.

For example, let pp' (Fig. 623) be a given point on the surface of a right circular cone whose axis vo, $v'o'$ is inclined to the planes of projection. (In general only one projection of P will be given and the other will have to be found as explained in Art. 228, p. 264). Draw the projections of a sphere inscribed in the cone. Join the vertex vv' to pp' and find the point rr' where the line vp, $v'p'$ touches the sphere. A tangent plane to the sphere at rr' will be a tangent plane to the given cone and the line vp, $v'p'$ will be the line of contact between the cone and plane. The tangent plane to the sphere at rr' will be perpendicular to the radius or', $o'r'$ and its traces may be determined as explained in (Art. 183, p. 217).

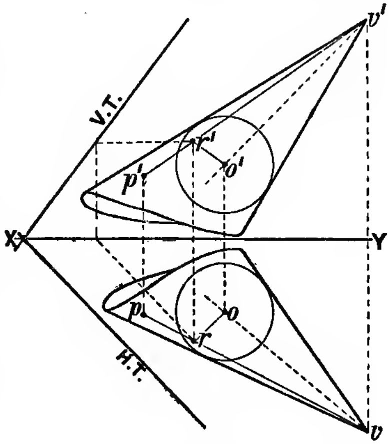


FIG. 623.

Further examples of special constructions to avoid drawing the non-circular trace of a right circular cone, to which a tangent plane is required, will be found in subsequent articles of this chapter. It should however be noted that in many cases the drawing of the non-circular trace of the cone is the most straightforward construction to adopt, and, particularly when the trace is an ellipse, it will often be found quicker and more accurate to draw the trace than to adopt more or less elaborate constructions involving straight lines and circles only.

281. Planes Tangential to a Given Cone and having a Given Inclination.—Let θ be the given inclination of the planes which are to be tangential to the cone.

Referring to Fig. 624, the cone is given by its vertex vv' and its horizontal trace. Determine a right circular cone having its vertex at vv', its base on the horizontal plane, and it base angle equal to θ. Planes tangential to these two cones are the planes required. In Fig. 624 there are four such planes, their horizontal traces being common tangents to the bases or horizontal traces of the cones. The vertical traces of the planes are found from the condition that the planes contain the common vertex of the two cones. The vertical trace of the fourth plane is outside the limits of the figure.

In Fig. 625 the given cone is a right circular cone whose axis vo, $v'o'$ is inclined to both planes of projection. The horizontal trace of this cone may be found as in Art. 227, p. 263, and the construction just given for Fig. 624 applied. A much simpler construction however is as follows. Draw a sphere inscribed in the given cone, its centre being oo'. Determine a cone enveloping this sphere, having its base

on the horizontal plane, and its base angle equal to θ. uu' is the vertex of this cone. Determine also a right circular cone having its

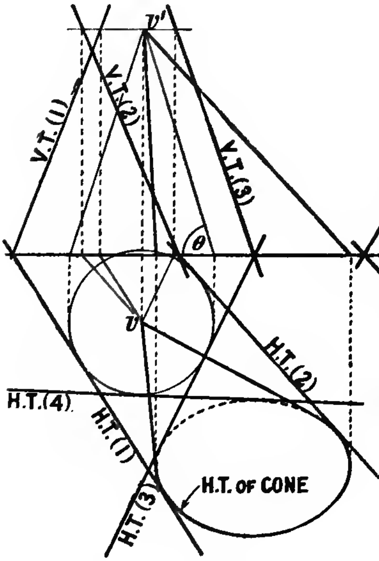


FIG. 624.

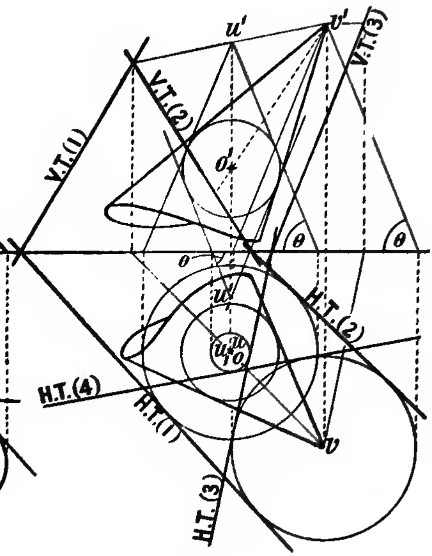


FIG. 625.

vertex at vv', its base on the horizontal plane, and its base angle equal to θ. The two planes (1) and (2) tangential to these two auxiliary cones are two of the planes required, their horizontal traces being common tangents to the circles which are the bases or horizontal traces of the auxiliary cones.

If the auxiliary cone enveloping the sphere be taken with its vertex u_1u_1' below the centre of the sphere instead of above it, the other auxiliary cone being as before, two other tangent planes (3) and (4) to the given cone and having the inclination θ are found. In this case however it is the pair of tangents to the traces of the auxiliary cones which cross one another between the circles which must be taken. The vertical trace of plane (4) is not shown.

282. Tangent Plane to a Cylinder at a Given Point on its Surface.—Let the cylinder be given by its horizontal trace and the direction of its generating line (Fig. 626), and let the given point on its surface be given by its plan p. Through p draw pqr parallel to the plan of the direction of the generating line, cutting the horizontal trace of the cylinder at q and r. If P is on the under surface of the cylinder pq will be the plan of the generating line through P and q will be its horizontal trace. If P is on the upper surface of the cylinder pr will be the plan of the generating line through P, and r will be its horizontal trace. The elevations of these generating lines are found as shown.

A tangent to the horizontal trace of the cylinder at q will be the horizontal trace of the plane which is tangential to the surface of the cylinder at P when P is on the under surface, and a tangent to the horizontal trace of the cylinder at r will be the horizontal trace of the plane which is tangential to the surface of the cylinder at P when P is on the upper surface. The generating lines PQ and PR will be the lines of contact of these planes with the surface of the cylinder, and by making use of this the vertical traces of the planes may be found.

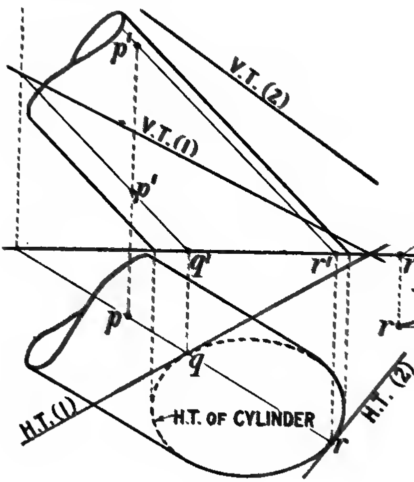


FIG. 626.

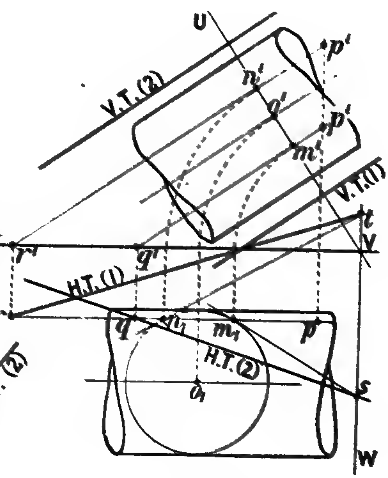


FIG. 627.

If the given cylinder is a right circular cylinder its trace on one of the planes of projection is readily found and the method just described may then be applied. The drawing of the trace of the cylinder, when that trace is not a circle, may be avoided as follows. Draw an elevation of the cylinder (Fig. 627) on a vertical plane parallel to its axis. Take a plane UVW at right angles to the axis of the cylinder cutting that axis at O. The section of the cylinder by this plane is a circle whose centre is O. Draw the rabatment of this circle on the horizontal plane as shown. The plan of the generating lines through the two possible positions of P, P being given by its plan p as before, cuts the rabatment of the circle at m_1 and n_1. Draw tangents to the rabatment of the circle at m_1 and n_1 to meet VW at s and t respectively. q and r being the horizontal traces of the generating lines through M and N, lines qs and rt will be the horizontal traces of the planes required. The traces of the planes on any vertical plane of projection may be found from the condition that each plane contains a generating line through one of the two possible positions of P.

The theory of the foregoing construction is that MS and NT, being

tangents to a section of the cylinder at points on the generating lines through the two possible positions of P, must lie on the planes, one on each, which are tangential to the cylinder along these generating lines.

283. Tangent Planes to a Cylinder through a given external Point.—Through the given point draw a line parallel to the generating line of the cylinder. Let Q be the trace of this line on one of the planes of projection. Through Q draw the tangents to the trace of the cylinder on the plane of projection containing Q. These tangents will be the traces on one of the planes of projection of the planes required and the other traces may be found from the condition that the planes contain the given point.

If the cylinder is a right circular cylinder the drawing of its trace, when that trace is not a circle, may be avoided by a slight modification of the construction shown in Fig. 627, p. 321. Let the line through the given point parallel to the generating line of the cylinder intersect the plane UVW at L and the horizontal plane of projection at K. Find l_1 the rabatment of L on the horizontal plane. From l_1 draw tangents to the rabatment of the circle which is the section of the cylinder by the plane UVW. Let these tangents intersect VW at *s* and *t*. K*s* and K*t* are the horizontal traces of the planes required and the vertical traces may be found from the condition that the planes contain the given point.

284. Planes Tangential to a Cylinder and Parallel to a given Straight Line.—Let AB be the given straight line.

Let the cylinder be given by its trace on one of the planes of projection and the direction of its generating line. Through any point C in AB draw CD parallel to the generating line of the cylinder. Determine the traces of the plane containing AB and CD. The planes required will be parallel to this plane. Tangents to the trace of the cylinder parallel to the corresponding trace of the plane containing AB and CD will be the traces on one of the planes of projection of the planes required. The other traces may be found from the condition that the required planes are parallel to the plane containing AB and CD.

If the cylinder is a right circular cylinder, the drawing of its trace, when that trace is not a circle, may be avoided by a modification of the construction shown in Fig. 627, p. 321. As in the case just considered, draw CD to intersect AB and be parallel to the generating line of the cylinder and find the traces of the plane containing AB and CD. Find the intersection of this plane with the plane UVW. Let EF be this intersection. Draw the rabatment e_1f_1 of EF on the horizontal plane. Draw parallel to e_1f_1 tangents to the rabatment of the circle which is the section of the cylinder by the plane UVW. Let these tangents meet VW at *s* and *t*. Lines through *s* and *t* parallel to the horizontal trace of the plane containing AB and CD are the horizontal traces of the planes required. The vertical traces may be found from the condition that the required planes are parallel to the plane containing AB and CD.

285. Planes Tangential to a Cylinder and having a given

Inclination.—Let θ be the given inclination, say to the horizontal plane.

In Fig. 628 the cylinder is given by its horizontal trace and the direction of its generating lines. Take ab, $a'b'$ a generating line, a being its horizontal trace. Take a point vv' in ab, $a'b'$ as the vertex of a right circular cone whose base is on the horizontal plane and whose base angle is equal to θ. A tangent aL to the base of this cone will be the horizontal trace of a plane tangential to the cone and containing the generating line ab, $a'b'$. The vertical trace LM of this plane is found from the condition that the plane contains the point vv'. This plane will also have the given inclination θ. Tangents to the horizontal trace of the cylinder parallel to aL will be the horizontal traces of two of the planes required and their vertical traces will be parallel to LM.

Two other planes (not shown) fulfilling the given conditions may be found by using the tangent aN to the base of the cone in the same way that aL was used.

The planes found as above will evidently have the required inclination and they will also contain each a generating line of the cylinder whose horizontal trace is at the point where the horizontal trace of the plane touches the horizontal trace of the cylinder. In Fig. 628, cd, $c'd'$ and ef, $e'f'$ are the lines of contact with the cylinder of planes (1) and (2) respectively.

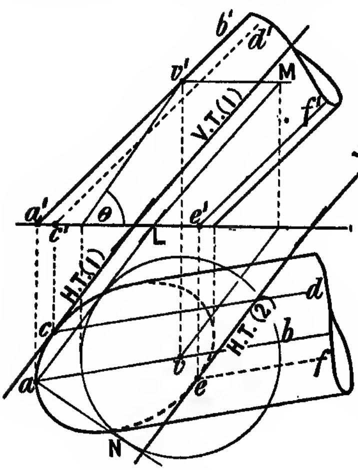


Fig. 628.

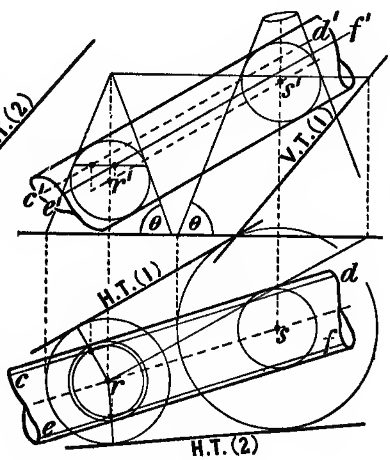


Fig. 629.

When the given cylinder is a right circular cylinder the construction shown in Fig. 629 may be used. In this case a trace of the cylinder is not required. Take rr' and ss' two points on the axis of the cylinder as the centres of two spheres inscribed in the cylinder. Envelop these spheres by cones whose bases are on the horizontal

plane and whose base angles are equal to θ. Tangent planes to these cones are two of the planes required. The vertical trace of the second plane falls outside the limits of the figure. cd, $c'd'$ and ef, $e'f'$ the lines of contact of the tangent planes with the cylinder pass through the points of contact of these planes and the spheres.

Two other planes satisfying the given conditions may be found by drawing two inverted cones enveloping the spheres and constructing the tangent planes to them. The base angles of these inverted cones must of course be equal to θ.

The given angle θ must not be less than the inclination of the generating lines of the cylinder.

286. Planes Tangential to a Sphere and containing a given Line.—*First solution.* O is the centre of the sphere and AB the given line. Draw an elevation of the sphere and given line on a vertical plane parallel to the line (Fig. 630). Take a plane perpendicular to AB and containing the centre of the sphere. LM the vertical trace of this plane will be at right angles to $a'b'$ and will be an edge view of the plane. This plane intersects the given line at C and the sphere in a great circle. The plan of this great circle, which is an ellipse, is shown but it need not be drawn. In the plane LM take the horizontal line OS through the centre of the sphere. Obtain the rabatment of the section of the sphere by the plane LM and also the point C by rotating the plane LM about OS until it becomes horizontal. The rabatment of the section of the sphere will be the circle which is the plan of the sphere and the rabatment of C will be c_1. Through c_1 draw the tangent c_1e_1 to the circle which is the plan of the sphere, e_1 being the point of contact. Restore the plane LM to its original position taking with it the point e_1 determining e' and e as shown. CE is a tangent to the section of the sphere by the plane LM, and the plane containing CE and AB will be a tangent plane to the sphere at E. The horizontal and vertical traces of the tangent plane will be perpendicular to oe and $o'e'$ respectively.

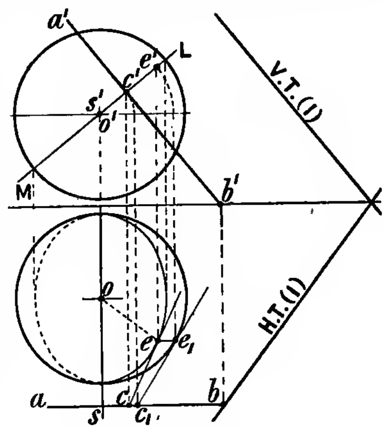


Fig. 630.

Since a second tangent can be drawn from c_1 to the circle, the above construction applied to this second tangent will lead to a second plane tangential to the sphere and containing AB. This second plane is not shown.

Second solution. Envelop the sphere by a cone having its vertex in the given line AB. The planes tangential to this cone and containing AB will also be tangential to the sphere. The horizontal

traces of the tangent planes required will pass through the horizontal trace of AB and will be tangential to the horizontal trace of the cone.

By selecting the point in AB for the vertex of the cone so that the axis of the cone is parallel to one of the planes of projection, the plane (M) of the circle of contact between the cone and the sphere is readily found. The base angle of the cone is the inclination of the planes required to the plane M, and the planes required may be found by the construction of Art. 193, p. 225.

287. Cones Enveloping Two Spheres.—If one sphere lies entirely outside the other and the spheres do not touch one another there will be two cones which will envelop both.

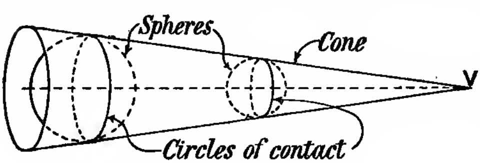


FIG. 631.

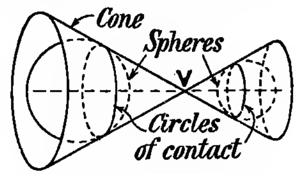


FIG. 632.

The projections of the cones are obtained by drawing the common tangents to the circles which are the projections of the spheres. These tangents intersect the projection of the line joining the centres of the spheres at the projections of the vertices of the cones. One cone has its vertex on the line joining the centres of the spheres produced beyond the smaller sphere (Fig. 631), while the other has its vertex between the spheres (Fig. 632). If the spheres touch one another externally, one of the cones becomes a plane tangential to the spheres at their point of contact. If the spheres cut one another there is only one enveloping cone.

288. Planes Tangential to Two Spheres and having a given Inclination.—Let the given inclination be θ to the horizontal plane. Envelop the two spheres by a cone. Find by Art. 281, p. 319, the planes tangential to this cone and having the given inclination.

The following construction follows readily from that shown in Fig. 629, p. 323. Envelop each sphere by a cone, axis vertical and base angle equal to θ. Planes tangential to these two cones are the planes required. There may be as many as eight planes satisfying the given conditions, depending on θ and the relative positions of the spheres. Denoting the cones by A and B, there are two planes for the case where A and B are upright, two when A and B are inverted, two when A is upright and B is inverted and two when A is inverted and B is upright.

289. Planes Tangential to Two Spheres and containing a given Point.—Envelop the two spheres by a cone. Find by Art. 278, p. 318, the planes tangential to this cone and containing the given point. Or, proceed as follows. Let V be the vertex of the cone enveloping the two spheres, and let P be the given point. Join PV. Find by Art. 286, p. 324, the planes tangential to one of the spheres

and containing the line PV. These planes will also be tangential to the other sphere.

290. Planes Tangential to Three Spheres.—Determine the vertex of a cone enveloping any two of the spheres, also the vertex of a cone enveloping another two. A plane containing these two vertices and touching any one of the spheres will also touch the other two.

If the spheres are entirely external to one another and no two touch one another there will be eight planes which will touch all three spheres. Two of these will have all the spheres on the same side, while the others will have one sphere on one side and two on the other.

291. Tangent Plane to a Surface of Revolution at a given Point on the Surface.—Let the axis of the surface (Figs. 633 and 634) be vertical, and let pp' be the given point on the surface.

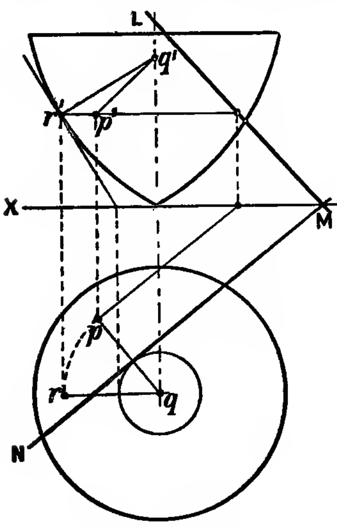


FIG. 633.

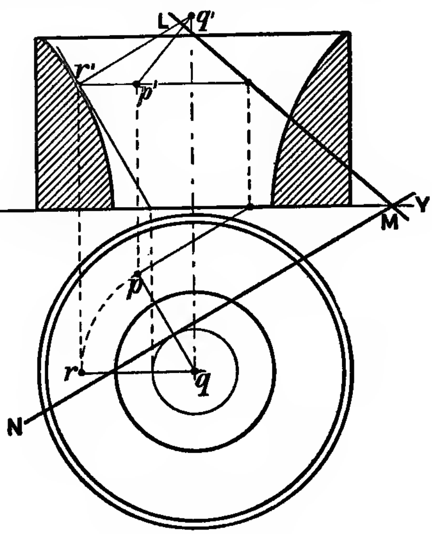


FIG. 634.

All tangent planes to the surface at points on it at the same level as pp' will have the same inclination, and the normals to the surface at these points will meet on the axis at the same point. Hence if $p'r'$ be drawn parallel to XY to meet the outline of the elevation (or plane generatrix) of the surface at r', and a line $q'r'$ be drawn perpendicular to the tangent at r' meeting the elevation of the axis at q', $q'p'$ will be the elevation of the normal to the surface at pp'. The plan of this normal will be the line joining p with the centre of the circle which is the plan of the surface.

A plane LMN through the point pp' and perpendicular to the line pq, $p'q'$ will be the tangent plane to the surface at pp'. This plane is also tangential to the right circular cone which envelops the surface

of revolution and has for its line of contact the horizontal circle through pp'. MN, the horizontal trace of the tangent plane LMN, is a tangent to the circle which is the horizontal trace of the above-mentioned cone.

In Fig. 633 the tangent plane meets the surface of revolution at one point only, while in Fig. 634 the tangent plane also cuts the surface.

292. Cylinder Enveloping a Surface of Revolution.—The vertical line ab, $a'b'$ (Fig. 635) is the axis of a surface of revolution

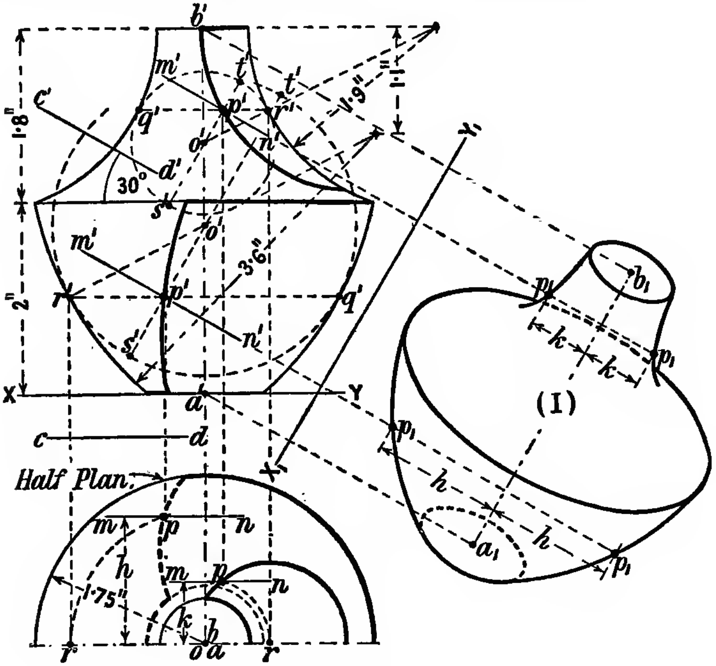


FIG. 635.

whose outline is given. cd, $c'd'$, is a line which is parallel to the vertical plane of projection but is inclined to the horizontal plane. It is required to determine the cylinder which envelops the surface of revolution and has its generatrices parallel to cd, $c'd'$. The cylinder is determined when its line of contact with the surface of revolution is found. The construction is shown for finding two points on this line of contact, one on the upper or concave part of the surface of revolution, and the other on the lower or convex part.

Take a point rr' on the meridian section of the surface of revolution which is parallel to the vertical plane of projection. Draw $r'o'$ the normal to the elevation of the outline of this meridian section at r',

and let $r'o'$ meet $a'b'$ at o'. With o' as centre and $o'r'$ as radius describe a circle. This circle is the elevation of a sphere inscribed in the surface of revolution. The line of contact of this sphere and the surface of revolution is a horizontal circle whose elevation is the straight line $q'r'$. A cylinder which envelops this sphere and has its axis parallel to cd, $c'd'$ will touch the sphere in a circle of which $s'o't'$, perpendicular to $c'd'$, is the elevation. Let $s'o't'$ intersect $q'r'$ at p', then p' is the elevation of a point on the line of contact of the surface of revolution and the required enveloping cylinder. The plan p of this point is found by an obvious construction which is shown.

A line mn, $m'n'$ through pp' parallel to cd, $c'd'$ is one of the generatrices of the cylinder which envelops the sphere and is tangential to the sphere at the point pp'; but the surface of the sphere is tangential to the surface of revolution at this point, therefore the line mn, $m'n'$ is a generatrix of the enveloping cylinder required.

The outline of a projection of the surface of revolution by projectors parallel to cd, $c'd'$ will be the projection of the line of contact of the

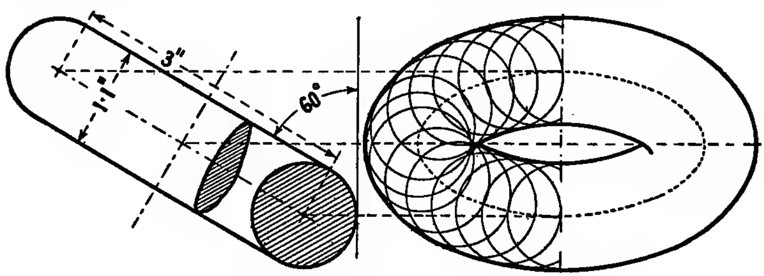


FIG. 636.

enveloping cylinder determined as above. Such a projection is shown at (I), on a plane perpendicular to the projectors.

The projection of a surface of revolution on a plane inclined to its axis may also be obtained directly by first drawing the projections of a sufficient number of spheres inscribed in the surface and then drawing the boundary line of all these projections. The "anchor ring" shown in Fig. 636 is a solid which is easily projected by this method. The anchor ring may be considered as generated by a sphere rotating about the axis of the ring, and the projections of this sphere in a sufficient number of positions being drawn, the curves bounding them form the projection of the ring.

293. Cone Enveloping a Surface of Revolution.—The vertical line ab, $a'b'$ (Fig. 637) is the axis of a surface of revolution whose outline is given. cc' is a given point at the same distance from the vertical plane of projection as ab, $a'b'$. It is required to determine a cone which envelops the surface of revolution and has its vertex at cc'. The cone is determined when its line of contact with the surface of revolution is found. The construction is shown for finding two points

on this line of contact, one on the upper or concave part of the surface of revolution, and the other on the lower or convex part.

As in the preceding Art. draw the elevation of a sphere inscribed in the surface of revolution, having for its line of contact the circle whose elevation is $q'r'$. A cone which envelops this sphere and has its vertex at the point cc' will touch the sphere in a circle of which the straight line $s't'$ is the elevation. The point p' where $s't'$ intersects $q'r'$ is the elevation of a point on the line of contact of the surface of revolution and the required enveloping cone. The plan p of this point is obtained by an obvious construction which is shown. A line mn, $m'n'$ through cc' and pp' is a generatrix of the cone which envelops

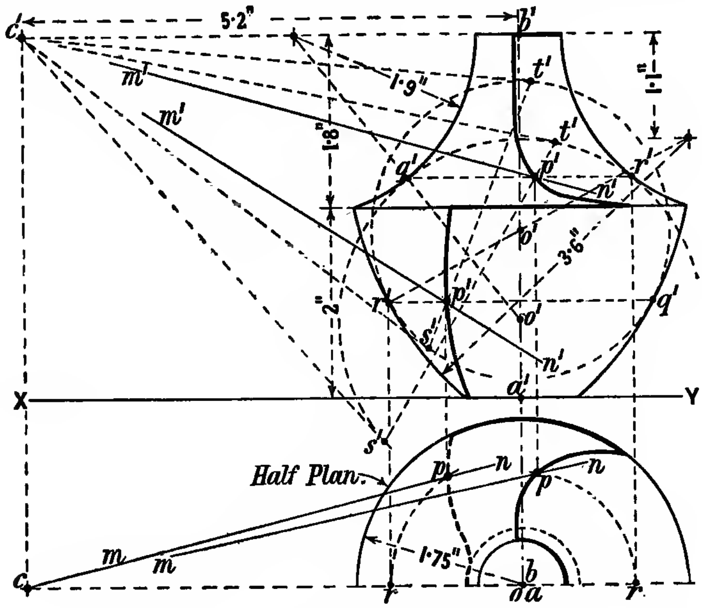


FIG. 637.

the sphere and is tangential to the sphere at pp'. But the surface of the sphere is tangential to the surface of revolution at this point, therefore the line mn, $m'n'$ is a generatrix of the enveloping cone required.

294. The Spheroid.—A *spheroid* may be generated by the revolution of an ellipse about one of its axes, and is called a *prolate* or *oblate* spheroid according as the axis of revolution is the major or minor axis of the ellipse.

A spheroid may also be generated by a circle of varying diameter which moves so that its plane is perpendicular to, and its centre in, a fixed straight line, the diameter of the circle being regulated by an ellipse which has one axis coinciding with the fixed straight line. The fixed straight line is the axis of the spheroid.

Fig. 638 shows the plan and elevation of a prolate spheroid whose axis is vertical. Eight meridian sections and seven circular sections at equal intervals are shown. A section by a plane LM perpendicular to the vertical plane of projection and inclined to the horizontal plane is also shown. All plane sections of a spheroid other than sections at right angles to its axis are ellipses.

Fig. 639 shows the same spheroid tilted over so that its axis is inclined at 45° to the horizontal plane but remains parallel to the vertical plane of projection. The same meridian and circular sections

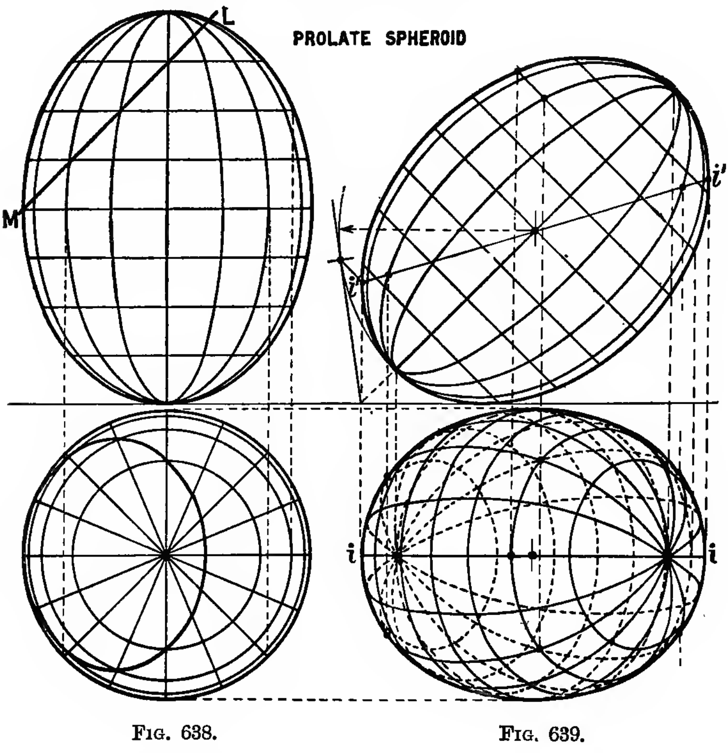


Fig. 638. Fig. 639.

which are shown in Fig. 638 are shown in Fig. 639. The student should draw these projections full size taking the major and minor axes of the generating ellipse 4 inches and 3 inches long respectively. The plan in Fig. 639 may be projected directly from the elevation in Fig. 638 by turning the ground line through 45°. This will save the labour of redrawing the elevation in the inclined position.

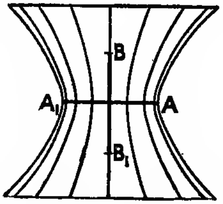
It should be noted that all the curves in the plan, Fig. 639, including the boundary line are ellipses. The boundary ellipse in the plan, Fig. 639, is the horizontal trace of a vertical cylinder enveloping the

spheroid, the straight line $i'i'$ being the elevation of the line of contact between the spheroid and the enveloping cylinder.

The axes of the various ellipses having been found the curves may be drawn by the paper trammel method.

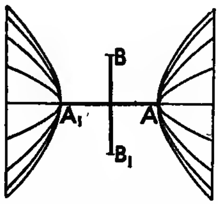
When the two axes of the revolving ellipse are equal the spheroid becomes a sphere.

295. The Hyperboloid of Revolution.—The *hyperboloid of revolution* may be generated by the revolution of an hyperbola about one of its axes. If the axis of revolution is the conjugate axis BB_1 of the hyperbola (Fig. 640) each branch of the hyperbola describes the same surface which is the *hyperboloid of revolution of one sheet*. If the axis of revolution is the transverse axis AA_1 of the hyperbola (Fig. 641) the two branches of the hyperbola describe separate surfaces and the two surfaces form the *hyperboloid of revolution of two sheets*. Of these two hyperboloids of revolution the one of one sheet is the more important, and when an hyperboloid of revolution is referred to the one of one sheet will be understood.



Hyperboloid of revolution of one sheet.

FIG. 640.



Hyperboloid of revolution of two sheets.

FIG. 641.

In Fig. 642, $a'a_1'$ is the transverse axis of an hyperbola whose plane is parallel to the vertical plane of projection and whose conjugate axis $b'b_1'$ is vertical, $m'o'm'$ and $n'o'n'$ are the asymptotes of the hyperbola, and f' and f_1' are the foci. As the hyperbola and its asymptotes revolve about the vertical conjugate axis, the hyperbola describes an hyperboloid of revolution and the asymptotes describe a cone which is asymptotic to the hyperboloid. Sixteen positions of the moving hyperbola are shown in the plan (u) and elevation (v).

Sections of the hyperboloid by planes perpendicular to the axis of revolution are circles, the smallest of which has a diameter equal to the transverse axis of the hyperbola and is called the *collar* or *throat* or *gorge* of the hyperboloid.

At (w) in Fig. 642 is shown a plan of the hyperboloid when the axis of revolution is inclined at 30° to the horizontal plane. In this new plan only the visible meridian and circular sections shown in the plan (u) and elevation (v) are shown.

A section of the hyperboloid by a plane is of the same kind as the section of the asymptotic cone by that plane or by a plane parallel to it. For example if a plane cuts the asymptotic cone in an ellipse all planes parallel to that plane will cut the hyperboloid in ellipses. The same is true for parabolic and hyperbolic sections, and it is obviously true for circular sections.

A plane containing the axis of revolution cuts the hyperboloid in an

hyperbola and the asymptotic cone in two straight lines which are the asymptotes of the hyperbola.

A plane parallel to the axis of revolution and tangential to the collar of the hyperboloid cuts the asymptotic cone in an hyperbola and the hyperboloid in two straight lines P and Q which intersect on the collar and which are the asymptotes of the hyperbola. These two straight lines P and Q are also parallel to the straight lines P_1 and Q_1 respectively in which the asymptotic cone is cut by a plane containing the axis of revolution and parallel to the plane of P and Q. Also all these straight lines P, P_1, Q and Q_1 are inclined at the same angle to

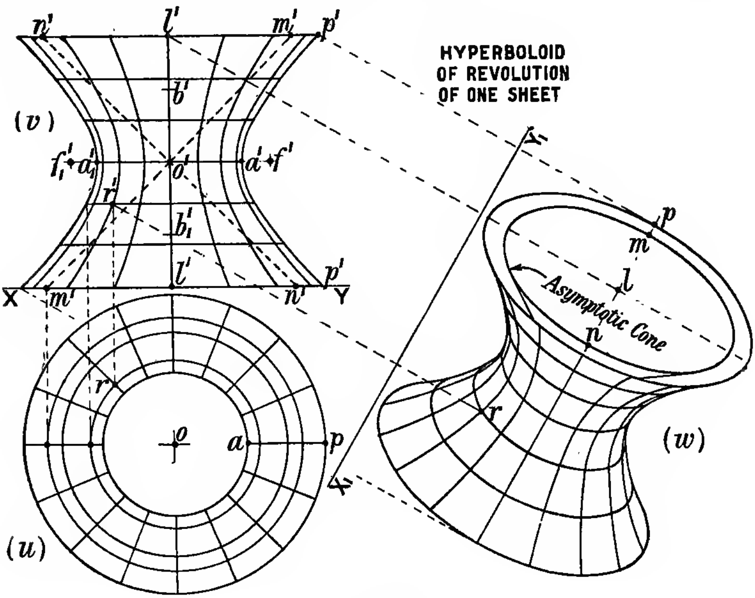


FIG. 642.

the axis of revolution but if P and P_1 slope forwards Q and Q_1 slope backwards.

It follows that from any point S on the surface of the hyperboloid two straight lines can be drawn on the surface, because two planes can be taken parallel to the axis of revolution and containing S and touching the collar and each of these planes will intersect the surface in a pair of straight lines and one of each pair will pass through S.

There are therefore two systems of straight lines which can be drawn on the surface of the hyperboloid. No two lines of the same system are in the same plane, but any line of the one system and any line of the other are in the same plane. The plane which contains a line of one system and a line of the other is the tangent plane to the surface at the point of intersection of the lines.

Since any line of one system and any line of the other are in the same plane it follows that any line of one system will intersect every line in the other system if the lines are produced far enough. Hence the hyperboloid of one sheet may be generated by the motion of a straight line which moves in contact with any three lines of either system.

The hyperboloid may also evidently be generated by the revolution of a line belonging to one or other of the systems above referred to, the axis of revolution being the axis of revolution already used. From this mode of generation the hyperboloid of revolution is also called the *twisted surface of revolution.* This mode of generation is illustrated by

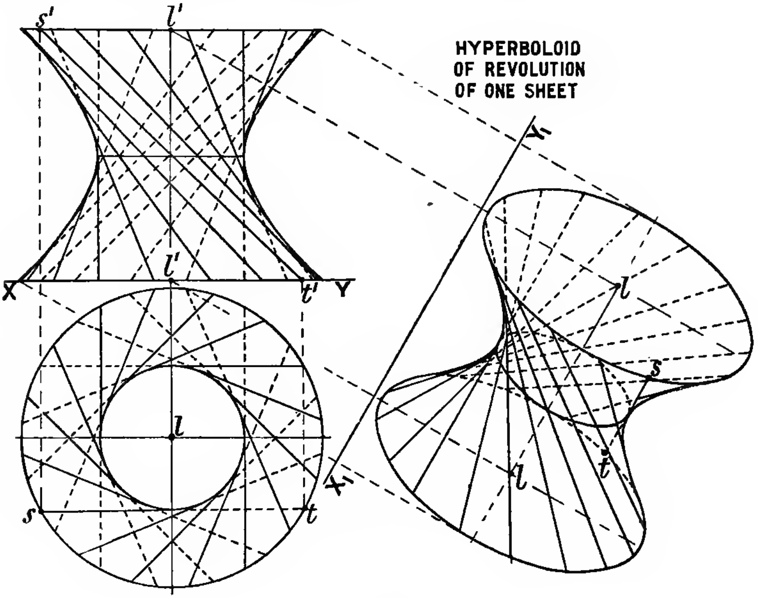


FIG. 643.

Fig. 643 which shows the same hyperboloid represented in Fig. 642, but instead of different positions of the revolving hyperbola different positions of the revolving straight line are shown. It may be left as an exercise for the student after drawing the views as shown in Fig. 643 to add the second system of straight lines on the hyperboloid.

The hyperboloid of two sheets cannot be generated by the motion of a straight line, it is therefore not a ruled surface.

296. Hyperboloids of Revolution in Rolling Contact.—Referring to the plan (u) and elevation (v) in Fig. 644, $o'a'$ is the elevation and the point o_1a is the plan of a vertical axis. $o'b'$ is the elevation and o_2b is the plan of another axis which is parallel to the vertical

plane of projection but is inclined to the horizontal plane. The point o' is the elevation and o_1o_2 is the plan of the common perpendicular to these two axes. $o'c'$ is the elevation and oc is the plan of a straight line which is parallel to the vertical plane of projection. $o'c'$ makes an angle a with $o'a'$ and an angle β with $o'b'$.

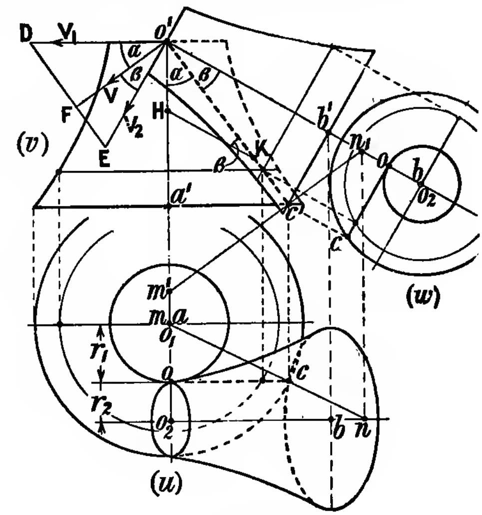


Fig. 644.

If an hyperboloid of revolution be described by the revolution of OC about the vertical axis and another hyperboloid be described by the revolution of OC about the inclined axis, these two hyperboloids will be in contact along the line OC, and if one hyperboloid be made to rotate about its axis the other will also rotate if the frictional resistance between the hyperboloids is sufficient.

The construction of the hyperboloids when the axes and the line of contact are given is clearly shown in Fig. 644. The view (w) is a projection of the inclined hyperboloid on a plane perpendicular to its axis.

The common normal to the surfaces of the hyperboloids at C will be at right angles to the tangent plane at C and this tangent plane will contain the line OC. Since the line OC is parallel to the vertical plane of projection the vertical trace of the tangent plane containing OC will be parallel to $o'c'$, therefore the elevation of the common normal will be perpendicular to $o'c'$. But this common normal must intersect the axis of each hyperboloid. Hence a line $m'c'n'$ at right angles to $o'c'$ and intersecting $o'a'$ at m' and $o'b'$ at n' will be the elevation of the common normal to the surfaces at C. The plan mcn of this normal is easily found as shown.

The relative angular velocities of the two hyperboloids, when one rotates the other, will now be determined. To facilitate reference denote the vertical hyperboloid by A and the inclined hyperboloid by B. Let ω_1 and ω_2 be the angular velocities of A and B respectively. Let V_1 be the linear velocity of the point O considered as a point on the throat of A, and let V_2 be the linear velocity of the point O considered as a point on the throat of B. Let r_1 and r_2 be the radii of the throats of A and B respectively.

Draw o'D, o'E, and o'F at right angles to $o'a'$, $o'b'$, and $o'c'$ respectively. Make o'D equal to V_1 and draw DFE at right angles to o'F meeting o'F at F and o'E at E. Then o'E will be equal to V_2, because

o'F and FD are evidently the components of V_1 along and perpendicular to OC respectively, and since there is rolling motion at O the component of V_2 at right angles to OC must also be equal to o'F. In addition to the rolling of the one hyperboloid on the other there is also a sliding motion along the line of contact and the velocity of this sliding is represented by DE.

$\frac{\omega_1}{\omega_2} = \frac{o'D}{r_1} \div \frac{o'E}{r_2} = \frac{o'D}{o'E} \cdot \frac{r_2}{r_1}$. But the triangle D$o'$E is similar to the triangle $m'o'n'$, therefore $\frac{o'D}{o'E} = \frac{o'm'}{o'n'}$. Also from the plan ($u$) $\frac{r_2}{r_1} = \frac{cn}{cm}$. But $\frac{cn}{cm} = \frac{c'n'}{c'm'}$. Hence $\frac{\omega_1}{\omega_2} = \frac{o'm'}{o'n'} \cdot \frac{c'n'}{c'm'} = \frac{c'n'}{o'n'} \div \frac{c'm'}{o'm'} = \frac{\sin\beta}{\sin\alpha}$.

On $o'a'$ make $o'H = \omega_1$, and draw HK parallel to $o'b'$ to meet $o'c'$ at K. Then angle $o'KH = \beta$, and $\frac{o'H}{HK} = \frac{\sin\beta}{\sin\alpha}$, but this has been shown to be equal to $\frac{\omega_1}{\omega_2}$, therefore since $o'H = \omega_1$, HK must be equal to ω_2. This gives the solution of the most common problem in connection with rolling hyperboloids which is, given the two axes and the relative angular velocities to find the line of contact and then construct the hyperboloids. For if on $o'a'$ o'H is made equal to ω_1 and HK is drawn parallel to $o'b'$ and made equal to ω_2 then o' being joined to K the elevation of the line of contact is found. To find the plan oc draw $m'c'n'$ perpendicular to $o'c'$ to meet $o'a'$ at m' and $o'b'$ at n'. A projector from n' to o_2b fixes n and mn being drawn a projector from c' to meet mn determines c. Then co parallel to o_2b is the plan required and the hyperboloids may be drawn as shown.

These hyperboloids are the pitch surfaces of what are called *skew bevel wheels*, which are used to transmit motion directly from one shaft to another when the axes of the shafts do not intersect and are not parallel to one another. In practice generally only comparatively short frusta of the hyperboloids are used as the pitch surfaces of the wheels as shown in Fig. 645. If the frusta are taken at the bases of the hyperboloids, as A and B, they are approximately conical, and if at the throats, as A_1 and B_1, they are approximately cylindrical. In these wheels the teeth have line contact.

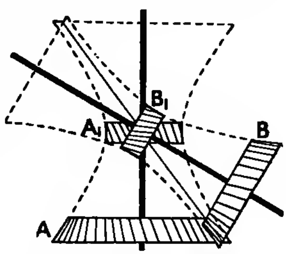


FIG. 645.

297. The Geometry of Cross-Rolls.—Referring again to Fig. 644, if the axes of the hyperboloids are given and it is also given that the line of contact OC is parallel to the axis of the inclined hyperboloid, then $o'c'$ would coincide with $o'b'$ and the inclined hyperboloid would become a cylinder. But the point nn' would coincide with the point cc' and the cylinder would have no diameter, that is, it would become a straight line only.

If it is desired that the hyperboloid which has become a straight line should be a cylinder of definite diameter, then another surface of revolution will have to take the place of the other hyperboloid. It will now be shown how this other surface of revolution may be determined.

Referring to Fig. 646, *bob* is the plan and $b'o_2'b'$ is the elevation of the axis of the cylinder which is assumed to be horizontal. *aoa* is the plan and o_1' is the elevation of the axis of the other surface or solid which is assumed to be horizontal and perpendicular to the vertical plane of projection. This other solid will be called *the roll*. o is the plan and $o_1'o'o_2'$ is the elevation of the common perpendicular to the two given axes. Making $o_2'o'$ equal to the radius of the cylinder determines $o_1'o'$ the radius of the roll at the throat. The plane of the throat circle of the roll intersects the cylinder in an ellipse which touches the throat circle at oo'. Any other plane parallel to the plane of the throat circle will intersect the cylinder in an ellipse and the roll in a circle, and the ellipse and circle will touch one another. Moreover all such sections of the cylinder will be exactly alike.

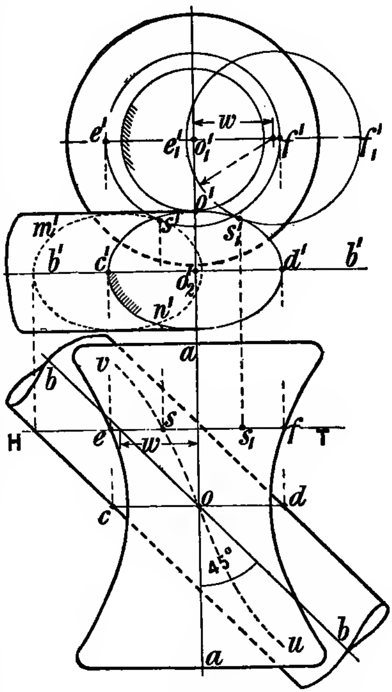


Fig. 646.

HT is the horizontal trace of a plane parallel to the plane of the throat circle of the roll. This plane intersects the cylinder in an ellipse of which $m'n'$ is the elevation. A circle with o_1' as centre drawn to touch the ellipse $m'n'$ is the elevation of the circular section of the roll by the plane HT, and this determines the points e and f on the horizontal meridian section of the roll. A projector from s', the point of contact of the ellipse and circle, to HT determines s a point on the plan *uov* of the curve of contact between the cylinder and roll.

The radius of the circular section of the roll at HT and the point of contact ss' may be found without drawing the ellipse $m'n'$. It is evident that if the ellipse $m'n'$ be moved to the right a distance equal to w on the plan it will then coincide with the ellipse $c'd'$ which is the elevation of the section of the cylinder by the plane of the throat circle of the roll. And if the circle $e'f'$ be moved an equal distance to

the right it will then be in the position $e_1'f_1'$ and will touch the ellipse $c'd'$ at s_1'. Hence the circle $e_1'f_1'$ may be drawn first and then the circle $e'f'$ which is equal to it. Projecting from s_1' to s_1 and making s_1s equal to w determines the point s. In like manner, by taking other positions for the plane HT any number of points on the horizontal meridian section of the roll and any number of points on the plan *uov* of the curve of contact may be determined. The ellipse $c'd'$ is therefore the only ellipse which need be drawn but it should be constructed as accurately as possible, to ensure a good result.

298. The Paraboloid of Revolution.—The *paraboloid of revolution* may be generated by the revolution of a parabola about its axis. Sections of the surface by planes parallel to or containing the axis are equal parabolas. Sections by planes perpendicular to the axis are circles. All other plane sections are ellipses.

299. The Ellipsoid.—The *ellipsoid* may be generated by a variable ellipse which moves so that its plane is parallel to a fixed plane and the extremities of its axes are on two fixed ellipses which have one common axis and which have their planes at right angles to one another and to the fixed plane.

Referring to Fig. 647, let the horizontal plane be the fixed plane. Let *oa*, *o'a'* and *ob*, *o'b'* be the semi-axes of one fixed ellipse whose plane is horizontal, and let *ob*, *o'b* and *oc*, *o'c'* be the semi-axes of the other fixed ellipse whose plane is parallel to the vertical plane of projection. A moving variable ellipse whose axes are horizontal chords of these two fixed ellipses generate an ellipsoid.

The plan of the horizontal fixed ellipse is the plan of the ellipsoid, and the elevation of the other fixed ellipse which is parallel to the vertical plane of projection is an elevation of the ellipsoid. *def* is the half plan and *d'e'f'* is the elevation of the moving variable ellipse in one position. *oe* on the half plan is equal to $o_1'e'$ on the part elevation (*v*) which is a projection on a vertical plane at right angles to the plane of the elevation (*u*).

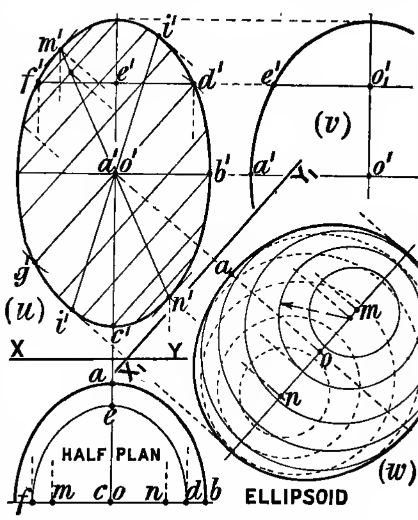


FIG. 647.

All plane sections of the ellipsoid are either ellipses or circles.

If $g'o'd'$, a diameter of the ellipse which is the elevation of the ellipsoid, be taken as the vertical trace of a plane which is perpendicular

to the vertical plane of projection, the elliptic section of the ellipsoid by this plane will have one semi-axis equal to $o'g'$ and the other semi-axis equal to oa. If $o'g'$ is equal to oa then the section is evidently a circle. This suggests the construction for finding the plane containing the centre of the ellipsoid and cutting the ellipsoid in a circle. There are evidently two such planes.

All plane sections parallel to one circular section are circles. In Fig. 647, $g'o'd'$ is the elevation of one circular section of the ellipsoid and the chords of the ellipse parallel to $g'o'd'$ are the elevations of other circular sections. The elevations of the centres of these circles lie on the diameter $m'o'n'$ which is conjugate to the diameter $g'o'd'$.

As the plane of a circular section moves away from the centre of the ellipsoid the circle gets smaller and smaller until the plane becomes tangential to the ellipsoid at mm' or nn' when the circle becomes a point which is called an *umbilic*. There are evidently four umbilics on an ellipsoid.

At (w) in Fig. 647 is shown a projection of the ellipsoid on a plane parallel to planes of circular sections. On this projection nine circular sections are shown.

The curve of contact between an ellipsoid and an enveloping cylinder or an enveloping cone is a plane section of the ellipsoid. The outline of the projection (w) in Fig. 647 is the trace of a cylinder which envelops the ellipsoid and is perpendicular to the plane of the projection; the diameter $i'o'i'$ of the ellipse (u) is the elevation of the curve of contact.

The tangent plane to the ellipsoid at any point on it contains the tangents at that point to any two plane sections of the ellipsoid through the point. When two of the three axes of an ellipsoid are equal it becomes a spheroid.

300. The Hyperboloid of One Sheet.—The *hyperboloid of one sheet* may be generated by a variable ellipse which moves so that its plane is always parallel to a fixed plane and has the extremities of its axes on two fixed hyperbolas whose planes are perpendicular to one another and to the fixed plane and which have a common conjugate axis.

Referring to Fig. 648, OA and OB are the semi-transverse axes of two fixed hyperbolas and OC is a common semi-conjugate axis. OA and OB are horizontal and OC is consequently vertical. OA is parallel to the plane of the elevation (v). The moving variable ellipse is in this case always horizontal, and it will be smallest when its plane coincides with AOB, and it is then the *throat ellipse* or *principal elliptic section* of the hyperboloid. In this article when "hyperboloid" is mentioned "hyperboloid of one sheet" will be understood.

The moving ellipse remains similar to the throat ellipse, hence if DD and EE are the axes of the moving ellipse in one position de is parallel to ab.

At (w) is shown an elevation of the hyperboloid on a plane

parallel to OB. The true form of one fixed hyperbola is shown in the elevation (*v*) and the true form of the other is shown in the elevation (*w*).

The asymptotic cone of the hyperboloid is generated by a variable ellipse which moves so that its plane is parallel to the throat ellipse and has the extremities of its axes on the asymptotes of the fixed hyperbolas. The vertex of this cone is at O. GG and HH are the axes of this moving variable ellipse in one position. This generating ellipse is similar to the throat ellipse, hence *gh* is parallel to *ab*. The ellipse which is the section of the asymptotic cone by a plane parallel

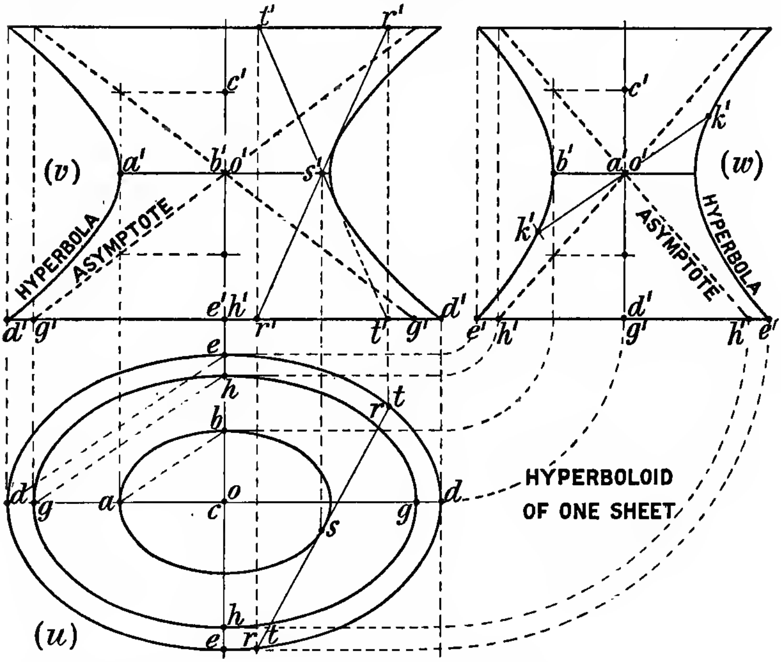


FIG. 648.

to the plane of the throat ellipse and at a distance from it equal to OC is evidently equal to the throat ellipse.

Sections of the hyperboloid and of the asymptotic cone by the same or by parallel planes are curves of the same kind.

If the transverse axes of the fixed hyperbolas are equal the hyperboloid becomes an hyperboloid of revolution.

The plane of a circular section of the hyperboloid may be found as follows. Let OA be greater than OB. Then on the elevation (*w*) take o' as centre and with a radius equal to *oa* describe an arc to cut the hyperbola $b'k'e'$ at k'; then $k'o'k'$ is the vertical trace of a plane, perpendicular to the plane of the elevation (*w*), which will cut the

hyperboloid in a circle whose radius is equal to $o'k'$. All sections by planes parallel to this plane will be circles. There are evidently two systems of parallel planes which will cut the hyperboloid in circles.

The hyperboloid of one sheet may also be generated by a moving variable hyperbola having a fixed conjugate axis, and having the extremities of its transverse axis on a fixed ellipse whose plane bisects at right angles the fixed conjugate axis of the moving hyperbola. The fixed ellipse is the throat ellipse of the hyperboloid.

Referring to the plan (u) and elevation (v) Fig. 648, the straight lines RSR and TST lie on the hyperboloid. These lines are in a vertical plane and their plans coincide and are tangential to the plan of the throat ellipse. The hyperboloid may therefore be generated by the motion of one or other of these straight lines. These two lines belong to two different systems. A line of one system will never meet any other line of that system but it will intersect every line of the other system if the lines are produced far enough. Hence if a straight line moves in contact with any three lines of one of the systems it will generate the hyperboloid.

301. The Hyperboloid of Two Sheets.—The *hyperboloid of two sheets* may be generated by a variable ellipse which moves so that its plane is always parallel to a fixed plane and has the extremities of its axes on two fixed hyperbolas whose planes are perpendicular to one another and to the fixed plane and which have a common transverse axis.

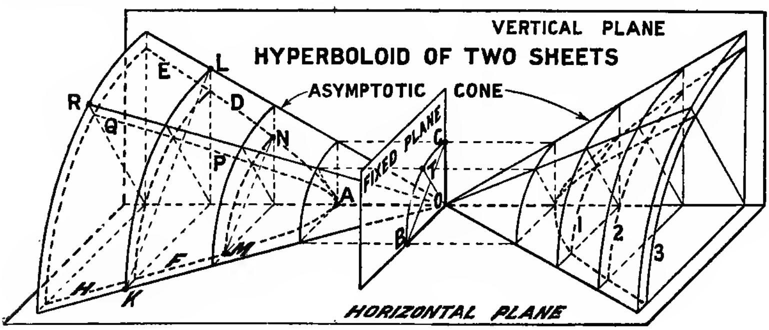


FIG. 649.

Referring to Fig. 649, which is a pictorial projection of one quarter of an hyperboloid of two sheets, OB and OC are the semi-conjugate axes of two fixed hyperbolas and OA is a common transverse axis. OA and OB are horizontal and OC is vertical. The plane containing OB and OC is the fixed plane referred to in the above definition. ADE is part of one branch of one fixed hyperbola having OA for its semi-transverse axis and OC for its semi-conjugate axis; the plane of this hyperbola is vertical. AFH is part of one branch of the other fixed hyperbola having OA for its semi-transverse axis and OB for its semi-

conjugate axis; the plane of this hyperbola is horizontal. The corresponding parts of the other branches of these fixed hyperbolas are shown to the right in Fig. 649. 1, 2, and 3 are three positions of the moving variable ellipse which is always parallel to the fixed plane BOC. The generating ellipse in any position is a section of the hyperboloid by a plane parallel to the plane BOC.

The asymptotic cone of the hyperboloid of two sheets is generated by a variable ellipse which moves so that its plane is parallel to the fixed plane BOC and has the extremities of its axes on the asymptotes of the fixed hyperbolas. The vertex of the cone is at O. This generating ellipse in any position is a section of the asymptotic cone by a plane parallel to the plane BOC.

All sections of the hyperboloid of two sheets, and of the asymptotic cone, by planes parallel to the plane BOC are ellipses similar to the ellipse having OB and OC as semi-axes. Straight lines such as KL and MN are therefore parallel to the straight line BC.

Any plane containing the common transverse axis of the fixed hyperbolas cuts the hyperboloid in an hyperbola, and the asymptotic cone in straight lines which are the asymptotes of the hyperbola. APQ is such an hyperbolic section, OR being an asymptote. The semi-conjugate axis of this hyperbolic section is OT the semi-diameter of the ellipse BTC which is in the plane of section.

The hyperboloid of two sheets may therefore be generated by a moving variable hyperbola having a fixed transverse axis, and having the extremities of its conjugate axis on a fixed ellipse whose plane bisects at right angles the fixed transverse axis of the moving hyperbola. The fixed ellipse is the ellipse whose axes are the conjugate axes of the fixed hyperbolas.

The hyperboloid of two sheets cannot be generated by a straight line and it is therefore not a ruled surface.

302. The Elliptic Paraboloid.—The *elliptic paraboloid* may be generated by a parabola which moves with its vertex in a fixed parabola, the two parabolas having their axes parallel, their planes at right angles to one another, and their concavities turned in the same direction.

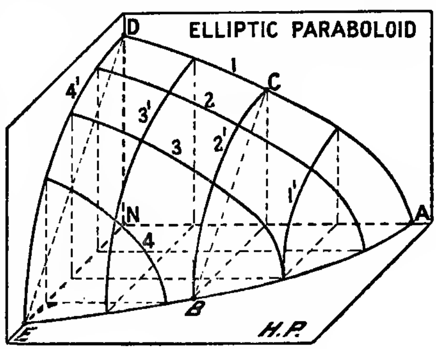


FIG. 650.

Fig. 650 is a pictorial projection of one quarter of an elliptic paraboloid. ABE is the fixed parabola of which AN is the axis and whose plane H.P. is horizontal. 1, 2, 3, and 4 are different positions of the moving parabola whose plane is vertical and whose axis is parallel to AN, and whose vertex is on the parabola ABE.

Sections of the elliptic paraboloid by planes perpendicular to AN are ellipses. Four such sections are shown at 1′, 2′, 3′, and 4′. These ellipses are similar to one another so that lines such as BC and ED are parallel. Hence the elliptic paraboloid may also be generated by a variable ellipse which moves with its centre on, and its plane perpendicular to, the common axis AN of two fixed parabolas ABE and ACD whose planes are perpendicular to one another, the extremities of the axes of the ellipse being on the parabolas.

303. The Hyberbolic Paraboloid.—The *hyperbolic paraboloid* may be generated by a parabola which moves with its vertex on a fixed parabola, the two parabolas having their axes parallel, their planes at right angles to one another, and their concavities turned in opposite directions.

Fig. 651 is a pictorial projection of one quarter of a hyperbolic paraboloid. APC is the fixed parabola of which the vertical line AN is the axis and whose plane is the vertical plane V.P. 1, 2, 3, etc. are different positions of the moving parabola whose plane is at right angles to the plane V.P. and whose axis is parallel to AN, and whose vertex is on the parabola APC.

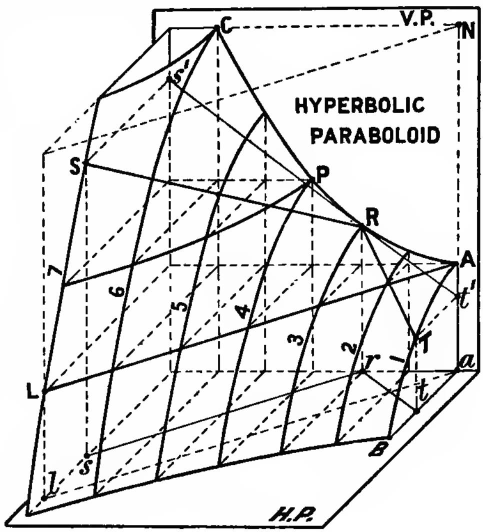


Fig. 651.

A section of the paraboloid by a horizontal plane through A, the vertex of the fixed parabola, is two straight lines of which AL is one. Vertical planes containing these two straight lines are asymptotic planes of the paraboloid. All other horizontal sections of the paraboloid are hyperbolas the asymptotes of which are the lines of intersection of the planes of section with the asymptotic planes. Horizontal sections are shown at the levels C, P, A, and B. It will be found that the hyperbolic sections above A are on opposite sides of the asymptotes to those below A.

The projections of the horizontal hyperbolic sections on a horizontal plane are hyperbolas of which the horizontal traces of the asymptotic planes are the asymptotes.

It follows that the hyperbolic paraboloid may be generated by a variable hyperbola which moves with its centre on, and its plane perpendicular to, the line aAN containing the axes of the two fixed

parabolas APC and ATB whose planes are at right angles to one another, whose asymptotes are in fixed planes, and having the extremities of their transverse axes on the corresponding fixed parabolas.

Sections of the hyperbolic paraboloid by planes parallel to the asymptotic planes are straight lines. RS and RT are two such sections. *rs* the projection of RS on the plane H.P. is parallel to *al* the horizontal trace of one of the asymptotic planes and it will be found that R*s'* the projection of RS on the plane V.P. is tangential to the parabola APC at R. Also, *rt* the projection of RT on the plane H.P. is parallel to the horizontal trace of the other asymptotic plane and R*t* the projection of RT on the plane V.P. is also tangential to the parabola APC at R. It will also be found that the projections of these straight lines RS and RT on the plane of the parabola ATB are tangential to that parabola.

Hence the hyperbolic paraboloid may also be generated by either of two systems of straight lines, one system being parallel to one fixed plane and the other parallel to another fixed plane. No two lines of the same system intersect, but each line of one system will intersect every line of the other system if they are produced far enough. These results lead to the following definition. The hyperbolic paraboloid may be generated by a straight line which moves parallel to a fixed plane and intersects two straight lines which are not parallel and do not intersect. From this mode of generation the hyperbolic paraboloid has been called the *twisted plane.*

304. Tortuous Curves.—A *tortuous curve* is one in which no definite part is in a plane.

If Q, P, and R be points on a tortuous curve, P being between Q and R, a plane may be found containing these three points. If Q and R be moved nearer to P then as Q and R move up to P the ultimate position of the plane containing Q, P, and R is the *osculating plane* of the tortuous curve at P, and the ultimate position of the straight line joining Q and R is the *tangent* to the tortuous curve at P. The tangent at P is evidently in the osculating plane at P.

A plane through a point P on a tortuous curve at right angles to the tangent at P is called the *normal plane* to the curve at P. Any straight line passing through P and lying in the normal plane to the curve at P is a *normal* to the curve at P. The line of intersection of the normal and osculating planes of the curve at P is called the *principal normal* of the curve at P. The normal through P which is perpendicular to the osculating plane is called the *binormal* to the curve at P.

The projection of the tangent at a point P of a tortuous or a plane curve is the tangent to the projection of the curve at *p* the projection of P.

Exercises XXIII

1. A circle 4 inches in diameter is in the V.P. A vertical straight line is 4 inches in front of the V.P. and its elevation touches the elevation of the circle. A surface is generated by a horizontal straight line which moves in contact with the given line and circle. Draw the elevation of the section of the surface by a plane parallel to the V.P. and 2 inches in front of it.

2. A right circular cone, base 4 inches in diameter, and altitude 4 inches, stands with its base on the H.P. A right circular cylinder 2 inches in diameter and 4 inches long stands with its base on the H.P. and its axis 3 inches away from the axis of the cone. A surface is generated by a horizontal straight line which moves in contact with the surface of the cone and the axis of the cylinder. Draw the elevation of the line of intersection of this surface with the surface of the cylinder, the plane of the elevation to be inclined at 45° to the plane of the axes of the cylinder and cone.

3. Three straight lines AB, CD, and EF are given by their projections in Fig. 652. A surface is generated by a straight line which moves in contact with each

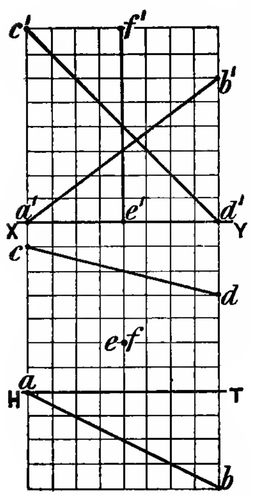


Fig. 652.

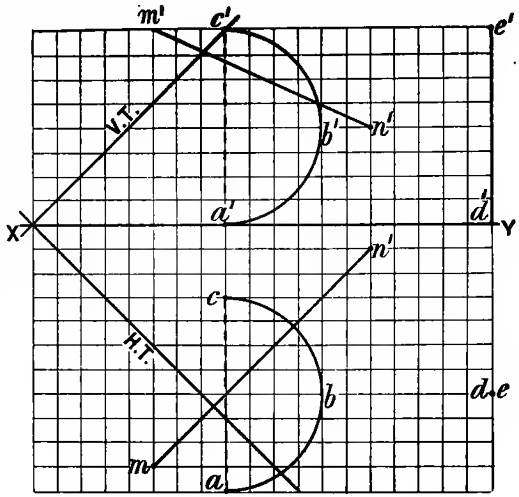


Fig. 653.

of the given lines. Draw the elevation of the intersection of this surface with the vertical plane whose horizontal trace is H.T. Take the squares of the squared background in Fig. 652 as of $\frac{1}{2}$ inch side.

4. The semicircles *abc* and *a'b'c'* (Fig. 653) are the projections of the half of an ellipse. *de* and *d'e'* are the projections of a vertical line. A horizontal straight line moves in contact with the curve ABC and the line DE. Draw the plan and elevation of the curve of intersection of this surface and the plane whose traces are given. Find also the point of intersection of the line *mn*, *m'n'* with the surface generated as above. Take the squares of the squared background in Fig. 653 as of $\frac{1}{4}$ inch side.

5. The circles *abcd* and *a'b'c'd'* (Fig. 654) are the plan and elevation respectively of an ellipse. A conical surface is generated by a straight line which moves in contact with the ellipse and passes through the fixed point *vv'*. Determine the horizontal trace of the conical surface. Show also the traces of the planes which are tangential to the conical surface and contain the point *rr'*.

6. $v'a'b'$ (Fig. 655) is the elevation of a right circular cone and vs is the plan of its axis. $v'r'$ is the elevation of a straight line on the front surface of the cone.

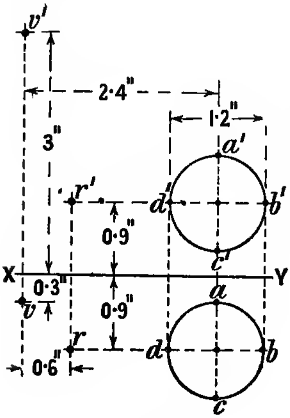


Fig. 654.

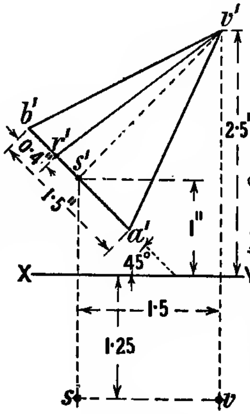


Fig. 655.

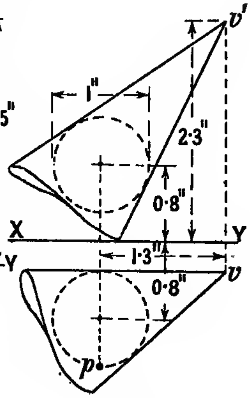


Fig. 656.

Without drawing an ellipse determine the traces of a plane which touches the cone along the line VR.

7. The plan and elevation of a cone enveloping a sphere are given in Fig. 656. Determine the traces of a plane inclined at 60° to the horizontal plane and tangential to the cone.

8. Determine the traces of a plane tangential to the cone given in Fig. 656 and inclined at 45° to the vertical plane of projection.

9. Referring to Fig. 656, p is the plan of a point lying on the upper surface of the given cone. Draw the traces of the plane which is tangential to the cone at the point P.

10. A right circular cone whose vertical angle is 50° lies with its slant side on the horizontal plane and the plan of its axis perpendicular to XY. Draw the traces of a plane which touches the cone in a line whose inclination to the horizontal plane is 40°.

11. A right circular cone, base 2·5 inches diameter and axis 2·5 inches long, has its base on the vertical plane of projection and its axis 1·5 inches above the horizontal plane. Draw the traces of the four planes which are tangential to this cone and inclined to the horizontal plane at 60°.

12. The plan of the vertex of a cone is on the circumference of its horizontal trace which is a circle 2 inches in diameter. The height of the vertex above the horizontal plane is 2·5 inches. Draw the scale of slope of a plane tangential to this cone and inclined at 60° to the horizontal plane.

13. The circle ab (Fig. 657) is the horizontal trace of a cylinder and c_0d_{20} is the indexed plan of its axis. p is the plan of a point on the upper surface of the cylinder. Draw the traces of the plane which is tangential to the cylinder at P.

14. Draw the traces of a plane which is tangential to the cylinder of the preceding exercise and inclined at 70° to the horizontal plane.

15. Draw the traces of a plane which is tangential to the cylinder of exercise 13 and which contains the line m_0n_0 (Fig. 657).

16. a_6b_{20} (Fig. 658) is the indexed plan of the axis of a right circular cylinder 1·5 inches in diameter. r is the plan of a point on the lower surface of the cylinder. Draw the scale of slope of a plane which is tangential to the cylinder at R.

17. Draw the scale of slope of a plane which is tangential to the cylinder of the preceding exercise and which contains the point s_{33} (Fig. 658).

18. Fig. 659 is the figured plan of a straight line AB and a sphere whose centre is C. Draw the traces of the planes which contain the line and are tangential to the sphere.

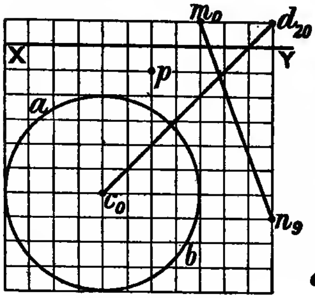


FIG. 657.

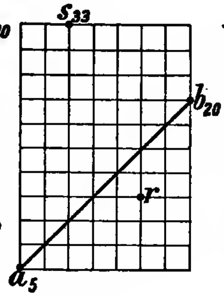


FIG. 658.

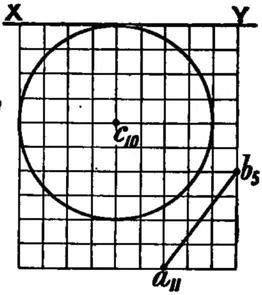


FIG. 659.

In reproducing the above diagrams the sides of the small squares are to be taken equal to a quarter of an inch. The unit for the indices is 0·1 *inch.*

19. Draw two circles, one 2 inches and the other 1 inch in diameter. The centres of the circles to be 2 inches apart. These circles are the plans of two spheres. The centre of the smaller sphere is 1 inch and the centre of the larger sphere is 2·5 inches above the horizontal plane. Draw the scales of slope of all the planes which are tangential to the two spheres and are inclined at 70° to the horizontal plane.

20. AB is a straight line whose plan is 3 inches long. A is one inch and B is 2 inches above the horizontal plane. Represent a plane which passes between A and B, is inclined at 60° to the horizontal plane, and is at perpendicular distances of 1 inch and 1·5 inches from A and B respectively.

21. *vh* and *hc* are two straight lines at right angles to one another and each 3 inches long. V is the vertex and VH the axis of a right circular cone whose vertical angle is 40°. H is on, and V is three inches above the horizontal plane. C is the centre of a sphere 2 inches in diameter which rests on the horizontal plane. Represent a plane which is tangential to the cone and sphere.

22. *abc* is a triangle. $ab = 1·5$ inches, $bc = 1·3$ inches, and $ac = 2·5$ inches, A is 2·5 inches, B is 1 inch, and C is 2·4 inches above the horizontal plane. AB is the axis of a right circular cylinder 1·6 inches in diameter. C is the centre of a sphere 1·4 inches in diameter. Draw the scale of slope of a plane which passes over the cylinder and under the sphere and is tangential to both.

23. A, B, and C are the centres of three spheres whose radii are, 0·8 inch, 0·4 inch, and 0·6 inch respectively. $ab = 1·7$ inches, $bc = 1·3$ inches, and $ac = 1·2$ inches. The heights of A, B, and C above the horizontal plane are, 0·8 inch, 0·6 inch, and 1·6 inches respectively. Represent a plane which passes over the spheres A and C and under the sphere B and is tangential to all three.

24. The elevation of a solid of revolution, whose axis MN is parallel to the vertical plane of projection, is given in Fig. 660. Draw the horizontal trace of a vertical cylinder which envelops this solid, and show the elevation of the curve of contact. s' is the elevation of a point on the front surface of the solid. Represent a plane tangential to the surface at S.

25. The circle *ab* (Fig. 661) is the plan of a prolate spheroid whose axis is vertical and 3 inches long, its lower end being on the horizontal plane. *cd* is the plan of a straight line, C being 4 inches above and D on the horizontal plane. A cylinder whose axis is parallel to CD envelops the spheroid. Show in plan and elevation the curve of contact between the cylinder and the spheroid. *r* is the plan of a point on the lower surface of the spheroid. Draw the traces of the plane which is tangential to the spheroid at R.

26. Taking the spheroid and the point C as in the preceding exercise, draw in plan and elevation the curve of contact between the spheroid and a cone

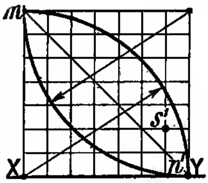


FIG. 660.

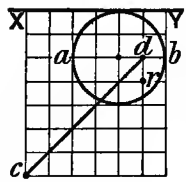


FIG. 661.

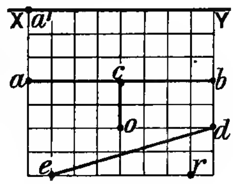


FIG. 662.

In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

enveloping it, the vertex of the cone being at C. Draw also the traces of a plane containing the point C, touching the spheroid, and inclined at 70° to the horizontal plane.

27. In Fig. 663, *aa*, *bb*, and *cc* are the plans and *a'a'*, *b'b'*, and *c'c'* are the elevations of three straight lines. *pp'* is a point on AA. Determine a point *qq'* on BB such that the line PQ shall be 1·5 inches long and sloping downwards from P to Q. Find a point *rr'* in CC such that PR + QR = 3·5 inches.

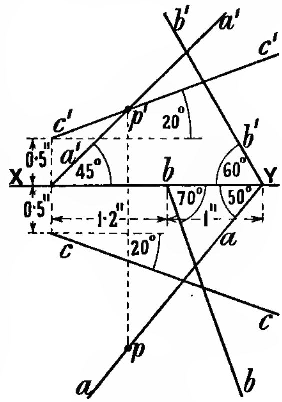


FIG. 663.

28. *acb* (Fig. 662) is the plan of a straight line inclined at 45° to the horizontal plane and sloping upwards from A to B. o is the plan of a vertical axis. A twisted surface of revolution is generated by the revolution of ACB about the vertical axis to which it is connected by a horizontal line of which *oc* is the plan for the position of ACB shown. Draw in plan and elevation 16 positions of the generating line at equal intervals. *r* is the plan of a point on the lower part of the surface generated. Draw the traces of the plane which is tangential to the twisted surface at R. *de* is the plan of a horizontal line whose height above the horizontal plane is 1·5 inches. Show in plan and elevation the points of intersection of DE with the twisted surface.

29. *ab*, and *cd* (Fig. 664) are the plans of two straight lines. The points B and D are on the horizontal plane. A is 3 inches and C is 2 inches above the horizontal plane. An hyperboloid is generated by the revolution of CD about AB. Show the true shape of the section of the hyperboloid by a vertical plane of which HT is the horizontal trace. *r* is the plan of a point on the lower surface of the hyperboloid. Taking H*d* as a ground line draw the traces of the plane which is tangential to the hyperboloid at R.

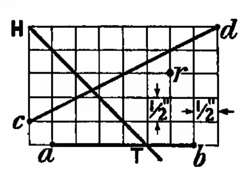


FIG. 664.

30. The axes of two hyperboloids A and B which are in rolling line contact are horizontal. The axis of B is 1·2 inches above that of A. The angular velocity of B is 1·5 times that of A. The ends of A are each 2 inches distant from its throat circle. Draw the plan of the hyperboloids and an elevation on a plane perpendicular to their line of contact.

31. The throat of a cross-roll (Fig. 646, p. 336) is 3·5 inches in diameter. The effective length of the roll is 7 inches. The cylinder is 2·5 inches in diameter and

its axis is inclined at 45° to the axis of the roll. Draw, half size, a plan of the roll and cylinder, their axes being horizontal, showing the curve of contact.

32. OA, OB, and OC are the semi-axes of an ellipsoid. OA = 1·75 inches, OB = 1·25 inches, and OC = 1 inch. OA is horizontal and OB is perpendicular to the vertical plane of projection. The centre O is 1 inch above the horizontal plane. The plan is shown in Fig. 665. Draw a projection of the ellipsoid on a plane parallel to the plane of a circular section.

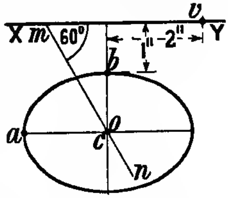


FIG. 665.

mn (Fig. 665) is the plan of a straight line which passes through O and is inclined at 45° to the horizontal plane. A cylinder enveloping the ellipsoid has its generating line parallel to MN. Show in plan and elevation the curve of contact between the cylinder and ellipsoid.

v (Fig. 665) is the plan of a point which is 3·5 inches above the horizontal plane. A cone having its vertex at V envelops the ellipsoid. Draw in plan and elevation the curve of contact between the cone and ellipsoid.

33. OA, OB, and OC are three straight lines mutually perpendicular, OC being vertical. OA = 1·25 inches, OB = 1 inch, and OC = 0·75 inch. OB and OC are the semi-conjugate axes of two hyperbolas having OA as a common transverse axis. These hyperbolas are axial sections of an hyperboloid of two sheets lying between two planes perpendicular to OA and each 4·25 inches from O. Draw a plan of both sheets of the hyperboloid and an elevation on a plane parallel to OA. Draw also an elevation on a plane perpendicular to OA. On each projection show the elliptic sections by planes perpendicular to OA at intervals of, say, 0·75 inch. Show also on each projection a number of axial sections.

34. An elliptic paraboloid is given in Fig. 666 by two elevations on planes at right angles to one another. Draw these elevations to the dimensions given, and

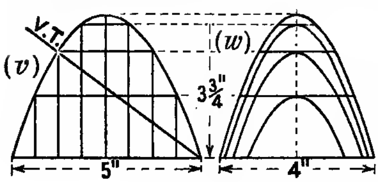


FIG. 666.

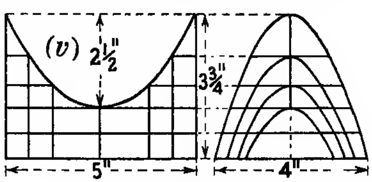


FIG. 667.

from the elevation (*v*) project a plan. Show on each projection the parabolic and elliptic sections indicated. V.T. is the vertical trace of a plane which is perpendicular to the plane of the elevation (*v*); show on the plan and on the elevation (*w*) the section of the paraboloid by this plane.

35. An hyperbolic paraboloid is given in Fig. 667 by two elevations on planes at right angles to one another. Draw these elevations to the dimensions given and from the elevation (*v*) project a plan. Show on each projection the parabolic and hyperbolic sections indicated. Show also on each projection a number of straight lines, say ten, which lie on the paraboloid, half the number to belong to one system and the other half to belong to the other system of straight generators of the paraboloid.

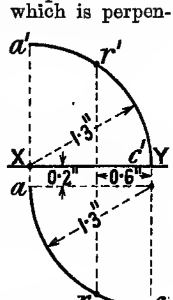


FIG. 668.

36. A tortuous curve ARC is shown in plan and elevation in Fig. 668. Draw the projections of the tangent to the curve at the point R. Show the traces of the normal and osculating planes of the curve at R. Draw also the projections of the principal normal and the binormal to the curve at R.

CHAPTER XXIV

DEVELOPMENTS

305. Development of a Surface.—A surface is said to be developed when it is laid out on a plane, and the figure so obtained is called the *development* of the surface. As has already been pointed out there are certain surfaces which cannot be developed, as for example the surface of a sphere, but approximate developments of such surfaces may be obtained by dividing them up into a number of parts. The determination of the developments of developable surfaces will first be studied and then the method of determining approximate developments of undevelopable surfaces will be considered.

In general when the surface of a prism, pyramid, cylinder, or cone is referred to, the bases or ends will be excluded, but these bases or ends, being plane figures, may easily be added to the development of the remainder of the surface, in order to complete the development, if required.

306. Development of the Surface of a Prism.—First consider the case of a right prism. Draw the plan and an elevation of the prism when the ends are horizontal as shown at (a) and (a') in Fig. 669. The vertical faces of the prism are rectangles and these placed side by side in a plane form the required development which is a rectangle. This development is most conveniently drawn as shown. On the horizontal line LM parts $\overline{1\,2}$, $\overline{2\,3}$, $\overline{3\,4}$, etc. are marked off equal to the sides $\overline{1\,2}$, $\overline{2\,3}$, $\overline{3\,4}$, etc. respectively of the base. Lines through the points 1, 2, 3, etc., on LM at right angles to LM and equal in length to the height of the prism are the positions of the vertical edges of the prism on the development LN.

Now suppose this prism to be converted into an oblique prism by having for the elevations of its ends the lines $c'd'$ and $e'f'$. The development of the surface lying between the ends of the new prism will be a portion of the rectangle LN obtained as follows. From the ends of the vertical edges of the new prism draw horizontals or parallels to LM to meet the lines on the development which are the positions of these edges on the development. For example, from r' the elevation of the upper end of the edge 3 draw the horizontal r'R to meet the vertical 3 on LN. Other points such as R are determined

in the same way and consecutive points being joined as shown, the required development is determined.

The student should now have no difficulty in following the construction shown for determining the part of the development which must be removed due to a hole in the prism, the elevation of the hole being the triangle $s'p'q'$. Observe that six additional vertical lines are required on the faces of the prism; these lines pass through the points where the horizontal internal edges of the hole meet the faces of the prism, and their elevations coincide in pairs. The positions of these additional lines on the development are easily found, and they contain important points in the complete development.

If any line or figure be drawn on the development of the surface of the prism, it is obvious how the elevation of the line or figure when

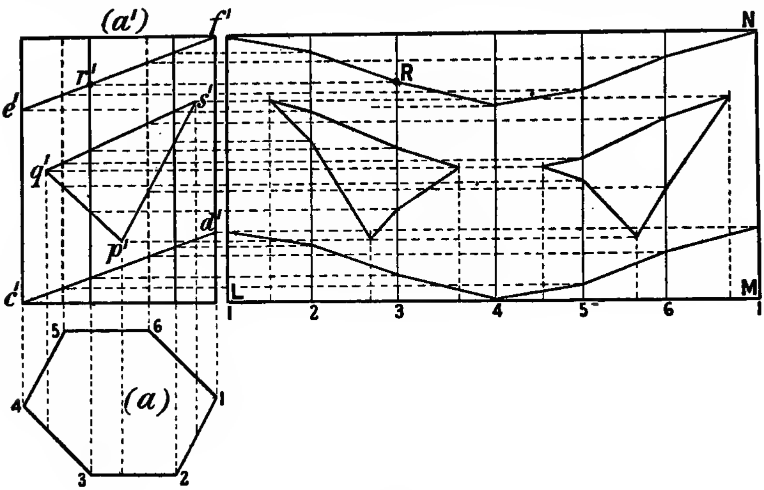


FIG. 669.

the development is wrapped round the solid, may be obtained by working backwards from the development to the elevation of the prism.

Again, given, by their projections, two points on the surface of the prism, the projections of the shortest line lying on the surface of the prism and joining the given points, are easily obtained when it is noticed that the required line will on the development be a straight line. It is necessary to observe however that there are two ways of joining the given points by lines whose developments are straight lines, one line goes round the surface in one direction and the other in the other, but one of these will generally be shorter than the other.

307. Development of the Surface of a Pyramid.—Excluding the base, the surface of a pyramid is made up of a number of triangles having a common vertex, and if the true forms of these triangles be

found and placed together so that those angular points which are made to coincide coincide when on the surface of the solid, the resulting figure will be the development of the surface of the pyramid. All the triangles which make up the development will have a common vertex.

The plan and elevation of a pyramid are given in Fig. 670, the base of the solid being horizontal. The true lengths of the sloping edges are conveniently found as follows. On the elevation of the base produced make v_1a_1, v_1b_1, v_1c_1, etc. equal to va, vb, vc, etc. respectively. Draw the vertical v_1V_1 equal to the altitude of the pyramid. Join a_1, b_1, c_1, etc. to V_1, then V_1a_1, V_1b_1, V_1c_1, etc. are obviously equal to the true lengths of VA, VB, VC, etc. respectively. The true forms

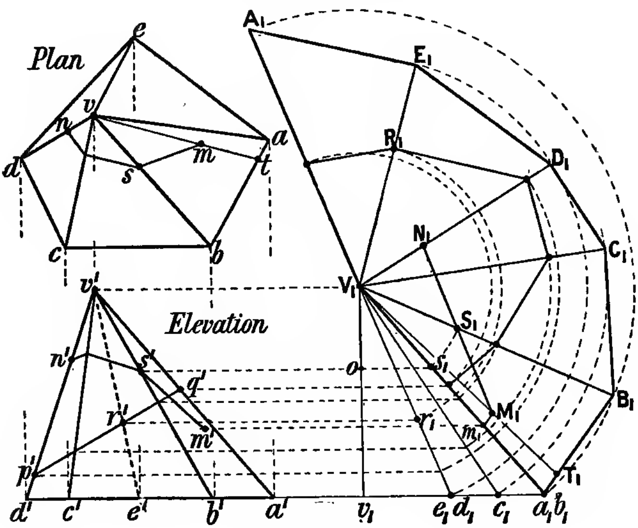


FIG. 670.

of the triangular faces of the pyramid may now be found and put together as shown to form the development $a_1B_1C_1D_1E_1A_1V_1a_1$.

Considering the line $p'q'$ on the elevation of the pyramid as the elevation of a plane section of the solid, the construction is shown for finding the boundary line between the developments of the surfaces of the two parts of the solid into which it is divided by the plane of section. Consider the angular point of the section which is on the edge VE and which has the point r' for its elevation. Draw the horizontal $r'r_1$ to meet V_1e_1 at r_1. With centre V_1 and radius V_1r_1 describe the arc r_1R_1 to meet V_1E_1 at R_1. R_1 is the position of the point R on the development. The positions of the other angular points of the section are determined in like manner.

A straight line $M_1S_1N_1$ is drawn on the development and the construction is shown for finding the plan and elevation of this line

when the development is wrapped round the solid. Consider the point S_1 where the straight line cuts V_1B_1. With centre V_1 and radius V_1S_1 describe an arc cutting V_1b_1 at s_1. Draw the horizontal s_1s' to meet $v'b'$ at s'. A vertical through s' to meet vb at s determines the plan of S. Or s may be found as follows. Let s_1s' meet V_1v_1 at o. Make vs equal to os_1. The latter construction for finding the plan of a point on one of the sloping edges of the pyramid is to be preferred when the plan of that edge is nearly perpendicular to the ground line. In determining the point m' observe that M_1m_1 is a line parallel to a_1B_1 and is not an arc of a circle with centre V_1.

308. Development of the Surface of a Cylinder.—Since a cylindrical surface is one described by a straight line which moves so that it is always parallel to a fixed line, it is evident that the development of a strip of the surface lying between two positions of the generating line which are not far apart will be a four-sided figure of which two opposite sides are straight and parallel and the other two opposite sides will be either straight or approximately straight. The two opposite sides which are either

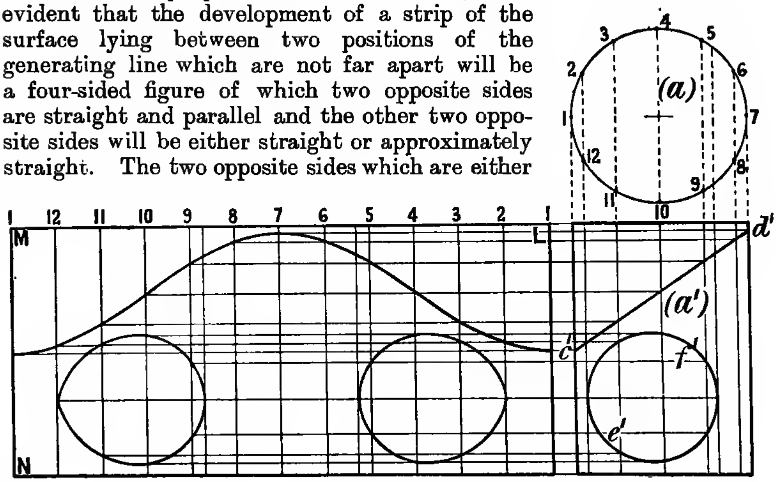


FIG. 671.

straight or approximately straight are equal to the arcs of the ends of the cylinder lying between the two positions of the generating line considered. This suggests the method of finding the development of the surface of a cylinder, which is, to divide the surface into a number of strips by a number of positions of the generating line and then to determine the true forms of these strips which added together give the required development.

In the case of a right cylinder the ends are perpendicular to the generating line and each of the strips mentioned above is a rectangle and all the strips when put together on a plane form a rectangle whose height is equal to the height of the cylinder and whose base has a length equal to the circumference of one end.

Referring to Fig. 671 (a) is the plan and (a') the elevation of a right circular cylinder whose axis is vertical. The circumference of the circle which is the plan of the cylinder is shown divided into

twelve equal parts at the points 1, 2, 3, etc. These points are the plans of twelve positions of the generating line of the surface of the cylinder and from these the elevations are projected. The straight line LM, in line with the elevation of one end of the cylinder is made equal to the circumference of one end and is then divided into twelve equal parts and numbered as shown. The length LM may be obtained by calculation but it will generally be sufficiently accurate to obtain it by stepping out with the dividers from L twelve divisions each equal to the chord of one of the twelve equal arcs on the plan (*a*). The rectangle LN in which MN is equal to the height of the cylinder is the development of the surface of the cylinder. If $c'd'$ is the elevation of a plane section of the cylinder, the form and position of the outline of this section on the development is found as in the case of the prism and is fully shown. Considering the circle $e'f'$ as the elevation of a

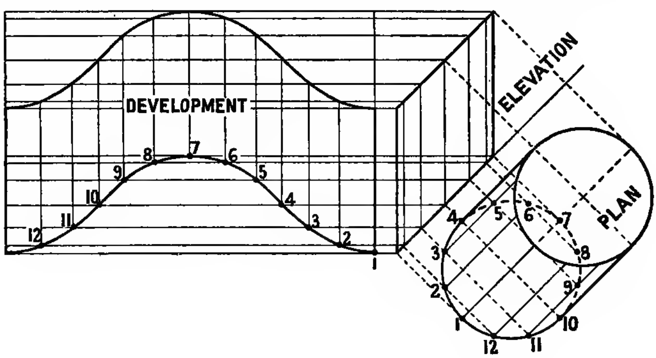


Fig. 672.

cylindrical hole in the given cylinder the forms and positions of the ends of this hole on the development are found as shown.

Fig. 672 shows how the development of the surface of an oblique cylinder having parallel circular ends may be conveniently obtained. The plan and elevation are first drawn as shown. The base circle is divided into twelve equal parts at the points, 1, 2, 3, etc. Considering these points as the lower ends of the lines which are positions of the generating line of the surface of the cylinder the elevations of these lines are then drawn. On the development these lines are not, however, at equal distances apart, but the distances 1 2, 2 3, 3 4, etc. on the development are each equal, very nearly, to one of the twelve equal chords on the base circle. The points 1, 2, 3, etc. on the development lie on the lines which are at right angles to the elevation of the axis of the cylinder and which are projected from the elevation as shown. The points, 1, 2, 3, etc. on the development are quickly obtained by stepping off with the dividers, starting at the point 1.

309. Development of the Surface of a Cone.—Since the

surface of a cone may be described by a straight line which, while moving, passes through a fixed point which is the vertex of the cone, it is evident that if two straight lines be taken which are two positions of the generating line including a small angle, the portion of the surface of the cone lying between these straight lines will when developed be a three sided figure of which these straight lines will be two sides and the third side or base will be nearly straight and of a length equal to the arc of the base of the cone lying between the two positions of the generating line considered. The surface of the cone may therefore be divided into a number of parts whose developments are very approximately triangles. The true lengths of the sides of these triangles are easily found and the triangles being drawn and placed together with their vertices at the same point and those sides coinciding which correspond to those which coincide on the surface of the cone, the development of the surface is determined.

Two examples will serve to illustrate the procedure in determining the development of the surface of a cone.

In the first example (Fig. 673) the cone is a right circular cone. The base is horizontal and its plan is therefore a circle whose centre v is the plan of the vertex of the cone. The elevation is the isosceles triangle $a'v'b'$. Since the line which describes the surface of this cone has a constant length equal to $v'a'$ it is evident that the development of the surface will be a sector of a circle $v'a'$FH whose radius is $v'a'$ and having the arc a'FH equal in length to the circumference of the base of the cone. The length of the arc a'FH is conveniently laid off with sufficient accuracy for practical purposes by dividing the circumference of the base of the cone into, say, twelve equal arcs and stepping the chord of one of these arcs out with the dividers on the arc a'FH twelve times.

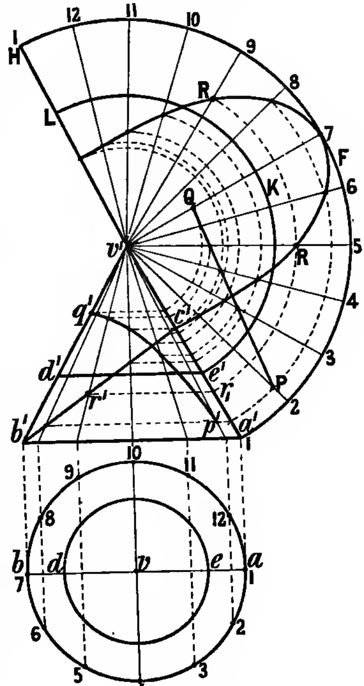


FIG. 673.

If a line on the surface of the cone is given, the form and position of this line on the development is found by a method similar to that used in the case of the pyramid. For example let $b'c'$ be the elevation of a plane section of the cone, it is required to show the form and position of the outline of the section on the development. Draw the

elevations of the lines which join the vertex of the cone to the points 1, 2, 3, etc. which divide the circumference of the base into, say, twelve equal parts. (The plans of these lines need not be drawn.) Join v' to the points 1, 2, 3, etc. which divide the arc a'FH into the same number of equal parts as the circumference of the base is divided into. Consider the point r' which is the elevation of the points on the section which lie on the lines from the vertex to the points 5 and 9 on the base. Draw $r'r_1$ parallel to $b'a'$ to meet $v'a'$ at r_1. $v'r_1$ is evidently the true distance of the points of which r' is the elevation from the vertex of the cone. Hence if with v' as centre and $v'r_1$ as radius an arc be described to cut the lines $v'5$ and $v'9$ on the development, points RR on the required curve are found. In like manner the points where the required curve cuts the other radial lines on the development are found and a fair curve through them is the curve required. The figure lying between this curve and the arc a'FH is the development of the surface of the frustum of the cone lying between the base and the plane of section. If the plane of section is parallel to the base and has for its elevation the line $d'e'$ then the development of the surface of the frustum is the figure lying between the arc a'FH and the arc e'KL struck from the centre v'. This figure is the development of a lamp shade whose elevation is $a'b'd'e'$.

Fig. 673 also shows how to determine $p'q'$ the elevation of a line on the surface of the cone which on the development is a straight line PQ. The construction is simply the converse of that already given for finding on the development a line whose elevation is given and which lies on the surface of the cone.

In the second example now to be considered the cone is an oblique cone. The plan[1] and elevation of the cone are given to the left in Fig. 674. The horizontal trace of the surface of this cone is a circle but the procedure would be the same if this horizontal trace were an ellipse or any other curve. Divide the horizontal trace of the surface of the cone into a number of equal parts, say twelve, at the points 1, 2, 3, etc. Draw the plans and elevations of the straight lines joining vv' the vertex of the cone to the points of division on the horizontal trace. The next step is to find the true lengths of these lines. This is conveniently done as follows. Let v_1 be the foot of the perpendicular from v' on XY. Set off on XY to the right of v_1 the distances $v_1 1$, $v_1 2$, $v_1 3$, etc. equal to the distances $v1$, $v2$, $v3$, etc. respectively on the plan. Join v' to the points 1, 2, 3, etc. on XY to the right of v_1. These lines give the true lengths of the lines joining the vertex of the cone to the points of division on the horizontal trace of its surface. With centre v' and these true lengths as radii describe arcs as shown. Now set the dividers to one of the equal divisions on the horizontal trace of the surface of the cone (shown on the plan) and starting at, say, the point A on the arc whose radius is the true length $v'1$ step out from arc to arc as shown and obtain the points 2, 3, 4, etc. which will lie on the curve ABC which together with the straight lines v'A and v'C form the outline of the development of the surface of the cone.

[1] To economise space the plan has been placed above the elevation in Fig. 674.

It is not absolutely necessary that the parts 1 2, 2 3, 3 4, etc. on the plan be equal and it may happen that in the solution of some problem on the cone, say, the determination of its intersection with another cone, plane sections through the vertex of the cone may have been used giving a series of lines whose horizontal traces do not divide the horizontal trace of the surface of the cone into equal parts. In such a case it is not necessary to draw a fresh set of lines arranged as in Fig. 674, but the parts 1 2, 2 3, 3 4, etc. on the curve ABC must be made equal respectively to the parts 1 2, 2 3, 3 4, etc. on the plan.

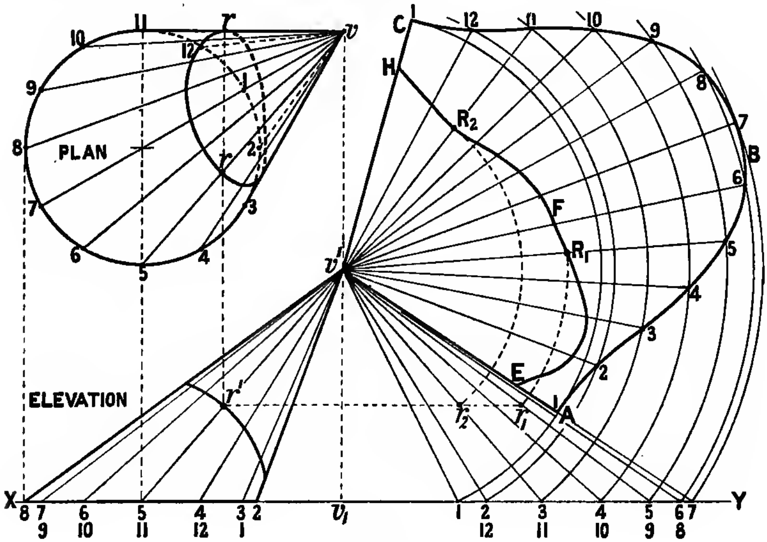


FIG. 674.

Fig. 674 also shows the curve EFH, which is the form and position of the outline on the development of a cylindrical section of the surface of the cone. The construction lines for two points R_1 and R_2 only on this curve are shown. The construction is obvious and needs no description.

310. Approximate Developments of Undevelopable Surfaces.—It has already been stated (Art. 273, p. 313) that a developable surface must be capable of generation by a straight line and that any two consecutive positions of the generating line must be in the same plane. A surface which is undevelopable may however be divided into parts of which approximate developments may be drawn, the degree of approximation being greater the more numerous the parts into which the surface is divided. Three examples will be taken to illustrate the method of determining approximate developments of undevelopable surfaces.

The sphere will be taken as the first example. The surface of a

sphere may be conveniently divided into *zones*. A *zone* of the surface of a sphere is the portion lying between two planes which are perpendicular to an axis of the sphere. The surface of a sphere may also be conveniently divided into *lunes*. A *lune* of the surface of a sphere is the portion lying between two planes which contain an axis of the sphere.

Fig. 675 shows in plan and elevation one-eighth of a sphere. The surface is divided into three zones. Each zone is assumed to be the surface of a frustum of a cone and its development A is found in the usual way as shown. The plan is shown divided into three equal sectors and these are the plans of three equal lunes. The figure B is the approximate development of the half of one of these equal lunes of

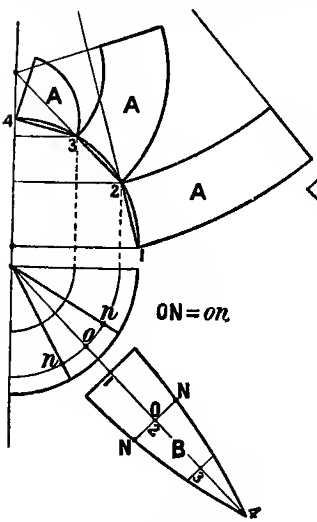


FIG. 675.

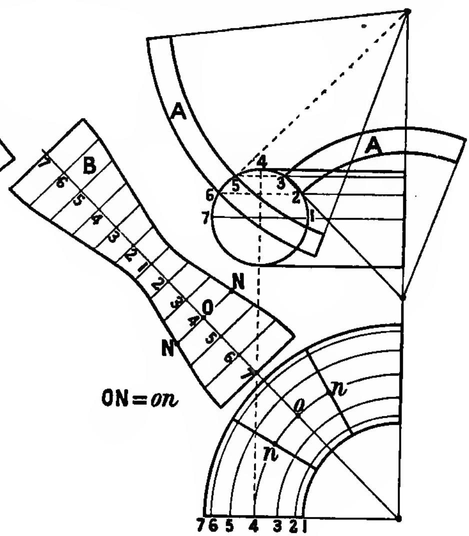


FIG. 676.

the surface of the sphere. In addition to the information given in Fig. 675, it is only necessary to state that the distances 1 2, 2 3, and 3 4 on the figure B are equal to the arcs 1 2, 2 3, and 3 4 on the elevation.

The second example (Fig. 676) is the quarter of an annulus. The surface is divided into zones and lunes which are developed as in the case of the sphere. Any surface of revolution may be treated in the same way.

In the last example (Fig. 677) the surface is one traced by a straight line which moves in contact with a straight line *ae*, *a'e'* and a curved line *fl*, *f'l'*. The straight line *ae*, *a'e'* is divided into a number of equal parts at the points *bb'*, *cc'*, and *dd'*. The curved line is divided into the same number of equal parts at the points *gg'*, *hh'*, and *kk'*. The

straight lines joining the points on the straight line to the points on the curved line as shown divide the surface into a number of three-sided figures whose true forms are found in the usual way and are put together to form the approximate development AELF.

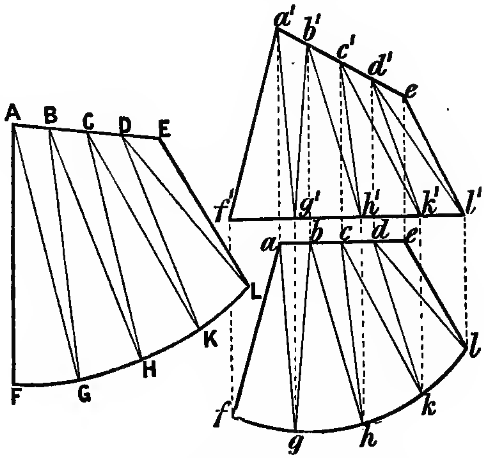


FIG. 677.

Exercises XXIV

Note. The student is advised to cut out the developments in the following exercises, after he has drawn them. He should then fold them up so as to exhibit the forms of the solids. Where a development has to be folded to give a sharp edge, the line corresponding to that edge on the development should be scratched with a needle or perforated at short intervals. Strips may also be left adjoining certain of the edges of the development to form overlaps for gumming together when the development is folded up to the form of the solid. In Fig. 678, which is the development of the surface of a cube, the strips referred to are shown at *a, a* . . .

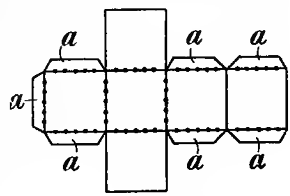


FIG. 678.

1. A plan and an elevation of a right prism are shown in Fig. 679. $p'q'$ is the elevation of a plane section of the solid. Draw the development of the surface of the prism and show on it the boundary line of the section. Draw also the elevation of the shortest line lying on the surface of the prism and joining the points M and N.

2. Two intersecting prisms are shown in Fig. 680, one being vertical and the other inclined. Draw the development of the surface of that part of the inclined prism which lies outside the other.

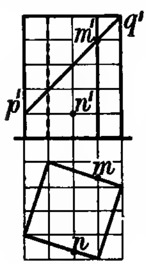


FIG. 679.

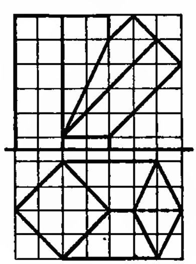


FIG. 680.

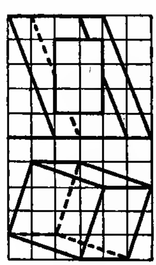


FIG. 681.

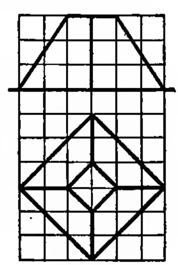


FIG. 682.

In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

3. An oblique prism standing on a square base is shown in Fig. 681. There is a rectangular hole in this prism which is shown in the elevation only. Draw the development of the surface of the prism showing on it the outline of the hole.

4. Draw the development of the frustum of a square pyramid shown in Fig. 682.

5. A pyramid is shown in Fig. 683. $p'q'$ is the elevation of a plane section of the solid. Draw the development of the surface of the pyramid with the outline of the section on it.

6. Fig. 684 shows a pyramid in plan and elevation. A square hole in the solid is shown in the elevation only. Draw the development of the surface of the pyramid with the outline of the hole on it.

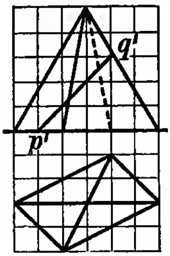


FIG. 683.

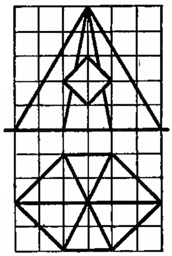


FIG. 684.

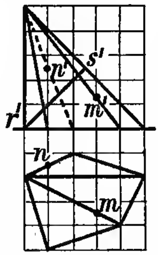


FIG. 685.

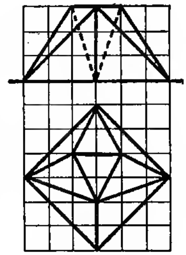


FIG. 686.

In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

7. Draw the development of the surface of the pyramid given in Fig. 685. $r's'$ is the elevation of a plane section of the solid. Show the outline of the section on the development. Draw, in plan and elevation, the shortest line lying on the surface of the pyramid and joining the points M and N.

8. A solid is shown in plan and elevation in Fig. 686. The base of the solid is a square and all the other faces are triangles. Draw the development of the surface of this solid.

9. The shaded part of Fig. 687 is the elevation of a portion of a right circular cylinder whose axis is vertical. Draw the development of the surface of this portion of the cylinder.

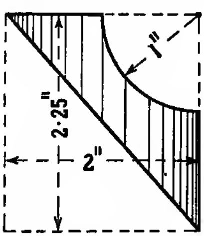


FIG. 687

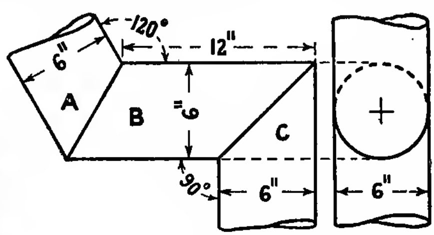


FIG. 688.

10. Fig. 688 shows two views of a sheet metal pipe B with two sheet metal branches A and C. Draw, one quarter of full size, the development of the surface of B.

11. Two pipes A and C (Fig. 689), of circular cross section and having their axes parallel, are connected by a third pipe B as shown. Draw to a scale of 1 inch to 2 feet the development of the surface of the pipe B.

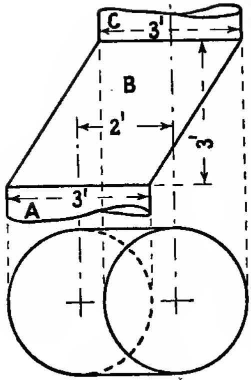


FIG. 689.

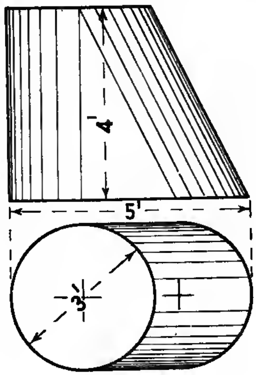


FIG. 690.

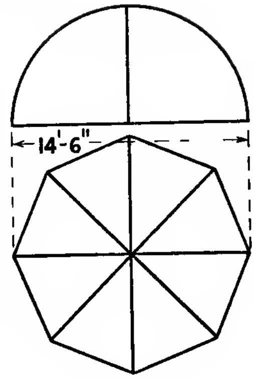


FIG. 691.

12. Draw to a scale of 1 inch to 2 feet the development of the surface of the part of a sheet metal uptake for a boiler shown in Fig. 690.

13. The plan and part elevation of a dome, horizontal sections of which are regular octagons, are shown in Fig. 691. Complete the elevation. Draw a development of one of the eight curved faces of the dome. Scale 1 inch to 5 feet. [B.E.]

14. Fig. 692 represents a pipe made from sheet metal. Draw, to a scale of $\frac{1}{4}$, a development of the pipe showing the shape of the sheet from which it is bent. [B.E.]

15. A right circular cone is cut by a plane which bisects its axis and is inclined at 45° to it. Draw the development of the surface of the frustum. Diameter of base of cone 3 inches, altitude 3 inches.

16. The elevation of a can is given in Fig. 693. Show the shapes to which the sheet metal must be cut, when flat, to form the two conical parts of the can. Omit the allowances for overlap at the seams.

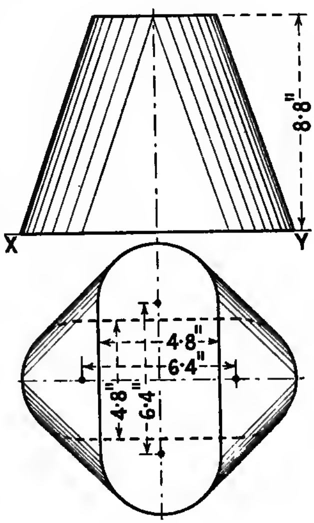


FIG. 692.

17. Fig. 694 shows the elevation of a right circular cone, whose axis is vertical, penetrating two circular cylinders whose axes are perpendicular to the axis of the cone. Draw the development of the surface of that part of the cone which lies between the two cylinders.

18. Fig. 695 shows the elevation of two frusta of two cones of revolution enveloping the same sphere. Draw the developments of the surfaces of the frusta.

19. Two vertical circular pipes of different diameters are connected by another which is conical, as shown in plan and elevation in Fig. 696. Draw the development of the surface of the conical pipe.

20. The solid shown in plan and elevation in Fig. 697 has a base which is a quadrant of a circle. Of the remaining six faces, four are plane triangles and two are conical surfaces. Draw the complete development of the surface of this solid.

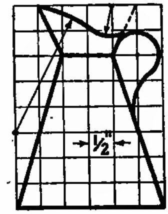


FIG. 693.

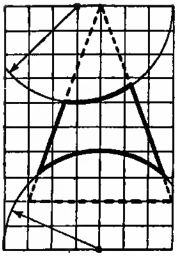


FIG. 694.

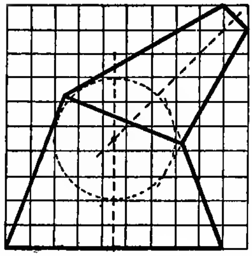


FIG. 695.

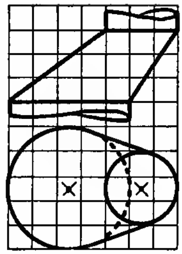


FIG. 696.

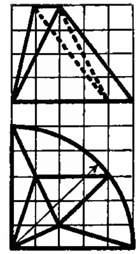


FIG. 697.

In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

21. A cone, base 2·7 inches diameter, height 2·35 inches, has its axis inclined at 40°. A curve is traced on the cone which, in development, would be a circle of 1 inch radius touching the base of the cone. Draw the plan of the cone, and of the curve traced on it, touching the base of the cone at its highest point. [B.E.]

22. The radius of a sphere is 1·5 inches. Draw the approximate development of a lune of the surface of this sphere. Angle of lune 30°.

23. A surface is described by the revolution of an ellipse about its minor axis. The major and minor axes of the ellipse are 3 inches and 2 inches long respectively. Draw the approximate development of a lune of this surface. The lune to lie between two planes containing the axis of revolution and including an angle of 30°.

24. An elbow pipe is 6 inches in diameter and the radius of its centre line is 8 inches. Draw the approximate development of a lune of the surface of this pipe. Angle of lune $22\frac{1}{2}$°. Scale $\frac{1}{4}$.

25. A sheet-metal hood is square at the top and circular at the bottom as shown in Fig. 698. Show the shape to which the flat sheet of metal must be cut to form the hood.

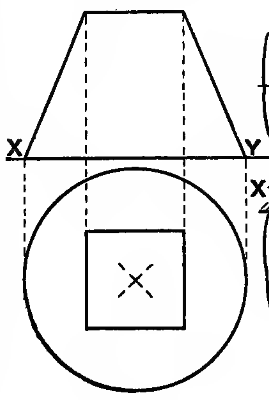


FIG. 698.

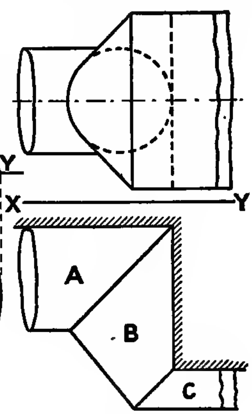


FIG. 699.

26. ABC (Fig. 699) is a sheet-metal pipe, the portions A and C of circular and rectangular section respectively. By developing B set out the shape to which the flat sheet of metal forming it must be cut. Omit all allowances for overlap at the seams. [B.E.]

CHAPTER XXV

HELICES AND SCREWS

311. The Helix.—The *helix* is the curve described by a point which moves with uniform velocity along a generating line of a right circular cylinder while the generating line revolves with uniform angular velocity about the axis of the cylinder.

The *axial pitch* of a helix is the distance between one turn of the helix and the next, measured parallel to the axis of the cylinder upon which it is traced. Or, the axial pitch is the distance travelled by the describing point along the cylinder while it moves once round the cylinder. The *normal pitch* of a helix is the distance between one turn and the next, measured along the shortest line on the surface of the cylinder. If several helices of the same pitch be traced on the surface of the same cylinder, at equal distances apart, the distance between two adjacent helices is called the *divided pitch*. When the term "pitch" is used without any qualification, "axial pitch" is understood. The *diameter of a helix* is the diameter of the cylinder upon which it is traced.

The construction for drawing the projection of a helix on a plane parallel to the axis of the cylinder follows at once from the definition of the curve and is shown in the right hand portion of Fig. 700. The axis of the cylinder is assumed to be perpendicular to the vertical plane of projection. Divide the circle which is the elevation of the cylinder into a number of equal parts, say twelve, at the points 1, 2, 3, etc.

It is evidont from the definition of a helix that if the generating point moves round any fraction of the circumference of the cylinder, it will at the same time move in the direction of the axis of the cylinder a distance equal to the same fraction of the pitch. Thus, if the point move round the cylinder a distance shown in the elevation by the arc 12, that is, through 1-12th of the circumference, it will at the same time move parallel to the axis a distance equal to 1-12th of the pitch. In like manner, in moving round another 1-12th of the circumference, it will move parallel to the axis another distance equal to 1-12th of the pitch. Hence the following simple construction.

Divide the pitch *ab* into as many equal parts as the circle in the elevation is divided into, in this case twelve, at the points 1, 2, 3, etc.

Through these points draw perpendiculars to ab to meet projectors from the points on the circle as shown. The points of intersection of these two sets of lines carrying the same numbers are points on the plan of the helix and a fair curve through them is the projection required. *aeb* is the plan of one turn of the helix and *bfd* is the plan of the next turn. The plan of the second turn of the helix may be obtained in the same way as the first, or it may be determined from the first by measuring from it, along the plans of the generating lines, a constant length equal to the pitch of the helix.

On the development of the surface of the cylinder the helix becomes a straight line. In Fig. 700 the straight line AEB is the development of one turn of the helix and the straight line CFD parallel to AEB is the development of the next turn. A straight line MN at right angles to AB and CD is the development of a helix at right angles to the one already considered. The perpendicular distance

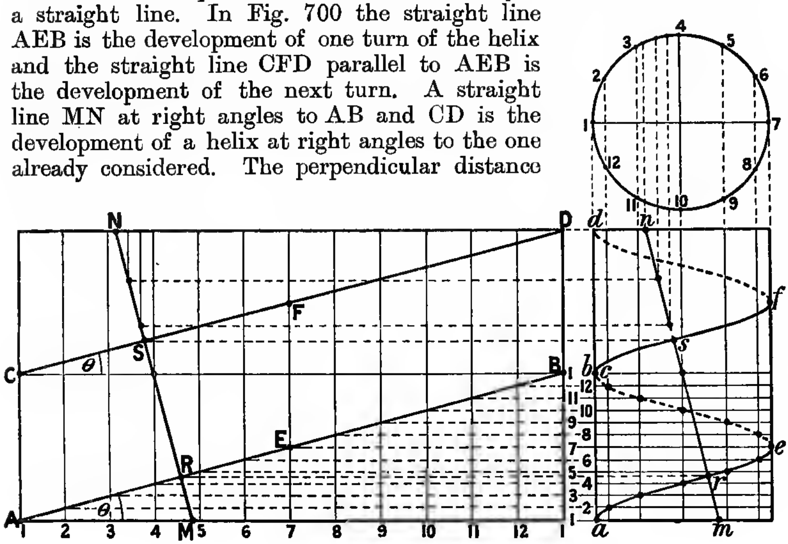


FIG. 700.

RS between the lines AB and CD is the normal pitch of the first helix.

The *inclination* of the helix or the *pitch angle* of the helix is the angle θ on the development in Fig. 700, and is the complement of the angle which the tangent to the helix at any point makes with the generating line of the cylinder through that point. If d is the diameter of the cylinder and p the pitch of the helix then $\tan\theta = \dfrac{p}{\pi d}$. If a second helix on the same cylinder is perpendicular to the first and p' is its pitch and θ' its pitch angle, then $\theta' = 90^\circ - \theta$ and $p' = \dfrac{\pi^2 d^2}{p}$.

The helix shown in Fig. 700 is *right-handed.* If the full and dotted parts of the plan of the helix shown in Fig. 700 be made dotted and full respectively the helix would become *left-handed.*

312. Helix of Increasing Pitch.—A point which moves round

a cylinder with uniform angular velocity and at the same time moves along the cylinder with an increasing velocity describes a curve which is generally called a *helix of increasing pitch.* The curve is however not a helix. The development of a helix of increasing pitch is a curved line while the development of a true helix is a straight line.

A helix of increasing pitch is shown in Fig. 701. In this example the describing point is supposed to move along the cylinder with

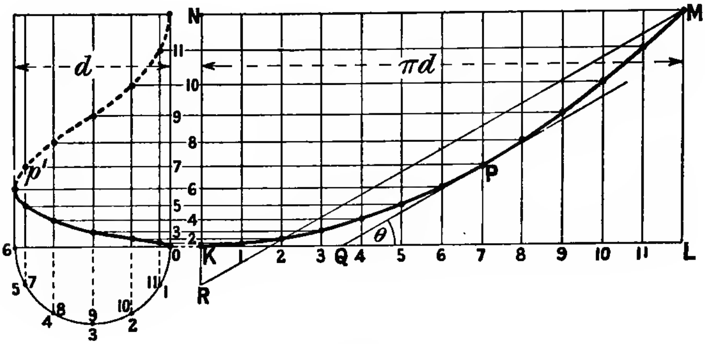


FIG. 701.

uniform acceleration while its angular velocity about the axis of the cylinder is constant.

The development of the curve is the parabola KPM, having KN for its axis and K for its vertex. If a tangent PQ be drawn to the parabola at P, then θ the inclination of PQ to KL is the pitch angle of the helix of increasing pitch at the point whose elevation is p' and the corresponding pitch is NR obtained by drawing MR parallel to PQ to meet NK at R.

313. Screw Surfaces.—A screw surface is generated by a straight line which slides with uniform velocity along a fixed straight line or axis with which it makes a constant angle, and at the same time revolves about that axis with uniform angular velocity. It is obvious that any point in the generating line describes a helix.

Fig. 702 shows the portion of a screw surface generated by a straight line which starts from the position ao in plan and $a'd'$ in elevation and makes half a revolution about a vertical axis. The generating line is shown in thirteen positions in plan and elevation. The outer end of the generating line describes the helix whose elevation is $a'b'c'$.

The curve $b's'c'$ is the elevation of the section of the screw surface by the vertical plane whose horizontal trace is LM, and ore is the plan of the section of the screw surface by the horizontal plane whose vertical trace is PQ. The constructions for one point ss' in the former section and one point rr' in the latter are shown.

It should be observed that the boundary line of the elevation of

the left-hand portion of the screw surface, apart from the curve $a'b'$ is not straight but is a curved line (not shown) which touches the elevations of different positions of the generating line. Also if the surface be extended upwards beyond $c'f'$ the boundary line of the right-hand portion of the screw surface will not be the straight line $c'f'$ but a curved line touching the elevations of different positions of the generating line.

Fig. 703 shows the surface generated by a quadrant of a circle which slides along a vertical axis with uniform velocity and at the same time revolves about that axis with uniform angular velocity. ao is the plan and $a'd'o'$ the elevation of the generating figure in its initial position. The generating figure makes half a revolution. The helices

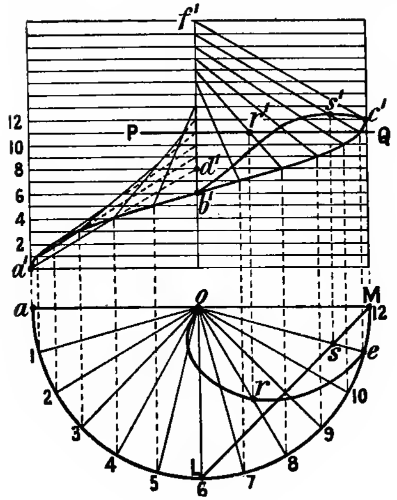


FIG. 702.

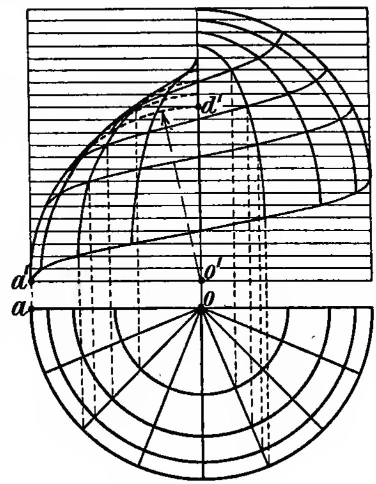


FIG. 703.

described by four points on the moving arc are shown. The generating figure is also shown in nine positions in plan and elevation. These two sets of contour lines give to the elevation a pictorial effect and represent the surface more clearly.

The remarks on the boundary line in Fig. 702 apply also to Fig. 703 except that in the latter Fig. the boundary line on the left-hand portion of the elevation has been added.

314. Screw Threads.—If the edges of a screw thread are sharp they form true helices. In practice the edges are often slightly rounded, as in the Whitworth standard V-thread, and this rounding can only be shown on the complete projection of the thread by shading or contouring. When the rounding of the edges is small it is generally neglected on drawings except in the case of a cross section.

Ordinary screws, such as are found on bolts, are generally

represented in a conventional way on technical drawings. The helices of such screws are of such small pitch compared with their diameter that their projections on a plane parallel to the axis of the screw are approximately straight lines. It is only when the pitch angle of the helix of a screw is comparatively large that the projection of the helix is drawn correctly. For conventional methods of representing screw threads the student is referred to any text-book on machine drawing.

Fig. 704 shows at (*a*) a projection of a right-handed V-threaded screw on a plane parallel to its axis. The section of the thread by a plane containing the axis is an equilateral triangle. The top and bottom edges of the thread are helices of the same pitch on two co-

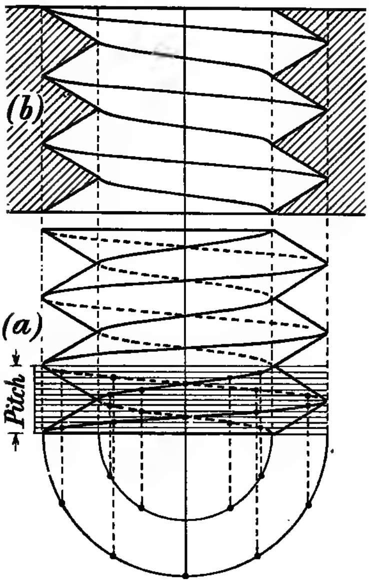


Fig. 704.

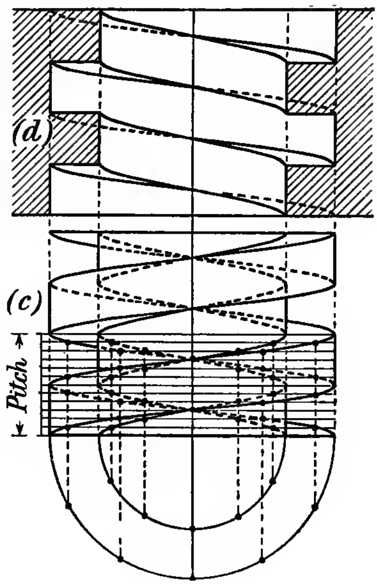


Fig. 705.

axial cylinders. Although the outer edge of this thread is quite sharp it will be observed that in the projection (*a*) it has at the right and left of the figure the appearance of being rounded. The boundary lines connecting the projections of the outer and inner helices at (*a*) are tangential to these projections and they are practically straight lines. A section of the nut for this screw by a plane containing its axis is shown at (*b*).

Fig. 705 shows at (*c*) a projection of a right-handed square-threaded screw on a plane parallel to its axis. The section of the thread by a plane containing the axis is a square. The top and bottom edges of the thread form four helices of the same pitch, two on one cylinder and two on

another, the cylinders being co-axial. A section of the nut for this screw by a plane containing its axis is shown at (*d*).

A screw thread of rectangular section is shown in Fig. 706. In this case the thread is thin compared with its depth and pitch.

A quadruple-threaded screw or worm is shown in Fig. 707. At the top left-hand corner of the figure is shown the form of the section of the threads by a plane containing the axis of the screw.

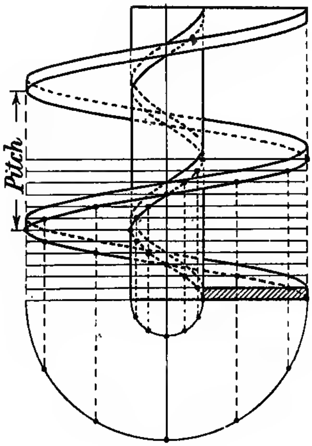


FIG. 706.

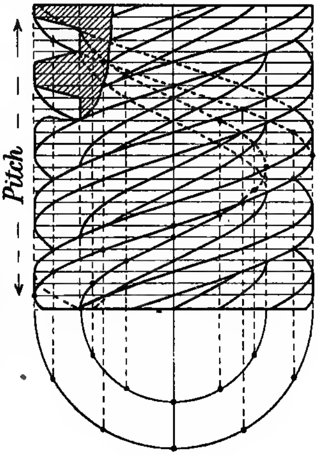


FIG. 707.

315. Pitch and Lead of a Multiple-threaded Screw.—It is usual to take the pitch of one of the threads of a multiple-threaded screw as the "pitch" of the screw, and the distance between the centres of two adjacent threads (measured parallel to the axis of the screw) is then called the "divided pitch" of the screw, as already stated for helices (Art. 311, p. 362). In America however it is a common practice to call the pitch of one of the threads of a multiple-threaded screw the "lead" of the screw and reserve the term "pitch" to what has been called the "divided pitch." The lead of a screw is of course the axial distance through which the nut moves for one revolution of the screw.

316. Helical Springs.—While a screw has a screw thread in one piece with a cylinder from which it projects, a *helical spring* is a screw thread without the attached cylinder. It therefore follows that in the case of the spring the absence of the cylinder will expose to view a greater part of the thread than is seen on a screw.

Fig. 708 shows a helical spring of which the section by a plane containing the axis of the spring is a square.

A helical spring made of round wire is represented in Fig. 709. The centre line of the wire forms a helix and the boundary lines of the projections of the spring are obtained by considering the spring as the

envelope of a sphere whose diameter is equal to that of the wire and which moves with its centre on the helix. On the half plan of the spring is shown a section by a horizontal plane whose vertical trace is LN. The outline of the section is tangential to the horizontal sections of the moving sphere by the given plane of section. A section of the spring by a plane containing the axis is nearly but not quite circular.

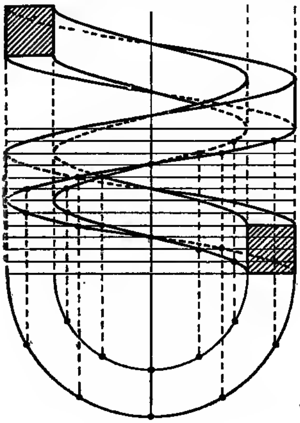


FIG. 708.

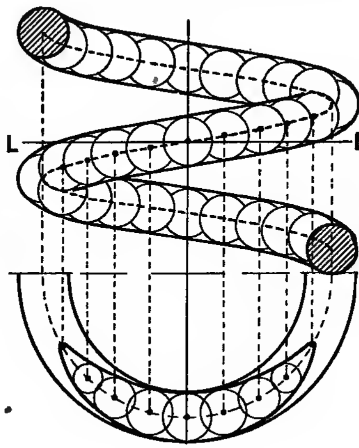


FIG. 709.

317. Axial and Normal Sections of Screw Threads.—By an axial section of a screw thread is meant a section by a plane containing the axis and by a normal section is meant a section by a plane at right angles to the central helix of the thread. When the pitch of a screw thread is small compared with its diameter there is little difference between an axial section and a normal section, but when the pitch is large the difference is considerable.

Fig. 710 shows a projection of a part of a screw thread or helical spring whose central helix has a large pitch angle, the projection being on a plane parallel to the axis. It is evident that where the projections of the various helices cut the projection of the axis these projections are straight and are inclined to the projection of the axis at angles which are the complements of the pitch angles of the helices. Also at the points considered the helices are parallel to the plane of projection.

Still referring to Fig. 710, the axial section of the thread is the square S, *oab* is the projection of the central helix of the thread, and *ot* is its tangent at *o*. L*o*N, at right angles to *ot* is the edge view of a plane which is perpendicular to the helix at O. The figure EFHK is the true form of the normal section at LN. The edges EF and KH are nearly straight but are arcs of ellipses, being parts of an oblique section of two cylindrical surfaces. The edges EK and FH are sections of screw surfaces and are very approximately straight lines. It will be

seen that the thickness of the thread tapers distinctly at a normal section.

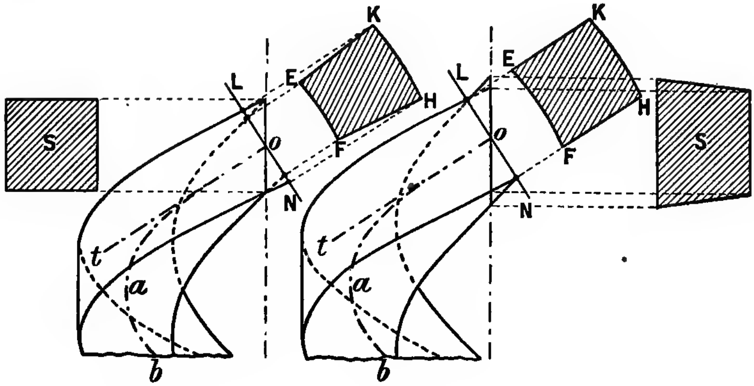


FIG. 710. FIG. 711.

Fig. 711 shows the construction for finding S the true form of an axial section when the normal section EFHK is of uniform width, that is, when the thickness of the thread at a normal section is uniform.

318. Handrails.—The making of handrails for stairs is considered to be one of the most difficult parts of the craft of the joiner and a knowledge of the geometry of handrails is essential to their correct and economical construction. The complete treatment of the subject of handrailing in all its technical details is beyond the scope of this work but the fundamental principles and their applications to a few examples may be studied here with advantage.

The construction of a straight handrail presents no difficulty. *Ramps* and *level easings* are also easily drawn and made. A *ramp* is the part of a handrail whose centre line is curved in a vertical plane only. A *level easing* is the part of a handrail whose centre line is curved in a horizontal plane only. A level easing whose centre line is a quarter of a circle is called a *level quarter*. The part of a handrail whose centre line is curved vertically and horizontally is called a *wreath*. A straight piece of handrail formed on the end of a wreath is called a *shank*. It is the construction of wreaths which is the difficult part of handrailing and their geometry will now be considered.

The geometry of the centre line of the wreath will first be studied. Referring to Fig. 712, *ab*, *a'b'* and *cd*, *c'd'* are the straight centre lines of two straight pieces of handrail which are to be connected by a wreath whose centre line is, in plan, a quarter of a circle *bc*. The lines *ab*, *a'b'* and *cd*, *c'd'* are called the central tangents of the wreath and in order that the wreath may be cut economically from a plank it is usual to arrange that these tangents shall intersect and therefore lie in a plane. In Fig. 712 the tangents *ab*, *a'b'* and *cd*, *c'd'* intersect at the

point tt' and in the example considered the tangents are equally inclined to the horizontal plane.

The centre line of the wreath lies on the surface of a vertical circular cylinder, and to the left of the elevation (U) is shown the development of the vertical surface containing the centre line of the wreath and the tangents, A_1B_1 and C_1D_1 being the developments of the tangents. The development of the centre line of the wreath is shown as a straight line B_1C_1, but it will be seen that A_1B_1 and C_1D_1 are not in the same straight line with B_1C_1. To produce a graceful centre line "easing curves" should be introduced between the centre line of the wreath and the tangents. These easing curves may be entirely on the surface of the cylinder or entirely outside of it or they may be partly on the cylinder and partly ontside of it.

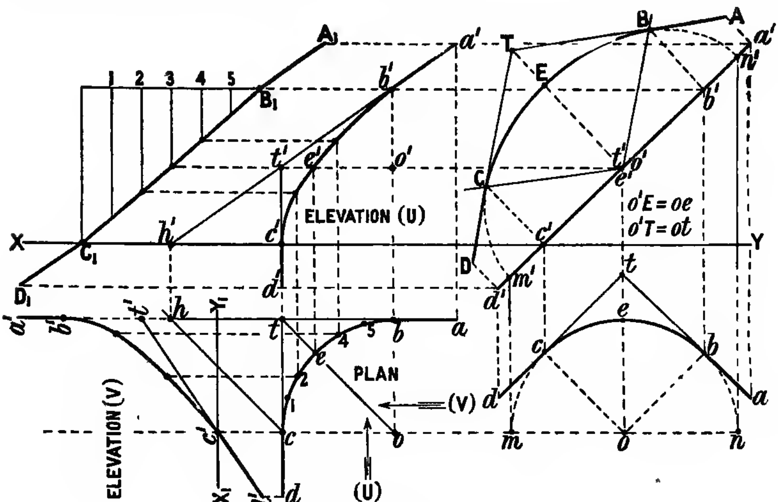


FIG. 712.

Taking the straight line B_1C_1 as the development of the centre line of the wreath the elevation of this centre line will be a helix obtained by the usual construction or by projectors from the plan and the development as shown. The plane containing the tangents ab, $a'b'$ and cd, $c'd'$ is an important one. If $a'b'$ be produced to meet XY at h' and a perpendicular $h'h$ be drawn to XY to meet ab produced at h, the horizontal trace of the tangent ab, $a'b'$ on a plane at the level of cc' is determined, and hc is the horizontal trace of the plane of the tangents ab, $a'b'$ and cd, $c'd'$. Any line in the plane of the tangents which is parallel to hc will be a horizontal or level line of the plane. When the tangents are equally inclined to the horizontal it is easy to prove that hc is parallel to to and that to bisects the angle boc. This is true whatever be the magnitude of the angle boc provided the tangents intersect and have equal inclinations to the horizontal.

An elevation of the tangents on a plane perpendicular to *hc* or *to* will show the two tangents in one straight line since this elevation is an edge view of the plane containing them. Such an elevation is shown to the right in Fig. 712, the plan having been redrawn and turned round for convenience so that *ot* is perpendicular to XY. The plane of the tangents will intersect the cylinder, on the surface of which the centre line of the wreath is situated, in an ellipse whose major axis is *m'n'* and whose semi-minor axis is the radius of the cylinder. The true form of this ellipse is shown with the tangents of the wreath in their proper positions in relation to the ellipse. The student who has reached this stage will not require to be told how to construct the ellipse and the lines connected with it shown to the right in Fig. 712. The line ABECD is the centre line of what is called the *face mould* of the wreath, the use of which will be explained later. Lines *o'*B and *o'*C drawn from the centre of the ellipse to the points B and C where the tangents meet the ellipse are called the *springing lines* of the wreath. The plane containing the tangents will not contain the true centre line of the wreath but in most cases the true centre line of the wreath will lie very near to this plane. In the case shown in Fig. 712 the true centre line of the wreath lies partly on one side of the plane of the tangents and partly on the other crossing it at the point *ee'*. In the right hand part of Fig. 712 the elevation of the true centre line of the wreath is not shown.

The case where the tangents are in vertical planes at right angles to one another and one of the tangents *cd*, *c'd'* is horizontal is illustrated in Fig. 713. The student should have no difficulty in dealing with this case after having mastered the more general case which has just been considered. It will be noticed that the plane of the tangents has *dct* for its horizontal trace and the elevation (U) is an elevation on a plane perpendicular to the plane of the tangents ABCD the centre line of the face mould is made up of the two tangents AB and CD and BC the quarter of an ellipse. B_1C_1 the development of the centre line of the wreath has been drawn with an easing curve joining it to C_1D_1 the development of the horizontal tangent, this easing curve being entirely on the development of the surface of the cylinder on which the central line of the wreath is situated.

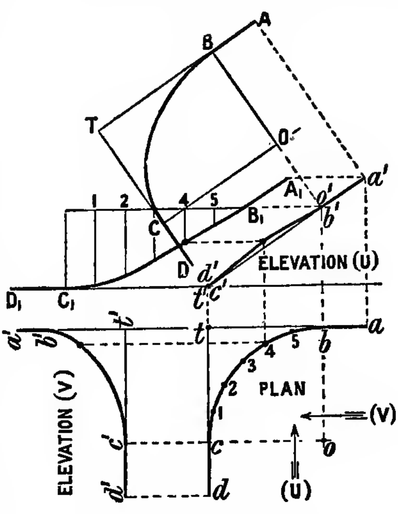


FIG. 713.

If the centre line of the face mould, determined as explained for Figs. 712 and 713, be taken as the true centre line of the wreath, the student should have no difficulty in drawing the development and any required projections of it. The construction of this new centre line is shown in Fig. 714 for the case illustrated in Fig. 712. The plan and the positions of the tangents *ab*, *a'b'* and *cd*, *c'd'* are supposed to be given as before. The plane of the tangents and the edge view of this plane shown to the right in Fig. 714 are determined exactly as in Fig. 712. From the edge view of the plane of the tangents and the plan beneath it the development $A_1B_1C_1D_1$ is projected as shown. The elevation (U) is projected from the plan and the development, or from the plan and the edge view as shown.

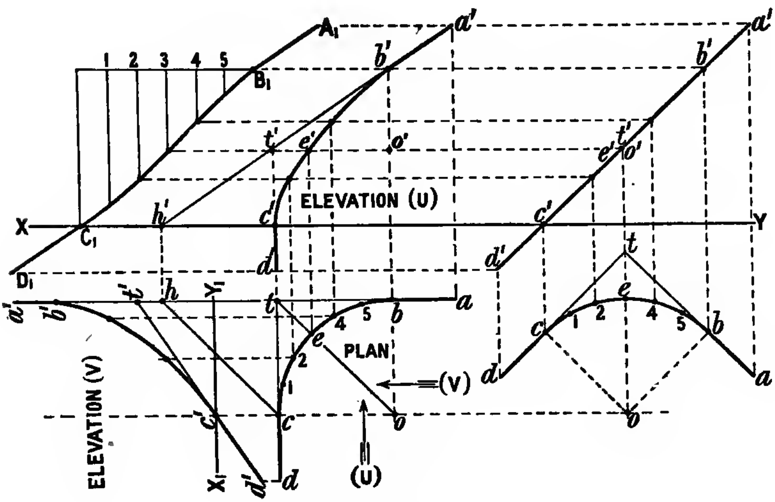


FIG. 714.

In making a wreath it is first formed of rectangular or approximately rectangular cross-section, and in this form it is called a *squared wreath*. The squared wreath is moulded to the required cross-section by hand with suitable tools. The geometry of the handrail is mainly the geometry of the squared wreath, and the correctness of the form of the finished wreath very largely depends on the skill and care with which the squared wreath is constructed.

The squared wreath resembles to some extent the thread of a square-threaded screw of large pitch. In one system of constructing squared wreaths the surfaces of the wreath are described by the sides of a rectangle which moves with its centre on the centre line of the wreath, the plane of the rectangle intersecting the centre line at right angles, and two sides of the rectangle are horizontal. The horizontal sides of the moving rectangle sweep out the upper and lower surfaces

of the wreath but even when the centre line of the wreath is a helix these surfaces are not true screw surfaces because the generating lines do not intersect the axis of the cylinder. Also, the other sides of the moving rectangle do not lie exactly on the cylindrical surfaces. It may however be assumed with sufficient accuracy for all practical purposes that the sloping sides of the moving rectangle sweep out portions of cylindrical surfaces which become the inner and outer vertical surfaces of the wreath. In that case the developments of these surfaces will be figures of uniform width. These developments of the inner and outer vertical surfaces of the wreath are called the inner and outer *falling moulds* of the wreath. The development of the vertical section of

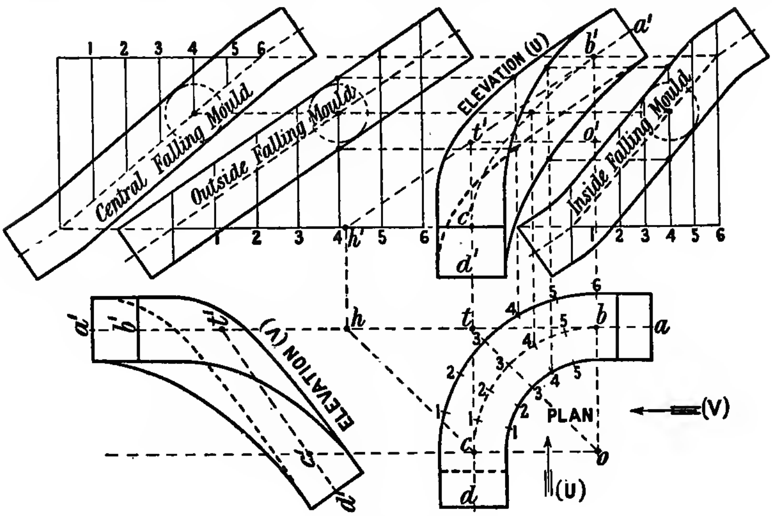


FIG. 715.

the wreath which contains its centre line is called the central falling mould.

The surface which contains the centre line and the centre lines of the inner and outer vertical surfaces of the wreath will be described by the horizontal centre line of the moving rectangle which, when produced, intersects the axis of the cylinder. This gives the construction for finding the centre lines of the inner and outer falling moulds from the centre line of the central falling mould.

Fig. 715 shows the constructions for determining the falling moulds and projections of the squared wreath whose centre line is the same as that shown in Fig. 712. The tangents ab, $a'b'$ and cd, $c'd'$ are first drawn intersecting at tt'. The shanks are then drawn in plan and elevation. The plan may then be completed. The centre line of the central falling mould is next drawn as in Fig. 712 and from this the centre lines of the inner and outer falling mould are determined as

shown. Upon these centre lines the falling moulds are constructed of uniform width. The elevation (U) may now be projected from the plan and the falling moulds as shown.

Fig. 716 shows an elevation of the wreath on a plane perpendicular to the plane of its central tangents and from this the face mould is shown projected. The *face mould* is the section of the wreath by the plane containing the central tangents. The determination of the centre line of the face mould has already been considered in connection with Figs. 712 and 713. The inner and outer curves of the face mould are the elliptic sections of the inner and outer cylindrical surfaces of the wreath by the plane containing the central tangents.

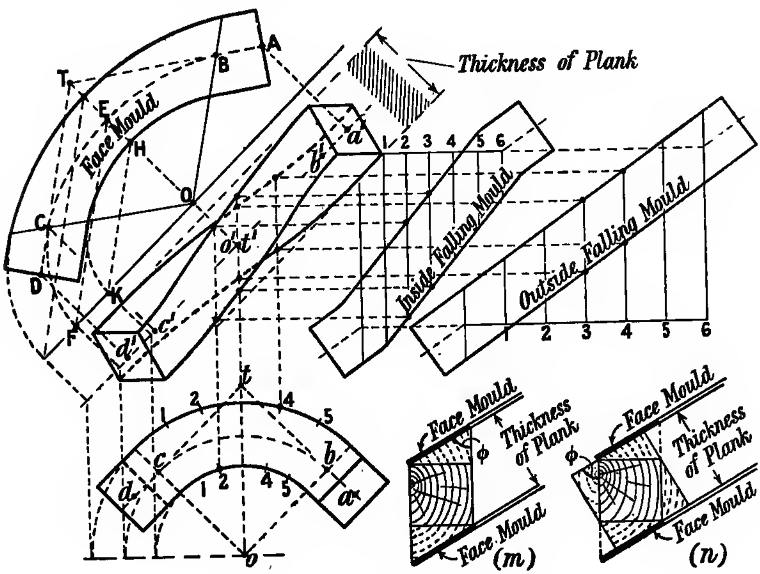


FIG. 716.

It is not necessary to draw the elevation of the wreath shown in Fig. 716 in order to determine the face mould. Having found the centre line ABECD, as previously explained, OE being the semi-minor and OF the semi-major axis of the ellipse of which the curve BEC is a part, make EH equal to half the horizontal width of the wreath. Join EF and draw HK parallel to EF. Then OH is the semi-minor and OK is the semi-major axis of the ellipse of which the inner curved portion of the face mould is a part. In like manner the semi-minor and semi-major axes of the ellipse of which the outer curved portion of the face mould is a part may be determined, and the elliptic arcs may then be drawn by the trammel method. The ends of the face mould are perpendicular to the tangents AB and CD, and the other straight

edges are tangents to the ellipses and are also parallel to the central tangents respectively, as shown.

On the elevation of the wreath in Fig. 716 is shown the thickness of the plank from which the wreath may be cut.

The wreath is cut from the plan either by the " bevel cut " shown at (m) Fig. 716, or by the " square cut " shown at (n) Fig. 716. The latter is considered to be the best as it involves less labour and requires less material.

The angle marked ϕ at (m) and (n) in Fig. 716 is called the " bevel " for that end of the wreath and this angle must be determined for each end of the wreath before the shanks can be formed and the " twist " given to the wreath. The angle ϕ is simply the angle between the vertical faces of a shank and the plane of the central tangents, and it may be conveniently found as follows. Referring to Fig. 717, at, $a't'$ and td, $t'd'$, intersecting at tt', are the central tangents of the wreath, ak being the ground line for these projections. kd is the horizontal trace and ka' is the vertical trace of the plane of these tangents. By the usual construction (Art. 164, p. 200) ϕ_A is the angle between the plane of the tangents and the vertical plane containing the tangent at, $a't'$. Taking td as a ground line t'_1d is a second elevation of the tangent td, $t'd'$ and it is also the new vertical trace of the plane of the tangents. The angle ϕ_D, which is the angle between the plane of the tangents and the vertical plane containing the tangent td, $t'd'$, can now be found as shown. If the given tangents are equally inclined to the horizontal it is evident that ϕ_A is equal to ϕ_D.

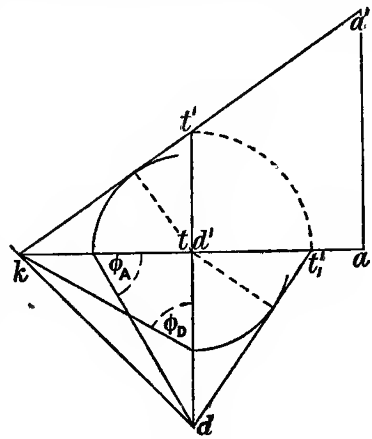


FIG. 717.

In Figs. 715 and 716 the central, inner, and outer falling moulds are all of the same width, measured at right angles to their centre lines, and the resulting wreath has everywhere an approximately rectangular cross-section, the cross-section being taken at right angles to the centre line of the wreath. In another system of construction, sections of the wreath by planes containing the axis of the cylinder are rectangles, and the sides of these rectangles which lie on the upper and lower surfaces of the wreath are horizontal; the falling moulds are then of different widths as will be seen from Figs. 718 and 719. Fig. 718 shows the falling moulds, on this second system, for the wreath illustrated by Fig. 715, and Fig. 719 shows the falling moulds, on this second system, for the case where the lower central tangent is horizontal. The central falling mould is constructed in each case as

before and from this the outer and inner falling moulds are projected as shown. It will be seen that at corresponding vertical lines the widths of the moulds, *measured vertically*, are the same. Having con-

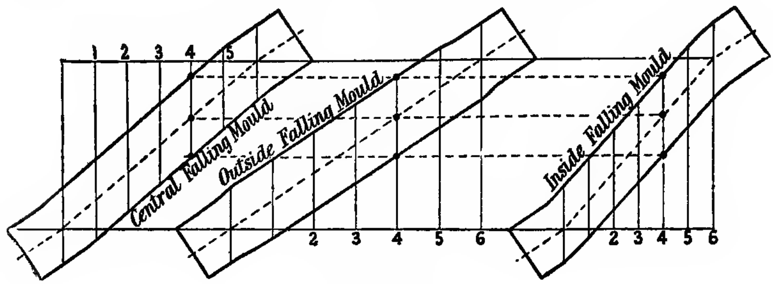


Fig. 718.

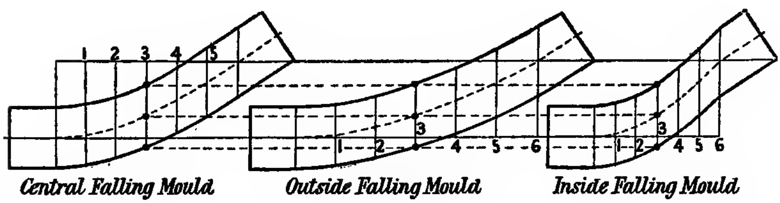


Fig. 719.

structed the falling moulds, any required projections of the wreath may be drawn as before. But it will easily be seen that only the central falling mould need be drawn for the purpose of obtaining the projections of the wreath.

319. Screw Propellers.—The geometry of a screw propeller blade and the construction of the working drawings of it will be more easily understood after a brief reference to the usual method of preparing the mould for a propeller in the foundry.

Referring to Fig. 720, ABC is a templet, which when laid out flat is a right-angled triangle having the right angle at B. The templet ABC, having AB vertical, forms part of a vertical cylindrical surface of which DE is the axis. If a straight line, intersecting the axis DE and making a constant angle with it, moves in contact with the sloping edge AC of the templet ABC it will generate a true screw surface. AE and CD are the extreme positions of the moving line and three intermediate positions are also shown.

In the foundry a rough bed of brickwork and loam is built up, the upper face of this bed roughly conforming to the screw surface required. A vertical spindle is fixed so that its axis coincides with DE. A sleeve attached to one end of a horizontal arm fits on the spindle and can slide or rotate freely on it. To the arm is attached a board, called

a *loam board*, the lower edge of which, guided by the sloping edge of the templet ABC, sweeps out the screw surface on the loam as the arm rotates about the axis of the spindle. The screw surface is then smoothed by hand. Sharp iron pins attached to the lower edge of the loam board at intervals cut helices FHJ, KLM, etc. on the screw surface of the loam. The work described so far is for one blade and this has to be repeated for each of the other blades if all the blades are cast together.

The boss of the propeller may be moulded from a pattern inserted in the usual way or it may be swept out of the loam with a suitable loam board. The mould thus far prepared is dried and blackwashed. The next step is to form with loam a piece of the same shape and size as the required blade. Strips of wood FHJN, PLRQ, etc. are

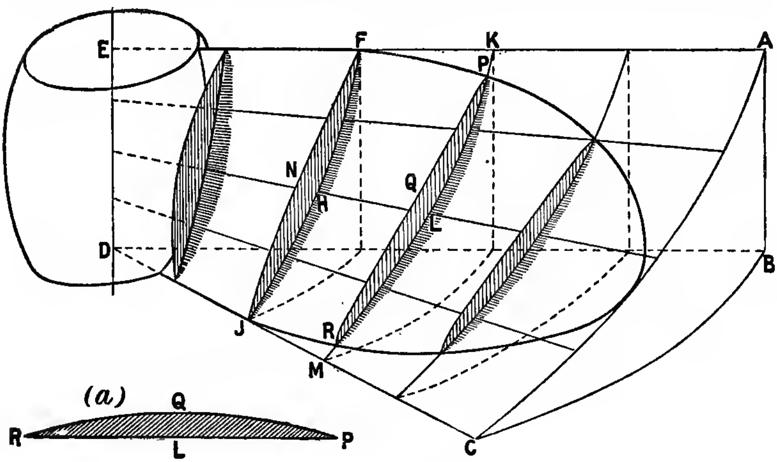


FIG. 720.

cut out to the shapes of the required cylindrical sections of the blade at different radii. One of these strips laid out flat is shown at (*a*). These strips or thickness pieces are bent and fixed on edge to the mould, their lower edges coinciding with the corresponding helices FHJ, KLM, etc. which have been previously traced on the mould.

The surface of the blade having been outlined on the mould this surface is covered with loam to the depth of the thickness pieces FHJN, PLRQ, etc. After drying and blackwashing, the upper part of the mould can now be built up on the lower part. The upper part of the mould is taken off and then smoothed and dried. The loam thickness piece is now removed and the finished halves of the mould are put together.

A screw surface cannot be developed, but the face of the blade of a screw propeller is generally a small portion of a complete convolution of a screw surface and it may be developed approximately.

It may here be pointed out that the helices on the face of the blade are approximately elliptic arcs each being a portion of a plane section of the surface of a cylinder whose axis is the axis of the screw, the inclination of the plane being the same as that of the helix. This is made use of in obtaining the projections of the blade from its approximate development in the manner to be shown presently.

The shape and area of the developed face of the blade, also the pitch of the screw surface and the general dimensions of the propeller are settled by the naval architect in consultation with the marine engineer. It will now be shown how the drawings of a propeller may be made, the necessary particulars being given.

The case where the face of the blade is a true screw surface gene-

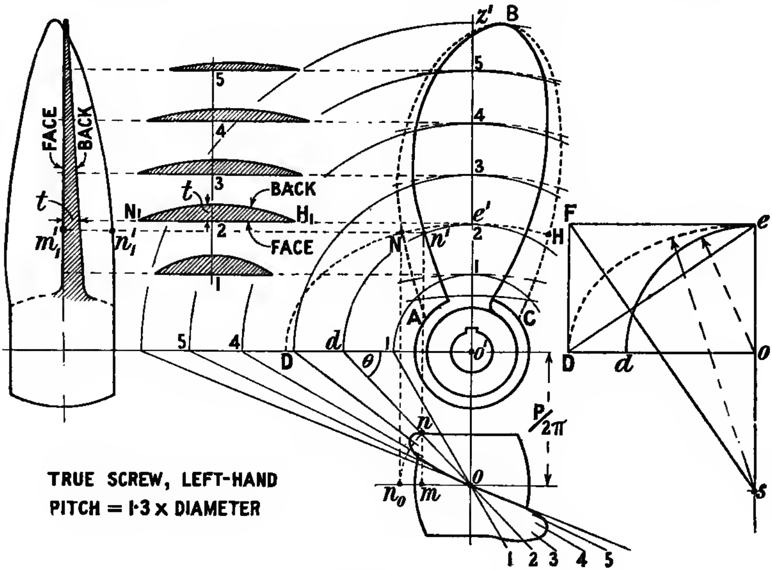


FIG. 721.

rated by a straight line moving at right angles to the axis will first be considered. Referring to Fig. 721, the dotted curve ABC is the given outline of the developed face of the blade drawn on the end elevation about the vertical line $o'z'$ which bisects the developed surface in the neighbourhood of the boss.

Take a point e' on $o'z'$ and with centre o' and radius $o'e'$ describe the arc $e'n'd$ cutting the horizontal through o' at d. Make $o'o$ equal to $\frac{P}{2\pi}$ where P is the pitch of the screw surface of the blade. Join od and produce it. The portion of this straight line od which falls within the plan of the blade will be, for all practical purposes, the plan of that part of the helix lying on the face of the blade at the radius $o'e'$.

Further, od is the horizontal trace of a vertical plane which cuts the surface of the cylinder of which $e'n'd$ is a part elevation, and the section is an ellipse an arc of which practically coincides with the portion of the helix just referred to. Obviously od is the length of the semi-major axis and $o'e'$ is the length of the semi-minor axis of this ellipse. Make o'D equal to od and taking o'D and $o'e'$ as the semi-major and semi-minor axes respectively construct D$\text{N}e'$ a part of the ellipse. This ellipse cuts the outline of the development of the face of the blade at N. A horizontal through N to cut the circle through e' at n' determines a point on the end elevation of the edge of the blade, and a vertical from n' to meet od determines the plan of this point. These points n' and n may be more accurately found as follows. From N drop a perpendicular to meet the horizontal through o at n_0. Make on equal to on_0. Draw the vertical nn' to meet the horizontal through N at n'.

Instead of drawing the ellipse it will be sufficient to draw e'N as a circular arc struck from the centre of curvature of the ellipse at e'. The construction for finding this centre of curvature has been given in Art 51, p. 48, and is shown to the right in Fig. 721. DF is parallel to $o'e'$ and e'F is parallel to o'D. Fs perpendicular to e'D meets $e'o'$ produced at s. A circular arc struck from the centre s with a radius se' will practically coincide with as much of the ellipse as lies within the development of the face of the blade.

Other points on the plan and end elevation of the edge of the blade are found in the same way. For the side elevation shown to the left in Fig. 721, make $m_1'n_1'$ equal to mn and similarly for other points.

On the elevation to the left in Fig. 721 is shown the *maximum thickness section* of the blade. This is not a true plane section of the blade, but its width at any distance from the axis of the screw is the maximum thickness of the blade at a radius equal to that distance.

The developed sections of the blade at different radii are shown between the two elevations. Referring to the second section from the axis, the straight base N_1H_1 is made equal in length to the elliptic arc $\text{N}e'\text{H}$. These developed sections give the sizes and shapes of the thickness pieces such as PLRQ (Fig. 720) referred to in describing the moulding of a propeller. These developed sections are generally superimposed on one of the elevations of the blade, but in Fig. 721 they have been moved to one side for the sake of clearness.

A second example is illustrated by Fig. 722 which shows a detachable blade with a flange for bolting to a boss keyed to the shaft. In this example the blade is set back by inclining the generating line of the screw surface to the axis at an angle less than a right angle. This is shown on the elevation to the left in Fig. 722 by the tilting over of the maximum thickness section. Points on the generating line will describe helices just as before but the various helices will be displaced in an axial direction each by a definite amount depending on its distance from the axis. For example, the helix at the radius $o'e'$ will be displaced by the amount $d_1'e_1'$. Hence the plan of this helix

will cut the plan of the axis at e such that oe is equal to $d_1'e_1'$. The slope θ of the helix may be obtained as before. but in Fig. 722 the triangle for finding θ is shown on the elevation instead of on the plan. *nes*, the plan of the helix on the face of the blade at radius $o'e'$ is drawn at right angles to e'T. The plan and the two elevations of the edge of the blade are otherwise drawn as before, the construction lines being clearly shown.

It will be seen that for the same diameter of screw the true length

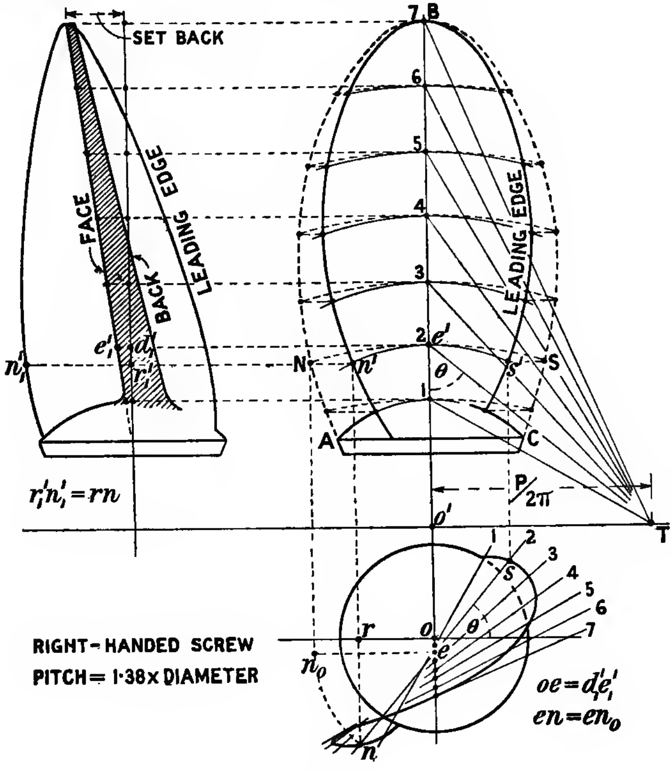


FIG. 722.

of the blade is slightly increased by giving it a set back, but this is generally neglected in applying the developed face of the blade to the elevation as shown in Fig. 722.

In the examples so far considered all the helices on the face of the blade have had the same pitch. Frequently however the helix at the root has a less pitch than the helix at the tip and the pitch increases uniformly from the root to the tip. The screw is then said to have a *radially increasing pitch.* Fig 723 shows clearly how the slopes of the

helices at different radii are obtained when the blade has no set back. After the slopes of the various helices have been found, the plan, shown in Fig. 723, and the elevations (not shown) are determined as before from the developed face of the blade.

For a screw which has a radially increasing pitch the moulder

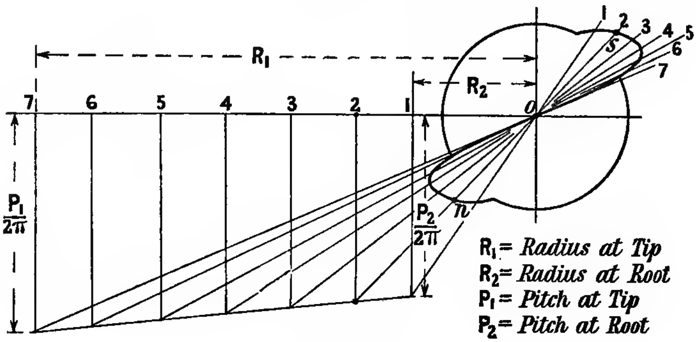


FIG. 723.

requires a templet for the root as well as one for the tip for the purpose of guiding the loam board in sweeping out the face of the blade. The loam board must also be connected to the arm in such a way that it can alter its inclination as the arm revolves, or the arm may be simply forked to embrace the central spindle.

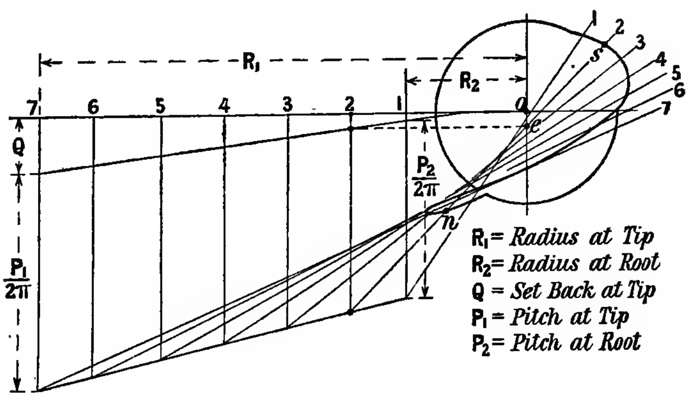


FIG. 724.

The slopes of the various helices will not be altered by giving the blade a set back and the construction given in Fig. 723 will therefore still apply in finding these slopes but the plans of the helices on the face of the blade will not now pass through the point o but through points determined as explained in connection with Fig. 722.

A convenient construction for finding at the same time the slope and the positions of the plans of the various helices on the face of the blade is clearly shown in Fig. 724 and needs no further description.

Still another modification of the design of the face of the blade of

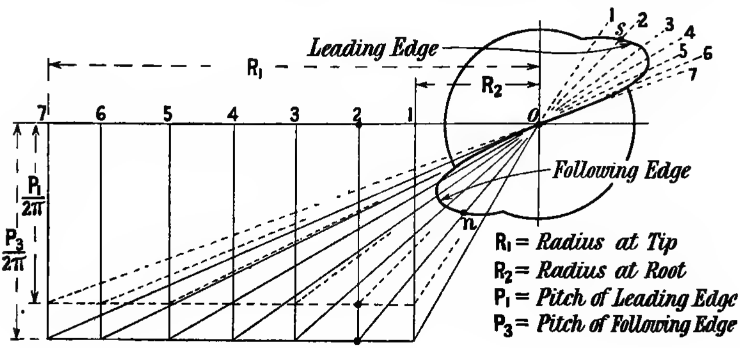


FIG. 725.

a screw propeller has to be mentioned. The part of the face of the blade towards the leading edge may have a less pitch than the remaining part towards the following edge. The face of the blade is then said to have an *axially increasing pitch*. The necessary modification in

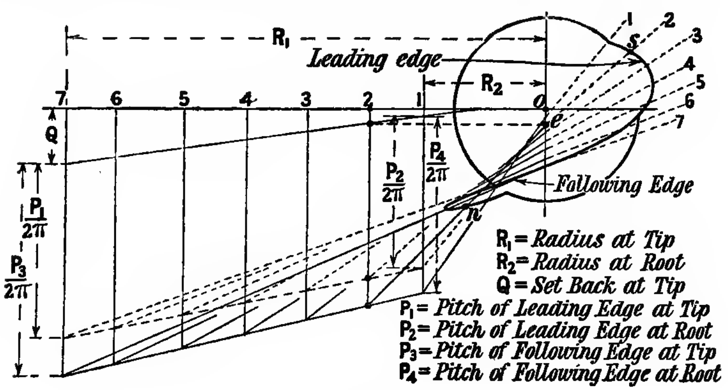


FIG. 726.

the construction of the plan of the blade is shown in Figs. 725 and 726. In Fig. 725 the face of the blade has an axially increasing pitch but a radially constant pitch and no set back. In Fig. 726 there is an axially increasing pitch, a radially increasing pitch, and also a set back.

Exercises XXV

1. A right-handed helix of 2 inches pitch, and a left-handed helix of 1 inch pitch are traced on a vertical cylinder 2 inches in diameter. Draw the elevation of two turns of the first helix and four turns of the second. The lower ends of the two helices to be at opposite extremities of a diameter of the cylinder which is perpendicular to the ground line.

2. Two helices are traced on the surface of a vertical cylinder 2 inches in diameter. The helices are at right angles to one another at their intersection and one of them, which is right-handed, has a pitch of 4 inches. Show as much of the two helices as is contained in 4 inches length of cylinder. Draw the elevations of the tangents to these helices at points which are 1 inch, 2 inches, and 3 inches above the lower end of the cylinder.

3. Show one turn of a helix of 3 inches pitch on a vertical cylinder 1 inch in diameter. Draw the locus of the horizontal trace of a moving tangent to the helix, and determine the true form of a vertical section of the surface described by the moving tangent, the plane of section to be at a distance of 1 inch from the axis of the cylinder.

4. A cylinder 1·5 inches in diameter and 3 inches long has three helices of 3 inches pitch traced on its surface at equal distances apart. Draw the plan of the cylinder and helices when the axis of the cylinder is inclined at 45° to the horizontal plane.

5. Work the example on the helix of increasing pitch illustrated by Fig. 701, p. 364, having given,—diameter of cylinder 2 inches, height of cylinder 3 inches. On the development draw the curve whose ordinates are equal to the pitch at each point of the helix of increasing pitch.

6. A straight line 1·5 inches long is at right angles to a fixed vertical axis and has one end on that axis. This straight line describes a screw surface of 3 inches pitch. Draw the elevation of one complete turn of this screw surface, showing on it the generating line in positions whose plans are at intervals of 15°. Show also the helices described by points on the generating line which divide it into three equal parts in addition to the helix described by the outer end.

7. Same as the preceding exercise except that the generating line is 2·4 inches long and is inclined at 45° to the axis.

8. Referring to the example illustrated by Fig. 703, p. 365, draw the elevation of one complete convolution of the surface generated by the quadrant of a circle, having given,—radius of quadrant 2 inches, axial advance per revolution 3 inches. Show the generating arc in positions whose plans are at intervals of 15°. Also, in addition to the helix described by the lower end of the generating arc, show the helices described by three intermediate points. Lastly draw the elevation of a section of the surface by a plane parallel to the vertical plane of projection and 0·5 inch in front of the axis.

9. An equilateral triangle of 2·5 inches side moves with one side on a vertical axis so that the other two sides describe screw surfaces of 1·5 inches pitch. Draw the elevation of the surfaces described by four revolutions of the triangle. On the lower half of the elevation show the generating lines in positions whose plans are at intervals of 15°. Show the helix which is the intersection of the two screw surfaces.

10. Same as the preceding exercise except that the pitch is 2·5 inches instead of 1·5 inches. [In this case the two screw surfaces do not intersect.]

11. Full size axial sections of various screw threads are shown in Fig. 727. In each the diameter over the threads is 3 inches, the screws are all right-handed and single threaded. Draw for each a projection on a plane parallel to the axis of the screw, showing as much of the screw as falls within a length of 4 inches measured parallel to the axis. Show also for each a section of the nut by a plane containing the axis. Height of nut in each case 3 inches.

12. A right-handed treble threaded worm 3·5 inches external diameter and

4 inches long has threads of the form and dimensions shown by the axial sectio (*d*) Fig. 727. Draw a projection of this worm on a plane parallel to its axis.

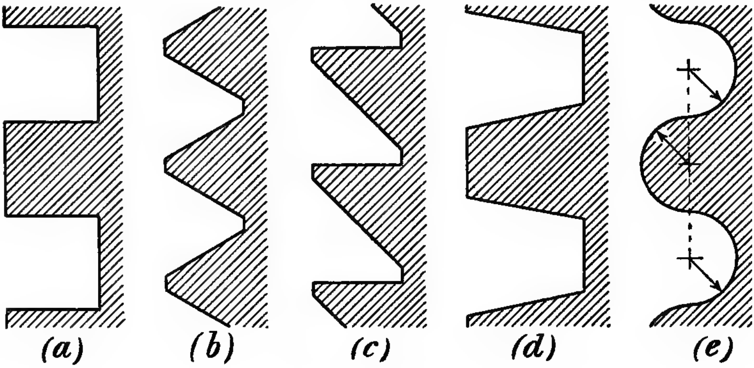


FIG. 727.

13. The following particulars relate to three helical springs.—No. 1. Axia section a rectangle 0·5 inch × 0·125 inch with the longer sides parallel to th axis. External diameter 2 inches. Pitch 1 inch. No. 2. Axial section a rect angle 0·5 inch × 0·125 inch with the longer side perpendicular to the axis External diameter 3 inches. Pitch 1 inch. No. 3. Normal section a circle 0· inch diameter. External diameter 3·5 inches. Pitch 1·75 inches.

14. A cylinder 3 inches in diameter and 4 inches long has a left-handed helical groove cut in it. A section of the groove by a plane at right angles to the axis of the cylinder is 1 inch wide and 1 inch deep, the sides being parallel to and the bottom at right angles to the mid-radial line of the section. Draw a projection of the cylinder and groove on a plane parallel to the axis of the cylinder and determine the true forms of axial and normal sections of the groove.

15. A portion of a $1\frac{1}{4}$ inch twist drill is shown in Fig. 728, consisting of a cylinder with a conical end, cut with two spiral grooves each of 12 inches pitch and of the form defined by the given sectional elevation. Complete the plan, showing the curve BB, and the helical grooves correctly projected for a distance of 6 inches from the line AA. [B.E.]

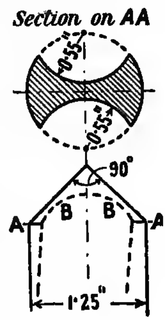


FIG. 728.

16. A circle $1\frac{1}{8}$ inches in diameter, whose plane is vertical, revolves with uniform angular velocity about a vertical axis in its plane, describing an annulus or anchor ring whose mean radius is $1\frac{7}{8}$ inches. While the circle is revolving a point P on its circumference moves round the circle with uniform velocity. The point P travels sixteen times round the circle while the circle revolves three times about the vertical axis. Draw the plan of the curve traced by the point P. The result is shown to a reduced scale in Fig. 729. Note that the curve is endless.

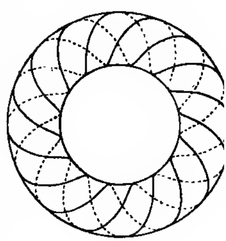


FIG. 729.

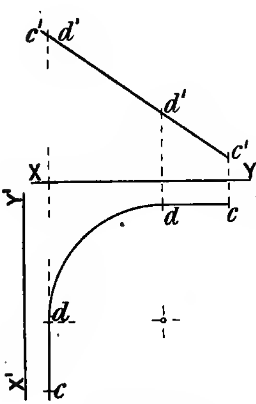
17. ABC is a right angled triangle. The sides AC and BC which contain the right angle are 3·5 inches and 1·5 inches respectively. The triangle revolves with uniform angular velocity about AC which is vertical. A point P starting at A moves along AB with uniform velocity an

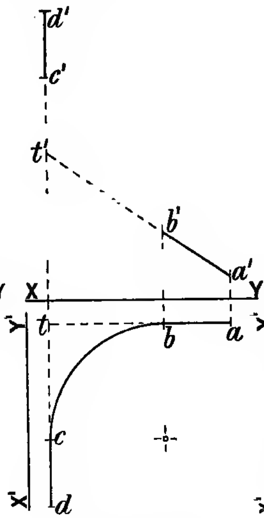
reaches B while AB makes two revolutions. Draw the plan and elevation of the path of P. Draw also the development of the surface described by AB and show on it the curve traced by P.

18. The radius of the base of a right cone is 1·25 inches, the axis, which is 4·5 inches long, being inclined at 40° to the H.P. The apex is above the base and pointing to the right. From the highest point in the base, a cord 0·25 inch in diameter is coiled round the cone in a right-handed spiral. The clear distance between the coils, measured in a straight line parallel to the surface of the cone is 1·25 inches. Draw the plan of the cone showing two turns of the cord. [B.E.]

19. The centre line CDDC of a portion of a handrail is shown in plan and elevation in Fig. 730, the parts CD, CD being straight, and the part DD being an elliptic arc. Draw the elevation of the centre line on X′Y′. Find the true shape of the figure CDDC. [B.E.]

20. Referring to Fig. 730, draw the development of the surface of which *cddc* is the plan and show on it the centre line CDDC.





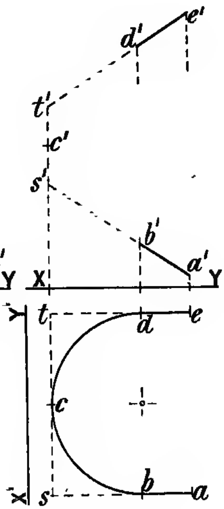


FIG. 730. FIG. 731. FIG. 732.

The above Figs. are to be reproduced double size.

21. The plan and part of an elevation of the centre line ABCD of a portion of a handrail are shown in Fig. 731. Assuming that the straight and curved parts are in the same plane complete the elevation and draw another elevation on X′Y′. Show also the true form of the centre line ABCD.

22. Fig. 732 shows the plan and part of an elevation of the centre line ABCDE of a portion of a handrail. The parts AB and DE are straight. ST is the tangent to the centre line at C and this tangent meets AB produced at S and ED produced at T. Assuming that BC is in the plane containing AS and SC and that CD is in the plane containing CT and TE, complete the elevation of the centre line and draw another elevation on X′Y′. Draw also the development of the vertical surface containing the centre line ABCDE showing on it that centre line.

23. Fig. 733 gives the plan and part of the elevation of a portion of a handrail of rectangular cross-section (as prepared for moulding); complete the elevation of the rail. Draw a second elevation of the rail on X′Y′. Set out the "face mould" for this rail. [B.E.]

24. Fig. 734 gives the plan and part of the elevation of a portion of a handrai of rectangular cross-section (as prepared for moulding); complete the elevation of the rail. Set out the "face mould" for this rail. Draw also the "falling mould" for the central vertical section CDDC of the rail. [B.E.]

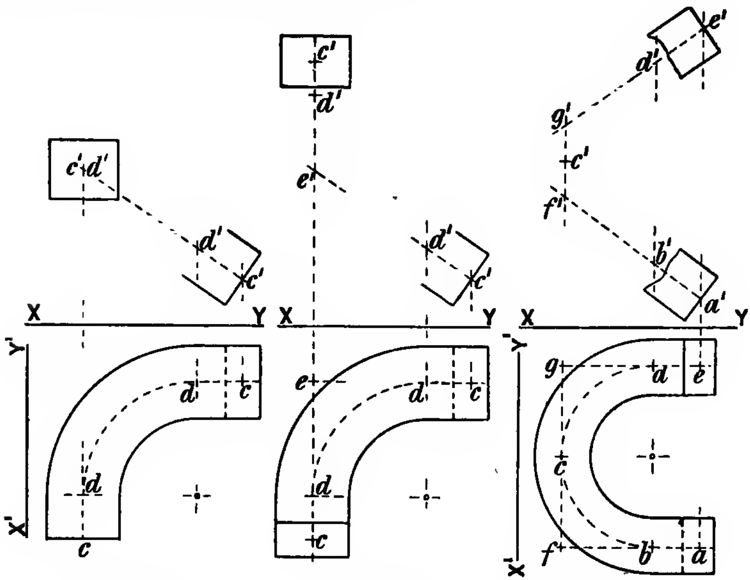


FIG. 733. FIG. 734. FIG. 735.

The above Figs. are to be reproduced double size.

25. Stair rail (Fig. 735). To be made in two parts, cut from planks and connected at C by a joint perpendicular to FG. A plan and part elevation are given. Complete the elevation of the centre line ABCDE and draw a second elevation of this line on X'Y'. Show the plan and two elevations of the joint at C. Draw the "face mould" for the wreath ABC, and set out the "bevels" for its ends.

Plot the "falling moulds" for the central section and the inner and outer surfaces of the wreath ABC. Complete the elevation of the wreath ABC. [B.E.]

26. In Fig. 736 are given the flange and the maximum thickness section of one blade of a three-bladed propeller; the dotted curve is the developed face of the blade. The face of the blade is a true right-handed screw surface whose pitch is one and a quarter times the diameter of the propeller. Draw the figure double size and then construct a plan and two elevations together with sections of the blade as shown in Fig. 721, p. 378.

27. Same as the preceding exercise except that the blade is to have a set back which, measured at the tip, is 0·06 of the diameter of the propeller; also the screw is to be left-handed instead of right-handed.

28. The flange and the maximum thickness section of one of the four detachable blades of a screw propeller are shown in Fig. 737. The dotted curve is the developed face of the blade. The diameter of the propeller is 19 ft. 6 in. at the tip and 5 ft. 2 in. at the root. The pitch of the face towards the leading edge increases from 19 ft. at the root to 21 ft. at the tip, and the pitch of the face towards the following edge increases from 19 ft. at the root to 23 feet at the tip.

Draw the plan and two elevations of this blade to a scale of one inch to a foot, and determine the developed sections of the blade by cylindrical surfaces of $3\frac{1}{2}$, $4\frac{1}{2}$, $5\frac{1}{2}$, $6\frac{1}{2}$, $7\frac{1}{2}$ and $8\frac{1}{2}$ ft. radii.

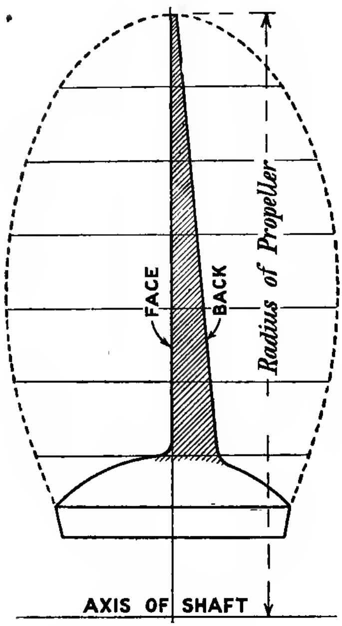


FIG. 736.

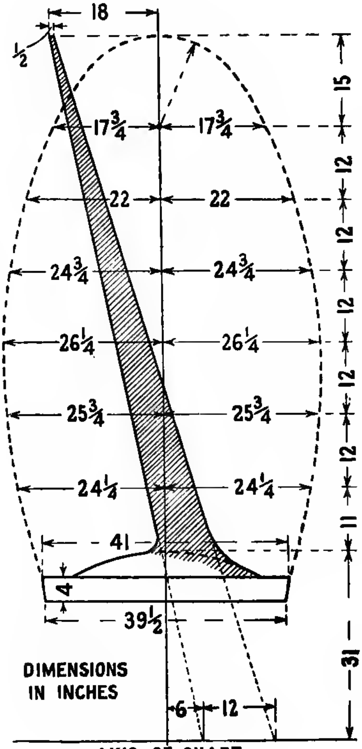


FIG. 737.

CHAPTER XXVI

INTERSECTION OF SURFACES

320. General Method.—The general method of finding the intersection of two surfaces is as follows. Let A and B denote two given surfaces whose intersection with one another is required. Cut the surfaces A and B by a third surface C, the latter surface being so chosen and employed that the projections of its intersections with A and B are lines, such as straight lines or circles, which can easily be drawn. Let A′ denote the line of intersection of C and A, and let B′ denote the line of intersection of C and B. Let the lines A′ and B′ meet at a point P. Then the point P lying on the intersection of C and A, lies on A. Also since the point P lies on the intersection of C and B, it lies on B. The point P therefore lies on A and B, and therefore it must be a point on the intersection of A and B. By moving the surface C into different positions, or by cutting A and B by other surfaces similar to C, any number of points on the intersection of A and B may be determined.

On the intersection of two surfaces there are generally certain important points which should be first determined. For example if the projection of the intersection meets a boundary line of the projection of one of the surfaces, the meeting point will be an important one to determine and in the case of a curved surface this will be a tangent point.

In moving the cutting surface in one direction from a position which gives points on the required intersection to a position which gives no such points, there will be an intermediate position which gives the last of the points obtained by moving the surface in that direction and these points are usually important.

As a general rule after the important points have been determined only a few others are necessary to fix the required line of intersection.

It is a good plan to number or letter the cutting surfaces in order, and the points determined by them, each point having the same number or letter as the cutting surface on which it is situated.

321. Intersection of Two Cylinders.—The general method of finding the intersection of the surfaces of two cylinders is to cut the surfaces by planes parallel to their axes or to their generating lines. These planes will intersect the surfaces of the cylinders in straight

lines, the intersection of which with one another will determine points on the intersection required.

EXAMPLE 1.—The horizontal trace of a vertical cylinder is an ellipse, 3·2 inches × 2·4 inches, the major axis being parallel to XY. A horizontal cylinder, 1·6 inches in diameter has its axis parallel to the vertical plane and 0·3 inch in front of the axis of the vertical cylinder. It is required to show the elevation of the intersection of the cylinders.

There will be two identical curves in this case and in Fig. 738 only enough of each of the cylinders is shown for determining one of these curves.

Cutting planes parallel to the axes of both cylinders will in this case be vertical planes parallel to the vertical plane of projection. HT is the horizontal trace of one of these planes. This plane will cut the vertical cylinder in two vertical straight lines, one of which will have the point a for its plan, and a perpendicular to XY from a for its elevation. This same plane will cut the horizontal cylinder in two horizontal lines of which ab will be the plan. To find the elevations of these lines, take an elevation of the horizontal cylinder on a plane perpendicular to its axis, this elevation will be a circle, but only half of it need be drawn, as shown. The cutting plane now being considered will have a trace on this new vertical plane which will coincide with its horizontal trace. The point b_1' where this trace cuts the circle, or semicircle, just mentioned will be the end elevation of AB one of the lines in which the horizontal cylinder is cut by the cutting plane HT, and the length bb_1' will be the vertical distance of these lines above and below the horizontal plane containing the axis of that cylinder. Hence the required elevations $a'b'$, $a'b'$ will be at distances equal to bb_1' above and below the elevation of the axis of the horizontal cylinder.

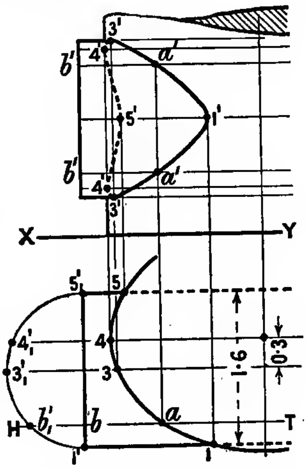


FIG. 738.

The intersections of $a'b'$ and $a'b'$ with the vertical line through a determine points on the intersection required.

In like manner, by taking other planes, other points can be found.

The most important points on the curves of intersection in this case are those determined by the following cutting planes. (1) The two planes which *touch* the horizontal cylinder, these determine the points 1′ and 5′ on the elevation. (2) The plane which contains the axis of the horizontal cylinder, this determines the points marked 3′. (3) The plane which contains the axis of the vertical cylinder, this determines the points marked 4′.

The foregoing example may also be worked by using horizontal cutting planes, and the student would do well to work the problem in this way.

EXAMPLE 2.—A vertical tube, external diameter 2·75 inches, internal diameter 1·5 inches, has a horizontal cylindrical hole bored through it 1·25 inches in diameter. The axis of the hole is 0·25 inch from the axis of the tube. It is required to draw an elevation on a vertical plane inclined at 35° to the axis of the hole.

First determine the intersection of the boring cylinder with the outside of the tube exactly as in the preceding example. The fact of the horizontal cylinder being inclined to the vertical plane will make no difference in the working, as the heights of all the lines in which the cutting planes intersect the horizontal cylinder will remain the same whatever be its inclination to the vertical plane.

Next in like manner the intersection of the boring cylinder with the interior surface of the tube is determined. It will be found that in this example this interior intersection consists of one line only.

The cutting planes which give the most important points on the curves of intersection in this example are the same as those in example 1, with the addition of the plane which touches the inner surface of the tube.

The result of this example is shown in Fig. 739, but the construction lines have been omitted.

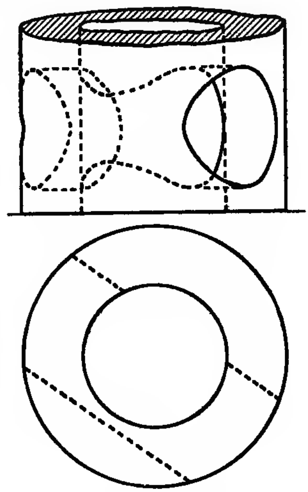


FIG. 739.

This example, like the preceding one, may also be worked by using horizontal cutting planes.

EXAMPLE 3 (Fig. 740).—*ab a'b'* is the axis of a cylinder whose horizontal trace is a circle 2·25 inches in diameter. *cd c'd'* is the axis of a cylinder whose horizontal trace is an ellipse (major axis 3 inches, minor axis 2 inches), whose minor axis is parallel to XY. It is required to show, in plan and elevation, the intersection of the surfaces.

In this example the planes which will intersect the surfaces of both cylinders in straight lines will be inclined to both planes of projection.

To find the directions of the traces of the cutting planes take a point *pp'* in *ab a'b'*. Through this point draw *pq p'q'* parallel to *cd c'd'*. The line *aq* which joins the horizontal traces of *ab a'b'* and *pq p'q'* will be the horizontal trace of a plane which contains the axis of one cylinder and is parallel to the axis of the other, and this plane, and planes parallel to it will, if they cut the cylinders at all, cut them in straight lines parallel to their respective axes. In this particular example the horizontal traces of these planes are parallel to XY.

Seven planes whose horizontal traces are shown, and numbered from 1 to 7, will determine all the important points, and no other planes need be taken.

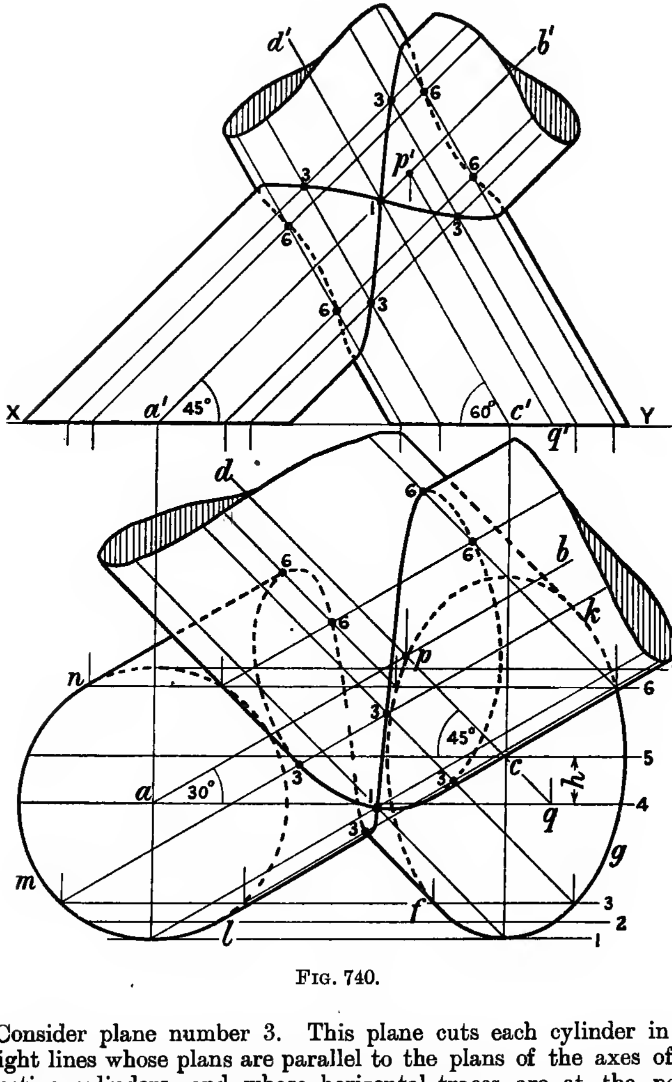


FIG. 740.

Consider plane number 3. This plane cuts each cylinder in two straight lines whose plans are parallel to the plans of the axes of the respective cylinders, and whose horizontal traces are at the points where the horizontal trace of the plane cuts the horizontal traces of the cylinders. The elevations of these lines will be parallel to the

elevations of the axes of the respective cylinders upon which they lie. These four lines intersect at four points in plan and in elevation and determine the four points on the intersection of the cylinders which have the number 3 attached to them.

It will be observed that plane number 1 touches both cylinders.

As already stated the horizontal traces of all the necessary cutting planes are shown, but the remaining construction lines are only shown for planes 1, 3, and 6, for the sake of making the figure clearer.

The portion of the curve which is seen in plan, and which must therefore be put in as a full line, is found from the intersection of those lines which lie on the upper surface of one cylinder with those which lie on the upper surface of the other, that is, from the intersection of the lines whose horizontal traces lie on the part *fgk* of the ellipse with those whose horizontal traces lie on the part *lmn* of the circle. The remainder of the curve in plan will be hidden by one or other or both of the cylinders, and will therefore be put in as a dotted line.

In like manner the part of the curve which is seen in the elevation is found from the intersection of the lines which lie on the front of one cylinder with those which lie on the front of the other, that is from the intersection of the lines whose horizontal traces lie on the halves of the horizontal traces of the cylinders which are furthest from XY.

Instead of using the horizontal traces of the cylinders and the horizontal traces of the cutting planes, the vertical traces may be employed if the vertical traces of the cylinders come within a convenient distance on the paper. When one cylinder has a vertical trace but no horizontal trace within a convenient distance, and the other has a horizontal trace but no vertical trace within a convenient distance, it is necessary to use both the horizontal and vertical traces of the cutting planes.

322. Intersection of Cylinder and Cone.—The general method of finding the intersection of a cylinder and cone is to cut the surfaces by planes parallel to the axis of the cylinder and passing through the vertex of the cone. These planes will cut the surface of the cylinder in straight lines parallel to its axis, and the surface of the cone in straight lines passing through its vertex. The intersection of the lines on the cylinder with the lines on the cone determine points on the intersection required.

Example 1 (Fig. 741). A right cone, base 2·9 inches diameter, axis 3·8 inches long, has its base horizontal. A cylinder 2 inches in diameter has its axis horizontal and 1·2 inches above the base, and 0·2 inch from the axis of the cone. It is required to draw the plan of the intersection of the surfaces and an elevation on a vertical plane inclined at 30° to the axis of the cylinder.

The general method which has just been described can easily be applied to this example, but in this case the cutting planes may be horizontal, because horizontal sections of the cone are circles and horizontal sections of the cylinder are straight lines. In Fig. 741, horizontal cutting planes have been taken.

Draw an elevation of the cylinder and cone on a plane perpendicular

to the axis of the cylinder. In Fig. 741 this elevation is shown brought round into the plane of the other elevation.

Consider the cutting plane of which L′M′ is the vertical trace. This plane intersects the cone in a circle of which the diameter is equal to $a_1'b_1'$. The plan of this circle is a circle concentric with the plan of the cone. This same cutting plane intersects the curved surface of the cylinder in two straight lines of which the points b_1' and c_1' are the end elevations. The plans of these lines are parallel to the plan of the axis of the cylinder, and at distances from it equal to half of $b_1'c_1'$. The points b and c where these lines on the plan meet the circle already mentioned, are the plans of points on the curve of intersection, and

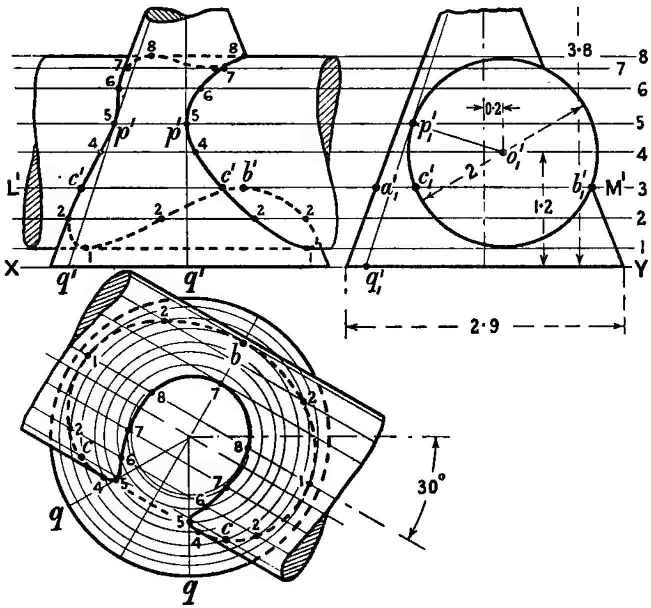


FIG. 741.

perpendiculars from b and c to XY to meet L′M′ at b' and c' determine the elevations of these points. In like manner any number of points on the curve of intersection may be found.

In Fig. 741 eight cutting planes are shown numbered from 1 to 8, and these should be sufficient.

The most important points are on the planes 1, 3, 4, 5, 7, and 8. The position of the plane number 5 is obtained as follows. $p_1'q_1'$ is the vertical trace of a plane which touches the cylinder and passes through the vertex of the cone, p_1' being the end elevation of the line of contact of this plane and the cylinder. $o_1'p_1'$ is of course perpendicular to $p_1'q_1'$. The vertical trace of plane number 5 passes through p_1'. The

lines in which the plane $p_1'q_1'$ cuts the cone are also shown and as these lines are tangents to the curve of intersection they assist in the correct drawing of the curve at important points.

EXAMPLE 2 (Fig. 742). *vv'* is the vertex and *va v'a'* the axis of a cone whose vertical angle is 30°. *bc b'c'* is the axis of a cylinder 2 inches in diameter. The dimensions which fix the positions of the axes of the cone and cylinder are given at the top right hand corner of

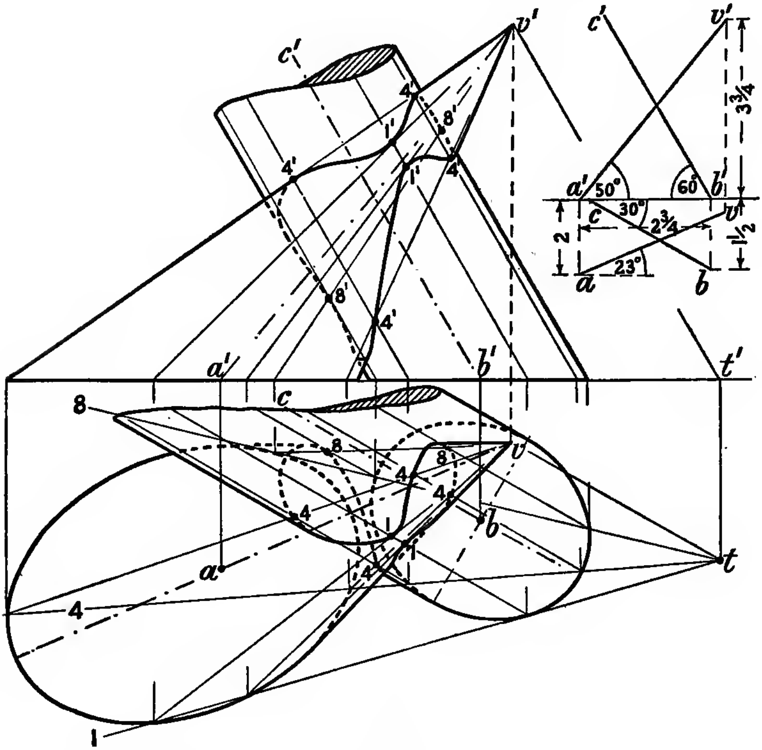


FIG. 742.

Fig. 742. It is required to show, in plan and elevation, the intersection of the surfaces of the cone and cylinder.

This example is worked by using cutting planes passing through the vertex of the cone and parallel to the axis of the cylinder which, as already stated, is the general method of finding the intersection of the surfaces of a cone and cylinder.

The first step is to determine the horizontal traces of the cylinder and cone by Arts. 219 and 226 respectively. The construction lines for this are omitted in Fig. 742.

All planes which are parallel to the axis of the cylinder and pass

through the vertex of the cone will contain the line *vt v't'* drawn through *vv'* parallel to *bc b'c'*. Hence the horizontal traces of all the cutting planes will pass through *t*, the horizontal trace of *vt v't'*.

There are eight planes required to determine all the important points but only three of these planes, numbered 1, 4, and 8, are shown in Fig. 742. Plane number 1 touches the cylinder and cuts the cone, and plane number 8 touches the cone and cuts the cylinder; each of these planes will therefore determine two points only on the intersection required. Plane number 4 cuts both surfaces and will determine four points on the intersection required. The omitted planes 2, 3, 5, 6, and 7 will each determine four points.

323. Intersection of Two Cones.—The general method in this

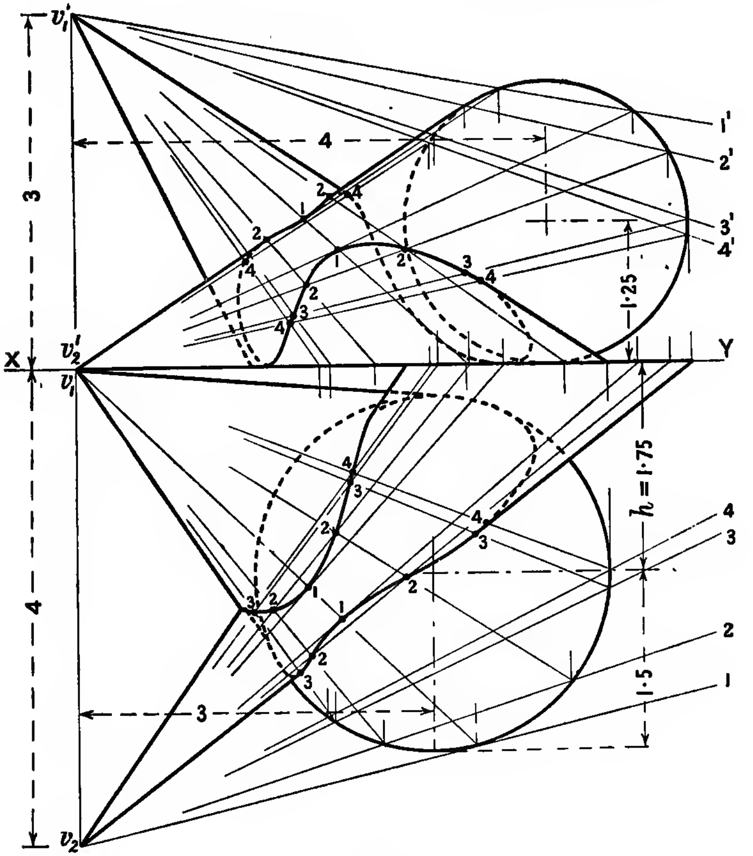


Fig. 743.

case is to use cutting planes passing through the vertex of each cone. These planes will cut the surface of each cone in straight lines, and

they will all contain the line joining the vertices of the cones. Hence the horizontal traces of all the cutting planes will pass through the horizontal trace of the line joining the vertices of the cones. Also, the vertical traces of the cutting planes will pass through the vertical trace of the line joining the vertices of the cones.

EXAMPLE (Fig. 743). v_1v_1' is the vertex of a cone whose horizontal trace is a circle 3 inches in diameter. v_2v_2' is the vertex of a cone whose vertical trace is a circle 2·5 inches in diameter. The other dimensions are given on the figure. It is required to show, in plan and elevation, the intersection of the surfaces of the two cones.

Seven cutting planes passing through the vertex of each cone will determine all the important points in this example, but in Fig. 743 only four of these planes, numbered 1, 2, 3, and 4, are shown. Both traces of each cutting plane are employed. All the horizontal traces of the cutting planes pass through v_2, and all the vertical traces pass through v_1'.

Consider plane number 1. This plane touches one cone and cuts the other. The horizontal trace of the plane is drawn first, and if the point where this trace meets the ground line comes within the paper the vertical trace is obtained by joining v_1' to this point. If however the point where the horizontal trace meets the ground line is off the paper, then the construction given in Art. 11, p. 9, may be used for drawing the vertical trace. The intersections of the line in which plane number 1 touches the one cone with the lines in which it cuts the surface of the other cone determine the points numbered 1 on the intersection required. In like manner the other points are determined.

324. Intersections of Cylinders and Cones enveloping the same Sphere.—If two cylinders envelop the same sphere their

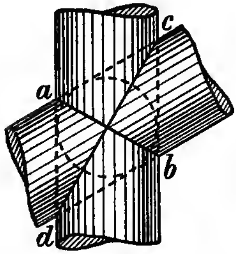


FIG. 744.

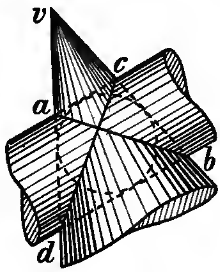


FIG. 745.

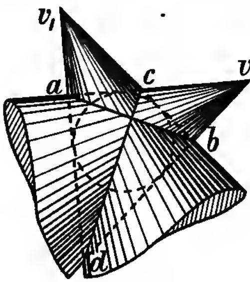


FIG. 746.

intersection will be plane sections of both and will therefore be ellipses. Fig. 744 shows a projection of two such cylinders on a plane parallel to their axes. The straight lines ab and cd form the projection of the intersection of the cylinders.

The same remarks apply to the cone and cylinder (Fig. 745), and to two cones (Fig. 746). Referring to Fig. 745, if the vertex of the

cone is placed on the curved surface of the cylinder, one of the ellipses will become a straight line. Referring to Fig. 746, it is obvious that if the cone v_2ad be turned round so as to make v_2d more nearly parallel to v_1a, the straight line cd will become more nearly parallel to v_1a and v_2b, and when v_2d is parallel to v_1a, cd will also be parallel to v_1a and v_2b, and the intersection cd will then be a parabolic section of each cone. Continuing the motion of the cone v_2ad in the same direction, the intersection cd will become a hyperbolic section of each cone.

325. Intersection of Cylinder and Sphere.—Since the surface of a sphere is a particular form of a surface of revolution this problem is a particular case of that discussed in Art. 326, p. 398. If

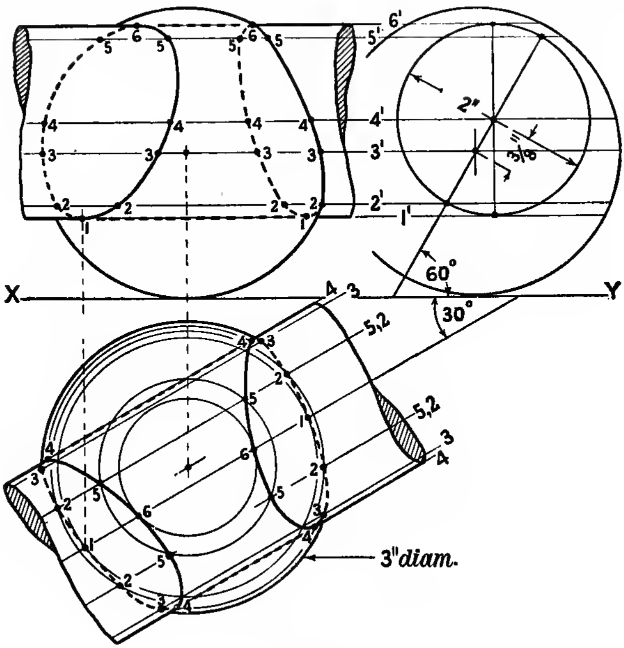


Fig. 747.

the cylinder is also a surface of revolution, that is, a right circular cylinder, then this problem is also a particular case of that considered in Art. 328, p. 401. But instead of using the method described in Art. 326, or the method of Art. 328 cutting planes parallel to the axis of the cylinder and perpendicular to one of the planes of projection may be employed, because such planes will cut the sphere in circles and the cylinder in straight lines.

If the axis of the cylinder is inclined to both planes of projection then an auxiliary elevation on a vertical plane parallel to the axis of

the cylinder should be drawn, and the cutting planes being vertical and parallel to the axis of the cylinder they will cut the sphere in circles which will appear as circles in the auxiliary elevation.

EXAMPLE (Fig. 747). The plan and elevation of a cylinder and sphere are given, the dimensions being marked on the figure. It is required to show, in plan and elevation, the intersection of the surfaces of the cylinder and sphere.

The method used here is that of cutting the surfaces by horizontal planes.

The upper right hand portion of Fig. 747 is a projection of the cylinder and sphere on a plane perpendicular to the axis of the cylinder and it is from this projection that the positions of the cutting planes which give the important points on the required intersection are determined.

326. Intersection of Cylinder and Surface of Revolution.—In all the problems hitherto considered on the intersection of surfaces, the auxiliary cutting surfaces which have been used, in order to find points on the required intersection, have been planes. The only simple plane sections of a surface of revolution are, in general, sections at right angles to its axis, which are always circles. Now it is evident that only in very particular cases would a plane which cuts the surface of revolution in a circle cut the surface of the cylinder in a circle or in straight lines. For instance, a plane which cuts the surface of revolution in a circle may cut the cylinder in an ellipse, and it would clearly be a laborious process to construct an ellipse for each cutting plane. This objection may however be got over in the present problem by using a tracing of the section of the cylinder by a plane which cuts the surface of revolution in a circle, in a manner to be explained at the end of this article.

Instead of cutting the given surfaces by planes they may be cut by the surfaces of cylinders and points on the required intersection obtained by means of circles and straight lines. Let a circle which is a section of the surface of revolution be taken and let a straight line move in contact with this circle and also remain parallel to the axis of the given cylinder. This moving line will describe the surface of a cylinder which intersects the surface of revolution in a circle and the surface of the given cylinder in straight lines, and the points in which these straight lines cut the circle will be points on the intersection required.

In order that the projections of the circular sections of the surface of revolution may be circles and straight lines the axis of revolution must be arranged perpendicular to one of the planes of projection.

Referring to Fig. 748, mn, $m'n'$ is the axis of a cylinder which intersects a surface of revolution whose axis is vertical. The construction for the outline of the elevation of the surface of revolution is shown at (c).

The ellipse which is the horizontal trace of the cylinder is determined as explained in Art. 219, p. 253.

ab, $a'b'$ is a circular section of the surface of revolution, oo' being the centre of this circle. ot, $o't'$ is a line through oo' parallel to mn, $m'n'$.

ot, *o′t′* is the axis of an auxiliary cylinder of which the circle *ab*, *a′b′* is one horizontal section, and all horizontal sections of this cylinder will be circles of the same diameter. *tt′* being the horizontal trace of the axis of this auxiliary cylinder the horizontal trace of its surface will be a circle whose centre is *t* and radius equal to *oa*. The horizontal trace

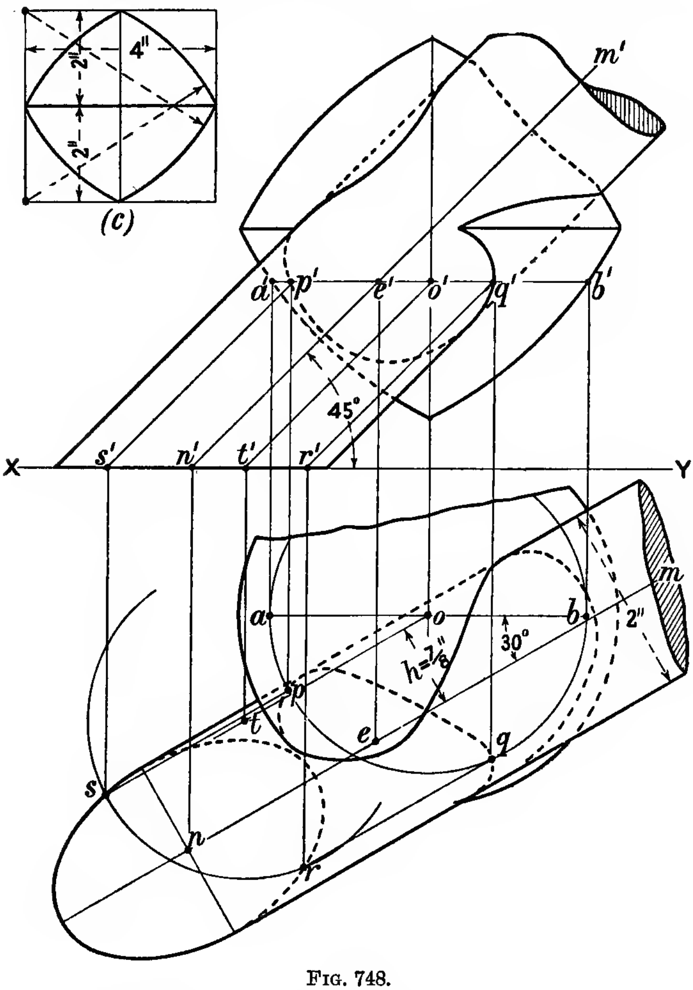


FIG. 748.

of this auxiliary cylinder intersects the horizontal trace of the given cylinder at *s* and *r*. The auxiliary cylinder intersects the given cylinder in straight lines *sp*, *s′p′* and *rq*, *r′q′* which are parallel to *mn*, *m′n′*. *pp′* and *qq′* the intersection of the lines *sp*, *s′p′* and *rq*, *r′q′* with

the circle *ab*, *a'b'* are points on the line of intersection of the given cylinder and surface of revolution. By taking other auxiliary cylinders, any number of points on the required intersection may be found.

The student should notice that each line of intersection of an auxiliary cylinder and the given cylinder intersects the corresponding circle on the surface of revolution in *one* point only although its plan may cut the plan of the circle in two points.

A convenient and practical method of solving the problem which has just been considered is to take horizontal sections of both the given surfaces. The sections of the cylinder are ellipses but these ellipses are all of the same size and if one of them be drawn and a tracing of it made, it only remains to draw a sufficient number of circular sections of the surface of revolution and apply the tracing of the ellipse to each to find points in the required intersection. The position of the centre of the ellipse corresponding to a particular circular section of the surface of revolution is where the plane of that section cuts the axis of the cylinder. For the circular section *ab*, *a'b'* (Fig. 748) *e* is the position of the centre of the ellipse in the plan and the tracing of the ellipse is placed so that the centre is at *e* and the major axls on *mn*. The points where the ellipse cuts the circle *ab* are points on the plan of the intersection required and these points may be pricked through. The elevations of the points are of course perpendicularly over their plans and on the elevation of the corresponding circular section.

327. Intersection of Cone and Surface of Revolution.—Placing the surface of revolution so that its axis is vertical, horizontal sections of it will be circles, but except in the special case where the cone is a right circular cone and its axis is vertical, horizontal sections of the cone will not be circles or straight lines and all the horizontal sections will be different. The tracing paper method which is applicable to the intersection of a cylinder and a surface of revolution and which was described in the latter part of the preceding Art., is therefore not suitable in the case of the intersection of a cone and a surface of revolution.

Auxiliary cones are taken which have their vertices at the vertex of the given cone and for their directrices they have circular sections of the surface of revolution. These auxiliary cones will intersect the given cone in straight lines and the intersection of these straight lines with the corresponding circles on the surface of revolution will determine points on the intersection required.

Referring to Fig. 749, *vn*, *v'n'* is the axis of a cone whose vertical angle is 30° and which intersects a surface of revolution whose axis is vertical. The dimensions are given on the figure.

The horizontal trace of the cone is determined as explained in Art. 226, p. 259. *ab*, *a'b'* is a circular section of the surface of revolution, *oo'* being the centre of this circle. *vt*, *v't'* is a line passing through *oo'*, the centre of the circle, and the vertex of the cone. *vt*, *v't'* is the axis of an auxiliary cone of which the circle *ab*, *a'b'* is one horizontal section. All horizontal sections of this auxiliary cone will be circles but they will be of different diameters. The point *tt'* being the horizontal trace

of the axis of the auxiliary cone it will be the centre of the circle which is the horizontal trace of that cone. The radius of this circle is equal to $t'm'$ where m' is found by joining v' to a' and producing it to meet XY as shown. The horizontal trace of the auxiliary cone and the horizontal trace of the given cone intersect at s and r, and the auxiliary cone intersects the given cone in straight lines vr, $v'r'$ and vs, $v's'$. pp' and

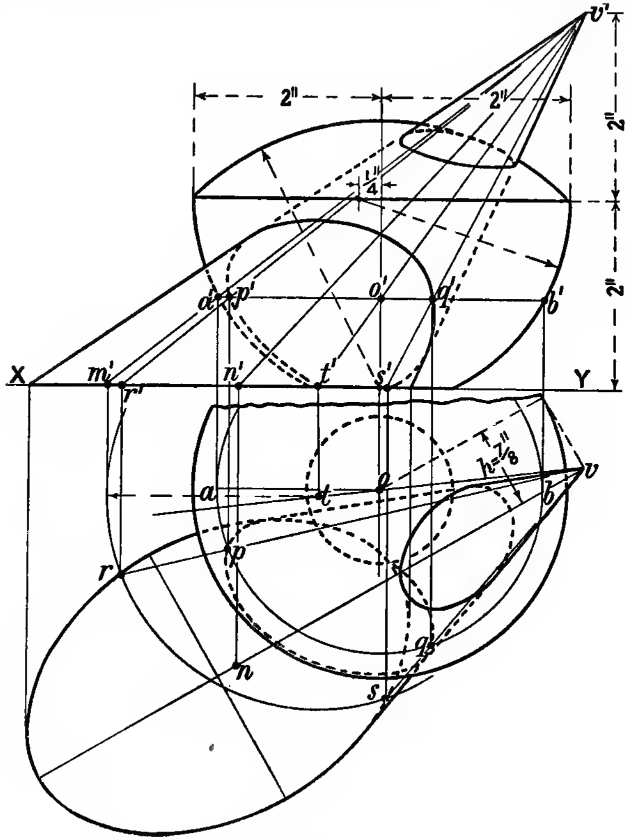


FIG. 749.

qq' the points of intersection of the lines vr, $v'r'$ and vs, $v's'$ with the circle ab, $a'b'$ are points on the intersection of the given cone and surface of revolution. By taking other auxiliary cones any number of points on the required intersection may be found.

328. Intersection of Two Surfaces of Revolution whose Axes are Parallel.—Since all sections of a surface of revolution by planes perpendicular to its axis are circles, it follows, that if the axes of two surfaces of revolution are parallel, a plane which is perpendicular

to the axis of one will be perpendicular to the axis of the other, and this plane, if it cuts both surfaces, will cut them in circles the intersection of which with one another determines points on the intersection required.

The surfaces of revolution should be arranged so that their axes are perpendicular to one of the planes of projection. The projections of the circles mentioned above will then be circles and straight lines which are easily drawn.

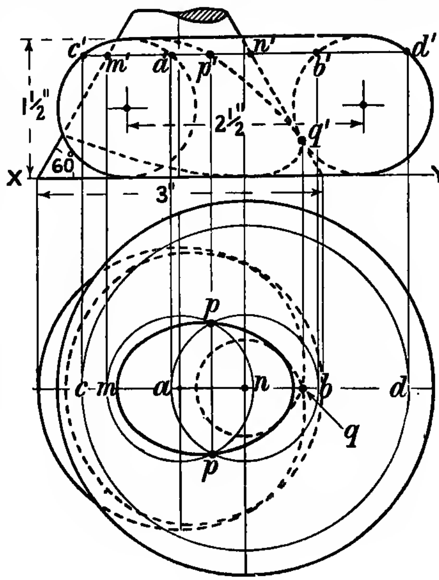


FIG. 750.

An example is shown in Fig. 750. An "anchor ring" whose surface is described by the revolution of a vertical circle about a vertical axis is shown penetrated by a cone of revolution whose axis is also vertical. One cutting plane is shown intersecting the anchor ring in two circles *ab*, *a'b'* and *cd*, *c'd'*. This same plane intersects the cone in the circle *mn*, *m'n'*. From the plan it is seen that the circle *mn* on the cone only intersects the circle *ab* on the anchor ring determining the two points *p* on the plan of the intersection of the cone and anchor ring. It will be seen that the cone touches the surface of the anchor ring at the point *qq'*.

329. Intersection of Two Surfaces of Revolution whose Axes Intersect.—Place the surfaces so that the axis of one of them is vertical and the axis of the other is parallel to the vertical plane of projection. Referring to Fig. 751, *vn v'n'* is the axis of a cone of revolution which intersects a surface described by the revolution of an arc of a circle about a vertical axis as shown. The axes of the two surfaces intersect at *oo'*. *b'a'd'*, the elevation of a sphere is shown, the centre of this sphere being at the intersection of the axes of the given surfaces of revolution. This sphere intersects the surface whose axis is vertical in a circle whose elevation is the horizontal line *a'b'* and whose plan is the circle *ab* having its centre at *o*. This sphere also intersects the other given surface of revolution in a circle whose elevation is the straight line *c'd'*. These two circles lie on the sphere and one being on one of the given surfaces of revolution and the other on the other, their points of intersection are points on the intersection required. *p'*, the point of intersection of *a'b'* and *c'd'*, is the elevation of the points of intersection of the two circles and their plans are determined by a

projector to intersect the circle ab. Observe that the plan of the circle of which $c'd'$ is the elevation, and which would be an ellipse, need not be drawn. Taking other spheres with their centres at oo' any number of points on the intersection of the given surfaces may be found.

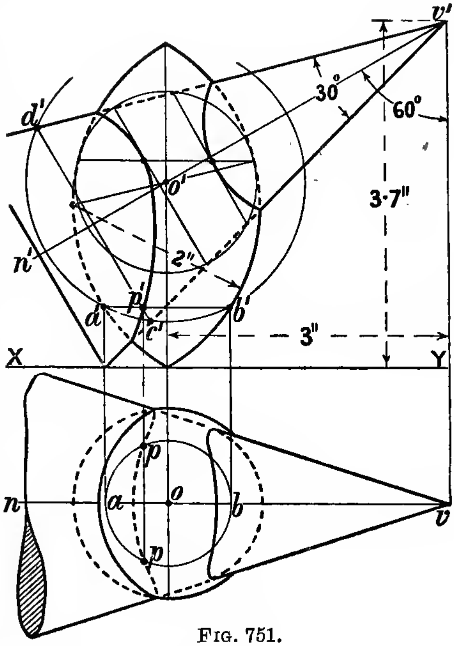


Fig. 751.

330. Intersection of Two Spheres.—Since the surface of a sphere is a surface of revolution the methods described in the two preceding articles for determining the intersection of two surfaces of revolution are applicable to the case of two spheres. But, since the intersection of two spheres is a circle, the projections of this circle may be found by a simpler method.

Referring to Fig. 752, o_1 and o_2 are the plans of the centres of two spheres whose radii are r_1 and r_2 respectively. An elevation is drawn on a ground line XY parallel to o_1o_2. The common chord $a'b'$ of the circles which are the elevations of the two spheres on the ground line XY is an elevation of the circle which is the intersection of the spheres. The plan of this circle is an ellipse whose minor axis is ab and whose major axis is equal to $a'b'$. From the plan and elevation shown an elevation on any other ground line is easily determined.

The intersection of the surfaces of the two spheres

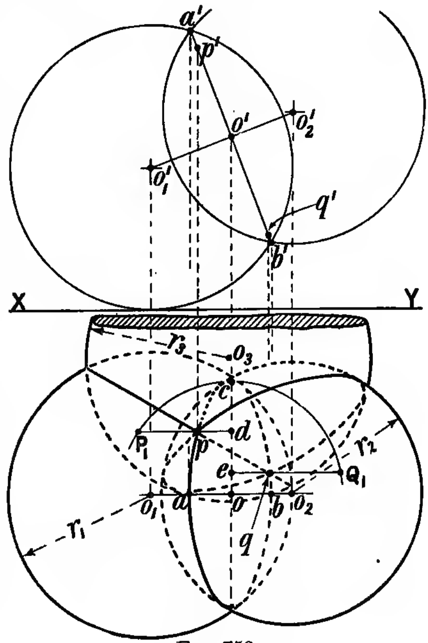


Fig. 752.

referred to above is the locus of a point whose distances from the points O_1 and O_2 are r_1 and r_2 respectively.

In like manner the intersection of a third sphere whose centre is O_3 and radius r_3 with the second sphere whose centre is O_2 and radius r_2 is the locus of a point whose distances from O_2 and O_3 are r_2 and r_3 respectively. The plan of this third sphere and the ellipse which is the plan of its intersection with the second sphere are shown in Fig. 752. The second ellipse is found in the same manner as the first from an elevation on a ground line parallel to o_2o_3. The circle which is the intersection of the first and second spheres intersects the circle which is the intersection of the second and third spheres at points whose plans are p and q. The points P and Q are evidently points whose distances from O_1, O_2 and O_3 are r_1, r_2 and r_3 respectively.

If $a'b'$ is inclined to XY at an angle which is nearly a right angle the positions of p' and q', which determine the distances of P and Q from the horizontal plane of projection, are best obtained as follows. With centre o and radius equal to $o'a'$ describe an arc of a circle P_1cQ_1. Through p and q draw pP_1 and qQ_1 perpendicular to oc to meet this arc at P_1 and Q_1 and oc at d and e respectively. Then $o'p'$ is equal to dP_1 and $o'q'$ is equal to eQ_1. The theory of this construction is obvious when it is observed that the arc P_1cQ_1 is part of the circle of intersection of the first and second spheres turned into a horizontal position about its horizontal diameter.

In Fig. 752, the centre of the third sphere is at the same level as the centre of the first sphere, hence the plane of the circle which is the intersection of the first and third spheres is vertical and the plan of the circle is a straight line.

331. Intersection of a Curved Surface with a Prism or a Pyramid.—Since the faces of a prism or a pyramid are portions of planes their intersections with any curved surface are plane sections or portions of plane sections of that surface and these may be determined by the methods described in Chapters XVIII and XXIII. If these plane sections meet one another, they will intersect at the points where the edges of the prism or the pyramid meet the curved surface. The latter points may be found separately by the method explained in Art. 275, p. 317.

332. Intersection of Prisms and Pyramids with one another.—Since the faces of prisms and pyramids are portions of planes, their intersections with one another are straight lines which are determined by the rules for the intersection of two planes (Art. 169, p. 204). The theory of the constructions for finding the intersections of prisms and pyramids with one another is therefore very simple, but in some cases the solution of the problem may be very complicated owing to the large number of lines which may be required, and considerable skill is necessary to obtain an accurate result. Moreover the student will find in some cases ample opportunity for selecting the simplest and most accurate method of finding the intersection of the different pairs of plane faces of the solids which may be given.

In a complicated example it may be desirable to proceed in some

systematic way such as follows. Denote the faces of one of the solids by A_1, B_1, C_1, etc., and the faces of the other by A_2, B_2, C_2, etc. Consider the faces A_1 and A_2. Find the intersection of these two faces if they do intersect, noting that only that part of the line of intersection of their planes which lies within both faces is required. Decide at once whether this line is visible or not in plan and in elevation, and line it in distinctly. Next consider the faces A_1 and B_2 in the same way, then A_1 and C_2 and so on until the intersections of the face A_1 of the one solid with each of the faces of the other solid have been found. In like manner deal with the face B_1 of the one solid and each of the faces of the other in turn, and so on until the intersections of all the faces have been found. Of course a face of the one solid may not intersect any of the faces of the other or it may only intersect one or two faces of the other. Very often an inspection of the figure will show which faces intersect especially after the intersections of one or two pairs of faces have been found.

The student is recommended to work out the three cases of the following exercise.—The square *ab* (Fig. 753), of 2 inches side, is the horizontal trace of a prism whose long edges are parallel to *ef*, *e′f′*. The square *cd*, of 2·5 inches side, is the horizontal trace of a prism whose long edges are parallel to *gk*, *g′k′*. *e* is the centre of the square *ab*, and *g* is the centre of the square *cd*. Show in plan and elevation the intersection of the surfaces of the prisms, (1) when $h = 0$, (2) when $h = 0·5$ inch, and (3) when *h* is such that the line joining *b* and *c* is parallel to XY.

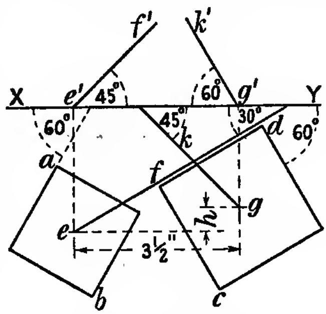


FIG. 753.

The solution of case (2), without the construction lines, is shown in Fig. 754.

Exercises XXVI

Note. *Where the surfaces in the following exercises are developable, the student should, in a selected number of cases, draw the developments and show on them the lines of intersection. He should also, in some cases, cut out the developments and construct models of the solids with them.*

1. The axes of two cylinders are horizontal and at right angles to one another. One cylinder is $2\frac{1}{2}$ inches in diameter, and the other is $2\frac{1}{4}$ inches in diameter. Draw the plan, and an elevation on a vertical plane inclined at 30° to the axis of the larger cylinder, showing the intersection of the surfaces of the cylinders, (*a*) when the axes intersect, (*b*) when the axis of the smaller cylinder is $\frac{1}{8}$ inch above the axis of the other, (c) when the axis of the smaller cylinder is $\frac{1}{4}$ inch above the axis of the other.

2. The same as the preceding exercise except that the angle between the axes is 60° instead of 90°.

3. A cylinder, 2·5 inches in diameter, has its axis perpendicular to the V.P. Another cylinder, 2 inches in diameter has the elevation of its axis inclined at

45° to XY. The angle between the plans of the axes is 90°. Draw the plan, showing the intersection of the surfaces of the cylinders, (*a*) when the axes intersect, (*b*) when the axes are 0·25 inch apart, (*c*) when the axes are 0·5 inch apart.

4. The same as the preceding exercise except that the angle between the plans of the axes is 60° instead of 90°.

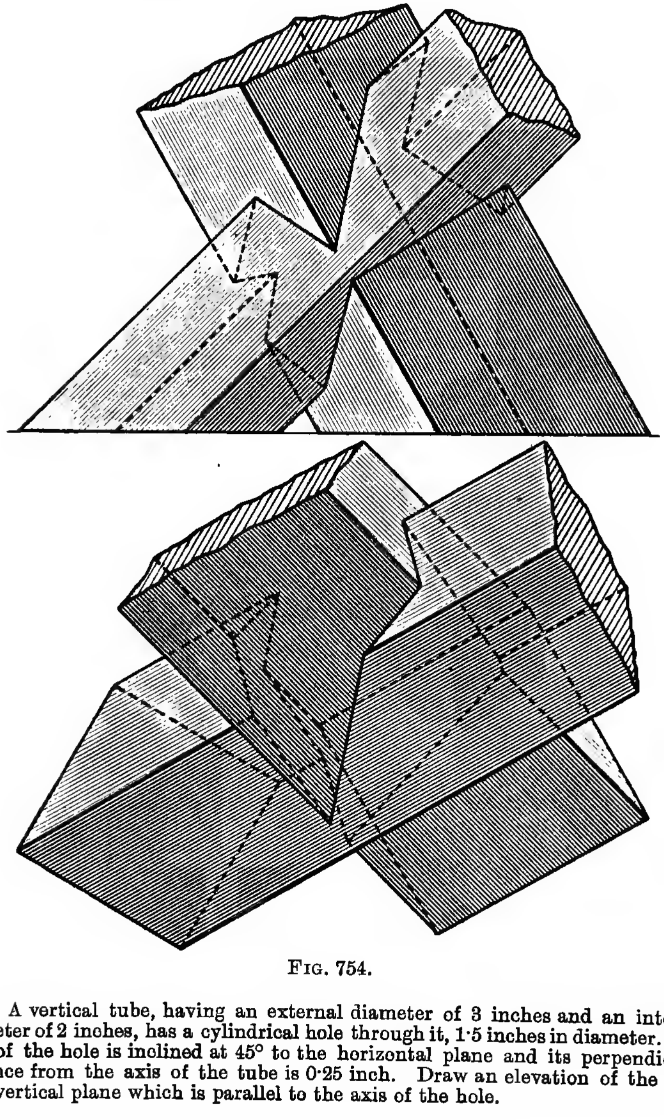


FIG. 754.

5. A vertical tube, having an external diameter of 3 inches and an internal diameter of 2 inches, has a cylindrical hole through it, 1·5 inches in diameter. The axis of the hole is inclined at 45° to the horizontal plane and its perpendicular distance from the axis of the tube is 0·25 inch. Draw an elevation of the tube on a vertical plane which is parallel to the axis of the hole.

6. The same as the preceding exercise except that the elevation is to be on a vertical plane which makes 45° with the plan of the axis of the hole.

7. AB (Fig. 755) is the axis of a cylinder whose horizontal trace is a circle 2·25 inches in diameter. CD is the axis of a cylinder whose horizontal trace is an ellipse (major axis 3 inches, minor axis 2 inches) whose minor axis is parallel to XY. The axes of the cylinders are parallel to the vertical plane. Show the intersection of the cylinders in plan and elevation in each of the following cases. (i) when $h = 0$, (ii) when $h = \frac{3}{8}$ inch, (iii) when $h = \frac{1}{2}$ inch.

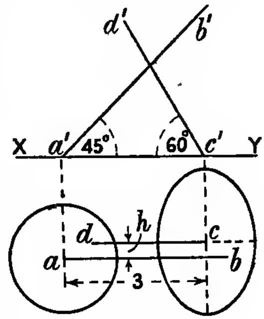


FIG. 755.

8. Same as example 3, p. 390, except that h (Fig. 740) is to be 0 instead of $\frac{3}{8}$ inch.

9. Same as example 3, p. 390, except that h (Fig. 740) is to be $\frac{1}{2}$ inch instead of $\frac{3}{8}$ inch.

10. The axis of a cylinder 2 inches in diameter is inclined at 60° to the ground. The axis of a second cylinder $2\frac{1}{2}$ inches in diameter is inclined at 45° to the ground. The angle between the plans of the axes is 80°, and the common perpendicular to the axes has a true length equal to k. Draw the plan, and an elevation on a vertical plane parallel to the axis of the first cylinder, showing the intersection of the surfaces, for each of the following cases. (i) $k = 0$, (ii) $k = 0{\cdot}25$ inch, (iii) $k = 0{\cdot}5$ inch.

11. *abd* is an equilateral triangle of 2·5 inches side. c is a point within the triangle, 1·75 inches from *b* and 1·25 inches from *d*. *a*, *b*, c, and *d* are the plans of points whose heights above the ground are 0·5 inch, 1 inch, 2·5 inches, and 1·5 inches respectively. A circular cylinder has its axis parallel to the line AD, and its surface contains the four points A, B, C, and D. A second circular cylinder has its axis parallel to the line BC, and its surface also contains the four points A, B, C, and D. Draw the plan showing the intersection of the surfaces of the cylinders.

12. A right cone having a base 3·5 inches in diameter, and an altitude of 3·5 inches, stands with its base on the ground. A cylinder, 2 inches in diameter lies on the ground and penetrates the cone. The axis of the cone is at a distance h from the axis of the cylinder. Draw the plan, and an elevation on a vertical plane inclined at 30° to the axis of the cylinder, showing the intersection of the surfaces, (i) when $h = 0$, (ii) when $h = 0{\cdot}25$ inch, and (iii) when h is such that the curved surface of the cone touches the curved surface of the cylinder.

13. A right circular cone passes through a cylindrical tube. The axis of the cone intersects the axis of the tube at right angles. Internal diameter of tube, 3 inches. Vertical angle of cone 15°. Diameter of cone at centre of tube 1·5 inches. Determine the development of the surface of that part of the cone which is within the tube.

14. The elevation of two horizontal cylinders is shown in Fig. 756, also the elevation of a right circular cone, whose axis is perpendicular to the axes of the cylinders. Determine the development of the surface of that part of the cone which lies between the cylinders.

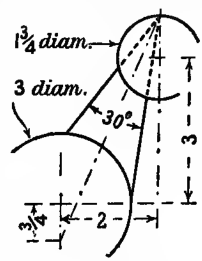


FIG. 756.

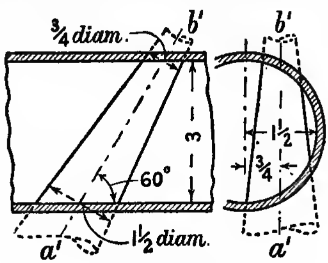


FIG. 757.

15. Determine the intersection of the given cone and hollow cylinder (Fig. 757), and draw the development of the surface of that part of the cone which is within the cylinder.

16. Same as example 2, p. 394, except that *av* is to make 25° with XY instead of 23°.

17. *abc* is an equilateral triangle of 4 inches side. A circle is described on *ab* as diameter, and another is described on *ac* as diameter. These circles are the horizontal traces of two cones. *c* is the plan of the apex of the cone of which the circle *ab* is the horizontal trace, and *b* is the plan of the apex of the other cone. Each apex is at a height of 4 inches above the horizontal plane. Show the plan of the intersection of the surfaces of the two cones and add an elevation on a ground line parallel to *ab*.

18. Same as the example on p. 396, except that the distance *h* in Fig. 743 is to be 2·25 inches instead of 1·75 inches.

19. Same as the example on p. 396, except that the distance *h* in Fig. 743 is to be such that the plane which contains the vertices of the two cones and is tangential to one of the cones shall also be tangential to the other cone.

20. v_1oa and v_2ob are two straight lines at right angles to one another. $v_1o = 2·5$ inches. $v_2o = 3$ inches. v_1oa is the plan of the axis of a cone, semi-vertical angle 25°, which lies with its slant side on the ground, v_1 being the plan of its vertex. v_2ob is the plan of the axis of a second cone also lying with its slant side on the ground, v_2 being the plan of its vertex. The axes of the cones intersect. Both cones are right circular cones. Draw the plan showing the intersection of the surfaces, and add an elevation on a vertical plane parallel to the axis of the first cone. *Note.* These cones will envelop the same sphere and their intersection will be two ellipses.

21. A circle 3 inches in diameter is the elevation of a sphere. Another circle 2 inches in diameter is the elevation of a cylinder. The centres of the circles lie on a line inclined at 60° to XY, and are at a distance *h* from one another. The horizontal plane which contains the axis of the cylinder is above the centre of the sphere. Show the plan of the intersection of the surfaces of the sphere and cylinder, (*a*) when $h = \frac{3}{8}$ inch, (*b*) when $h = \frac{5}{8}$ inch, (*c*) when $h = \frac{1}{2}$ inch.

22. A circle 3 inches in diameter is the elevation of a sphere. A line inclined at 40° to XY, and at a perpendicular distance of 0·5 inch from the centre of the circle is the elevation of the axis of a cylinder 1·5 inches in diameter which is parallel to the vertical plane. The cylinder penetrates the sphere and touches its surface internally. Draw in plan and elevation the complete curve of intersection of the surfaces.

23. A solid of revolution is generated by an ellipse, 4 inches by 2·5 inches, revolving about its major axis which is vertical. A cylinder 2 inches in diameter has its axis situated so that its plan and elevation are inclined at 45° to XY. The elevation of the axis of the cylinder passes through the centre of the elevation of the solid of revolution and the plan is at a perpendicular distance of 0·15 inch from the centre of the plan of the solid of revolution. Draw the plan and elevation of the two solids showing the intersection of their surfaces.

24. Referring to Fig. 748, p. 399, draw the plan and elevation of the given cylinder and surface of revolution, showing their intersection, keeping to the dimensions given except that: case I, $h = 0$; case II, $h = \frac{3}{8}$ inch; case III, $h = \frac{3}{4}$ inch.

25. Taking the particulars of exercise 23 except that the axis of the cylinder is to be made the axis of a cone having the elevation of its vertex at a distance of 4 inches from the centre of the elevation of the solid of revolution and having a vertical angle of 24°. Draw the plan and elevation of the solids showing the intersection of their surfaces.

26. Referring to Fig. 749, p. 401, draw the plan and elevation of the given cone and surface of revolution, showing their intersection, keeping to the dimensions given, except that: case I, $h = 0$; case II, $h = \frac{1}{2}$ inch; case III, $h = 1$ inch.

27. A cone of revolution, base 3 inches in diameter, axis 2·5 inches long, has its axis vertical. A cylinder of revolution 2 inches in diameter, has its axis parallel to the axis of the cone and 0·5 inch distant from it. Draw an elevation on a plane inclined at 45° to the plane containing the axis of the solids showing the line of intersection of their surfaces.

28. A circle, 3 inches in diameter, is the plan of a right circular cone, altitude

4 inches, standing on the ground. A circle, $2\frac{1}{2}$ inches in diameter, is the plan of a sphere also standing on the ground. The centres of these circles are at a distance h from one another. Show the plan of the solids and their intersection and an elevation on a vertical plane which makes 30° with the plane containing the axis of the cone and the centre of the sphere, (1) when $h = \frac{1}{4}$ inch, (2) when $h = \frac{3}{8}$ inch, and (3) when h is such that the curved surfaces of the solids touch one another.

29. Work the example illustrated by Fig. 750, p. 402, when the axis of the cone is moved until it is 0·5 inch distant from the axis of the anchor ring.

30. Work the example illustrated by Fig. 751, p. 403, when the vertical angle of the cone is increased until the straight boundary lines of the plan of the cone are tangential to the plan of the surface of revolution.

31. abc is a triangle, $ab = 2$ inches, $bc = 1{\cdot}25$ inches, $ca = 1{\cdot}5$ inches. a is the plan of a point which is 0·5 inch below the H.P. b and c are the plans of points which are 1·25 inches and 2 inches respectively above the H.P. Determine the plans of the two points which are 2·75 inches distant from each of the points A, B, and C, and state their distances from the H.P.

32. The plan of a sphere with a vertical triangular hole in it is shown in Fig. 758. Draw an elevation on a ground line parallel to ab.

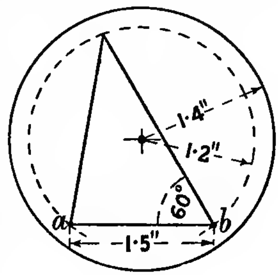


FIG. 758.

33. The equilateral triangle abc (Fig. 759) of $2\frac{1}{2}$ inches side is the horizontal trace of a pyramid of which v is the plan of the vertex. The height of the vertex above the H.P. is 4 inches. The square $defg$ of $2\frac{1}{2}$ inches side is the horizontal trace of a prism. Draw the plan and elevation of the pyramid and prism showing the intersection of their surfaces,—(1) when the long edges of the prism are vertical, (2) when the long edges of the prism are parallel to the V.P. and inclined at 60° to the H.P., sloping upwards from right to left in the elevation.

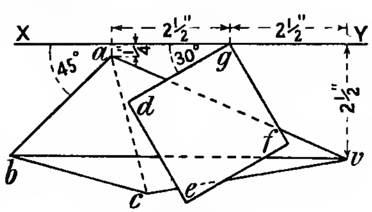


FIG. 759.

34. ab (Fig. 760) is a regular hexagon of $1\frac{1}{4}$ inches side. cde is an equilateral triangle of 2 inches side. The hexagon is the horizontal trace of a prism whose long edges are parallel to af, $a'f'$. The triangle is the horizontal trace of a prism whose long edges are

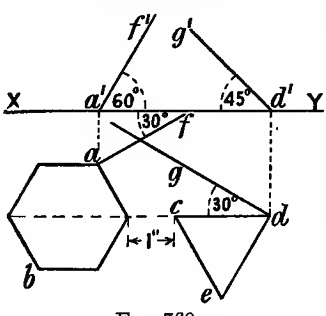


FIG. 760.

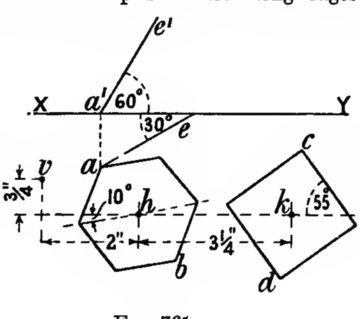


FIG. 761.

parallel to dg, $d'g'$. Draw the plan and elevation of these solids showing the intersection of their surfaces.

35. *ab* (Fig. 761) is a regular hexagon of $1\frac{1}{4}$ inches side and whose centre is *h*. *cd* is a square of 2 inches side and whose centre is *k*. The hexagon is the horizontal

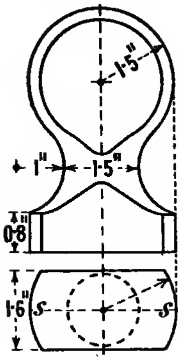


FIG. 762.

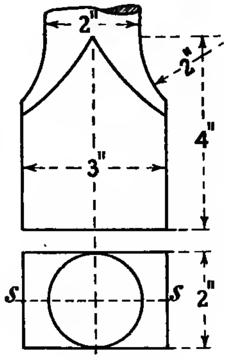


FIG. 763.

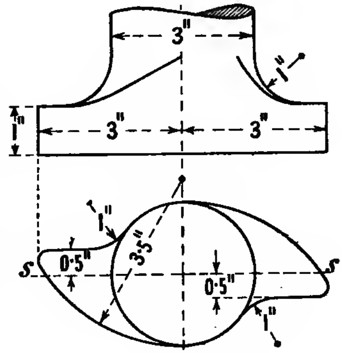


FIG. 764.

trace of a prism whose long edges are parallel to *ae*, *a'e'*. The square is the horizontal trace of a pyramid whose vertex V is 5 inches above the H.P. Draw the plan and elevation of these solids showing the intersection of their surfaces.

36. Certain solids are shown in Figs. 762, 763, and 764 in plan and elevation. For each solid draw the plan as shown and add an elevation on a ground line inclined at 45° to the centre line *ss* without drawing the elevation given. Use the dimensions marked but make no measurements from the illustrations given.

37. Draw the curve of intersection of the given cone (Fig. 765) with the helical surface (of uniform pitch) generated by the revolution of the horizontal line VH about the axis of the cone, the line descending to the base during one anti-clockwise turn.

The given point P will lie on the required curve. Determine the tangent to the curve at P. Also draw the normal and osculating planes at P. [B.E.]

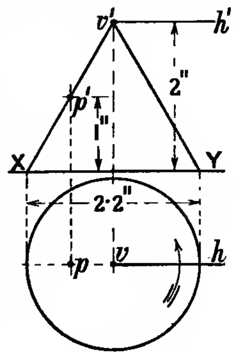


FIG. 765.

CHAPTER XXVII

PROJECTION OF SHADOWS

333. Theory of Shadows.—In a homogeneous medium light travels in straight lines, and in this chapter it will always be assumed that the medium through which the light passes is homogeneous. The rays of light may be parallel, or they may diverge from a point or converge to a point. If the light comes from a great distance, as from the sun, the rays are practically parallel. If the light comes from a point, which is practically the case when the luminous body is very small, the rays diverge from that point in all directions. Rays of light may be made to converge to a point by means of a reflector or a lens of suitable form.

If an opaque body be placed before a source of light, part of the surface of the body will be illuminated and the remainder left in darkness. Also a portion of the light from the luminous body will be intercepted, and a portion of the space behind the opaque body will be in darkness. This dark space behind the opaque body is called the *shadow* of that body. The surface which bounds the shadow is the *shadow surface*, and the line on the surface of the opaque body which separates the illumined from the unillumined part is the *shade line*. It is evident that the shadow surface is a ruled surface, and that its directing line is the shade line.

The intersection of the shadow surface with any other surface which it meets is the *cast shadow* of the opaque body on that surface, but generally the cast shadow is called simply the shadow.

The outline of the cast shadow of an opaque body is evidently the cast shadow of its shade line. In general the cast shadow is best determined by first finding the shade line, and then the cast shadow of that line; especially is this the case when the surface upon which the shadow is cast is other than a single plane.

It will always be assumed that the beam of light is large enough to embrace the whole of the object whose shadow is to be determined. It will also be assumed that any surface upon which a shadow is cast is an opaque surface. Hence a point can only have one shadow with one system of illumination.

334. Cast Shadow of a Point.—To determine the shadow cast by a given point on a given surface, draw the projections of the line which represents the ray of light which is intercepted by the point,

and determine the point of intersection of this line with the given surface upon which the shadow is cast. This point of intersection is the shadow required. The line which represents the ray of light may intersect the given surface at more than one point, but only that point of intersection which is nearest to the given point which casts the shadow is to be taken as the cast shadow of the point.

Three examples are shown at (*a*), (*b*), and (*c*), Fig. 766. In each case pp' is a given point and rr' is the ray of light intercepted by pp'.

At (*a*), L′MN is a given oblique plane and p_0p_0' is the shadow cast by pp' on this plane, p_0p_0' is found by using the vertical plane containing the ray rr'. The intersection of this plane with the given plane intersects rr' at p_0p_0'.

At (*b*) is shown the shadow cast by the point pp' on the surface of a vertical cylinder. The construction in this case is obvious.

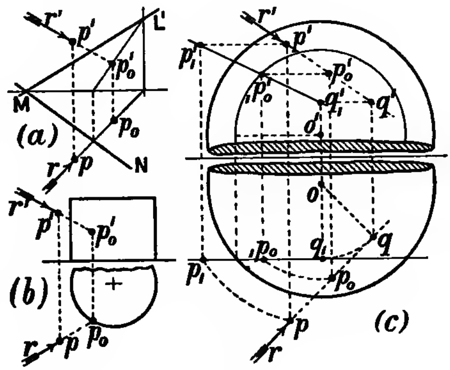


FIG. 766.

At (*c*), oo' is the centre of a sphere upon the surface of which the point pp' casts the shadow p_0p_0'. The vertical plane containing rr' is taken. This plane intersects the sphere in a circle. Turning this plane with the ray and circle in it about the vertical axis of the sphere until the plane is parallel to the vertical plane of projection, the intersection of the ray and circle in the new position is found. Turning this plane with the ray and circle and their point of intersection back to its original position, p_0p_0' is determined as shown.

335. Cast Shadow of a Line.—If the line is a straight line, then, whether the rays of light are parallel or proceed from a point, it is evident that all tho rays which meet the line are in the same plane. Hence the cast shadow of the line on any surface will be a portion of the intersection of this plane with the surface. The extremities of the cast shadow of the line will be the cast shadows of its extremities. The cast shadow of a straight line on a single plane is the straight line joining the cast shadows of the extremities of the line.

Fig. 767 shows the shadow cast by the straight line *ab*, *a′b′* on the plane L′MN and on the horizontal plane of projection, the rays of light being parallel to rr'. The traces of the plane containing the line *ab a′b′* and parallel to rr' are first determined. The required cast shadow is made up of a part of the horizontal trace of this plane and a part of the intersection of this plane with the plane L′MN as shown.

If the line whose cast shadow is required is a curved line, and the rays of light are parallel, all the rays which meet the line will lie on a

cylindrical surface; but if the rays all proceed from a point those which meet the curved line will lie on a conical surface. The intersection of the fore-mentioned cylindrical or conical surface with the surface upon which the shadow is cast is the cast shadow required.

Generally when the line is curved its cast shadow is determined by first finding, by Art. 334, the cast shadows of a sufficient number of points in the line and then drawing a fair curve through these. Fig. 768 shows the cast shadow of a curved line *ab a'b'* the rays of light being parallel to *rr'*. The shadow is cast partly on the surface of a horizontal cylinder and partly on the horizontal plane of projection.

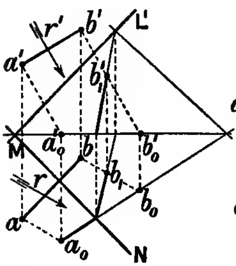


FIG. 767.

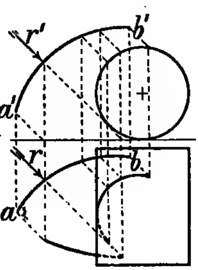


FIG. 768.

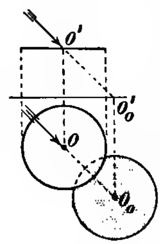


FIG. 769.

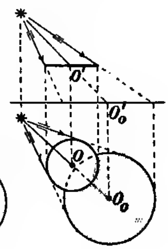


FIG. 770.

A simple case of importance is that of the cast shadow of a circle on a plane parallel to the plane of the circle. The cast shadow in this case is a circle whose centre is at the cast shadow of the centre of the original circle. If the rays of light are parallel (Fig. 769) the diameter of the cast shadow is equal to that of the original circle. If the rays of light proceed from a point (Fig. 770) the diameter of the cast shadow is greater than that of the original circle and is found by an obvious construction.

Two other simple cases of importance relate to the cast shadows of parallel lines on a plane. If the rays of light are parallel then the cast shadows of parallel lines on a plane are themselves parallel. If the rays of light proceed from a point then the cast shadows of parallel lines on a plane will, produced if necessary, meet at a point. The latter case is illustrated by Fig. 771 where *ab a'b'* and *cd c'd'* are parallel lines which cast their shadows on the inclined plane L'MN, the light coming from

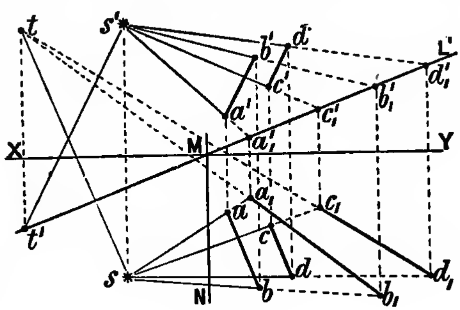


FIG. 771.

the point ss'. The point tt' to which the cast shadows converge is the trace on the plane L′MN of the line $st\ s't'$ which is parallel to the lines $ab\ a'b'$ and $cd\ c'd'$.

One other case may be mentioned. If a straight line casts a shadow, partly on one plane and partly on another plane parallel to the first, the shadow on the second plane is parallel to the shadow on the first.

336. Shadow of a Solid having Plane Faces.—When a solid having plane faces is placed in a beam of light which is large enough to embrace the whole of the solid, it is evident that one part of a face of the solid cannot be in light and another part in shade, unless the solid has re-entrant angles, in which case one part of the solid may cast a shadow on another part. Hence the shade line must be made up of edges of the solid. Those edges which make up the shade line can generally be determined by inspection. A particular edge is part of the shade line if a line representing a ray of light meeting a point on the edge in question does not enter the solid at that point.

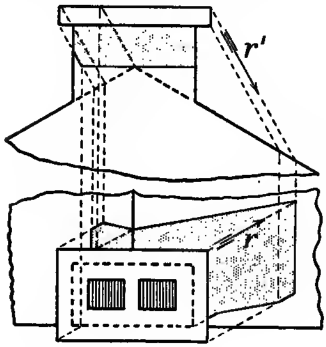


FIG. 772.

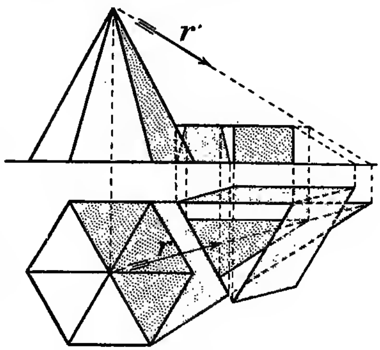


FIG. 773.

Having determined the shade line, its cast shadow, which is the outline of the cast shadow of the solid, may be determined by the constructions of the two preceding Arts.

Instead of first determining the shade line, the cast shadow of each edge of the solid may be found. The resulting figure is either a parallel or a conical projection of the solid according as the rays of light are parallel or proceed from a point. The boundary line of this projection is the cast shadow of the solid.

Fig. 772 shows the shadow cast by a short chimney on a roof and also the shadow cast by the coping on the chimney itself. The rays of light are parallel to rr'. All the necessary construction lines are shown.

Fig. 773 shows the shadows cast by a hexagonal pyramid and a triangular prism on the horizontal plane and also the shadow cast by the pyramid on the prism. The rays of light are parallel to rr'. All the necessary construction lines are shown.

337. Shadow of a Cylinder.—A cylinder which casts a shadow generally has its shade line made up of two straight lines and two curved lines. The straight lines are the lines of contact of the two tangent planes to the cylinder, these tangent planes are parallel to the rays of light or pass through the luminous point. Conceive a plane to contain these straight lines. This plane will divide each end of the cylinder into two segments, and the curved boundary line of one of these segments on each end will constitute the curved parts of the shade line. Those segments of the ends which must be taken in getting the curved parts of the shade line will be evident from inspection.

In the particular case where the rays of light are parallel to the axis of the cylinder the surface of the cylinder is itself the shadow surface, and in the particular case where the rays of light diverge from a point within the cylinder produced, the outline of that end of the cylinder which is nearest to the luminous point is the shade line.

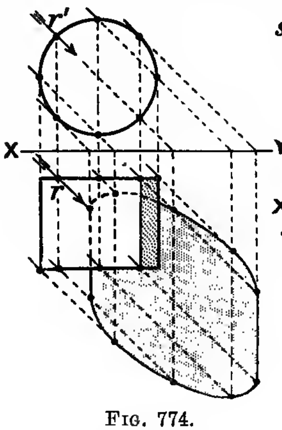


FIG. 774.

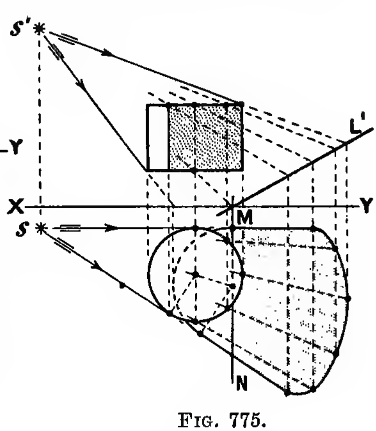


FIG. 775.

Having determined the shade line, its cast shadow, which is the outline of the cast shadow of the cylinder, may be determined by the constructions of Arts. 334 and 335.

Fig. 774 shows the shadow cast by a right circular cylinder on the horizontal plane. The axis of the cylinder is horizontal, and the rays of light are parallel to rr'. All the necessary construction lines are shown.

Fig. 775 shows the cast shadow of a right circular cylinder whose axis is vertical. The light proceeds from the point ss', and the shadow is cast partly on the horizontal plane and partly on the inclined plane L'MN. All the necessary construction lines are shown.

338. Shadow of a Cone.—A cone which casts a shadow generally has its shade line made up of two straight lines and one curved line. The straight lines are the lines of contact of two tangent planes to the cone, these tangent planes are parallel to the rays of light or

pass through the luminous point. A plane containing these straight lines will divide the base of the cone into two segments and the curved boundary line of one of these will be the curved part of the shade line. The segment of the base which must be taken in getting the curved part of the shade line will be evident from inspection.

In the case where the line which represents the ray of light through the vertex of the cone falls inside the cone, the shade line will consist simply of the whole outline of the base of the cone.

Figs. 776 and 777 show two cases of the shadow cast on the horizontal plane by a right circular cone, the axis of the cone being perpendicular to the vertical plane of projection. In Fig. 776 the rays of light are parallel to rr', while in Fig. 777 they diverge from the point ss'. tt' is the trace on the plane of the base of a line through the

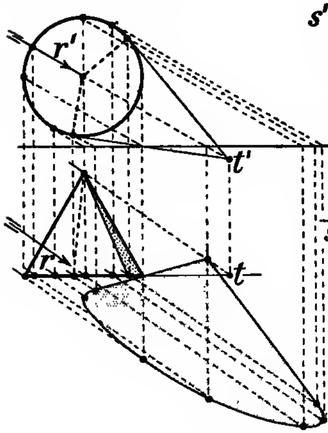


Fig. 776.

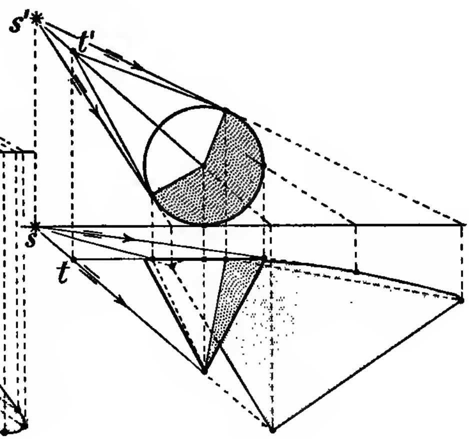


Fig. 777.

vertex of the cone parallel to rr' in Fig. 776 and through ss' in Fig. 777. Lines drawn from the vertex of the cone to the points of contact of the tangents from tt' to the base of the cone are the straight portions of the shade line. The remainder of the construction is obvious.

339. Shadow of a Sphere.—A sphere which casts a shadow, when the rays of light are parallel, will have for its shade line the circle of contact of the enveloping cylinder whose axis is parallel to the rays of light. This cylinder and its line of contact with the sphere are determined by the construction of Art. 221, p. 256.

If the rays of light proceed from a point, the sphere which casts a shadow will have for its shade line the circle of contact of the enveloping cone whose vertex is at the luminous point. This cone and its circle of contact with the sphere are determined by the construction of Art. 227, p. 263.

The intersection of the surface of the enveloping cylinder or

enveloping cone with any given surface will be the cast shadow of the sphere on that surface.

Two examples on the shadow cast by a sphere are illustrated by Figs. 778 and 779. The results only are shown, all the construction lines being omitted. The student should work out these examples to the dimensions given, which are in inches.

Fig. 778 shows the shadow cast by a sphere partly on the horizontal plane and partly on a concave cylindrical surface. The part of the

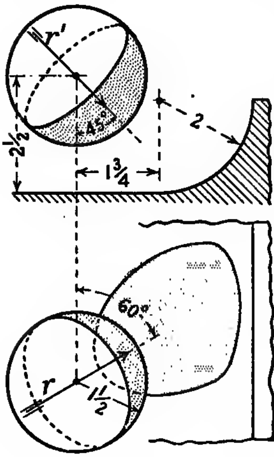


FIG. 778.

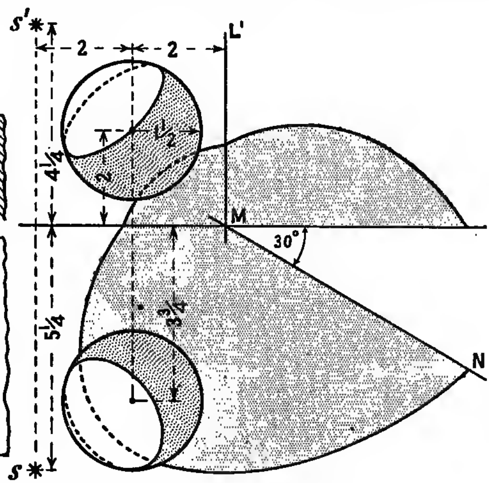


FIG. 779.

sphere in shade is also indicated. The rays of light are parallel to *rr'*.

Fig. 779 shows the shadow cast by a sphere partly on the horizontal plane, partly on the vertical plane of projection and partly on the vertical plane L'MN. The part of the sphere in shade is also indicated. The rays of light proceed from the point *ss'*.

340. Shadow of a Solid of Revolution.—If the rays of light are parallel determine, by Art. 292, p. 327, a cylinder to envelop the surface of the solid of revolution, the generatrices of the cylinder being parallel to the rays of light. The trace of this cylinder on the surface upon which the shadow is to be cast will determine the required cast shadow.

If the surface of the enveloping cylinder does not intersect the surface of the solid (it may touch at one part but cut at another) then the line of contact of the surface of the solid and the enveloping cylinder will be the line of separation between light and shade on the surface of the solid. If the surface of the enveloping cylinder also intersects the surface of the solid, then this intersection must be found in order to complete the shadow on the solid itself.

The solution of an example, which the student should work out carefully, is shown in Fig. 780. The solid, which is in the form of a vase, has its axis vertical. The rays of light are parallel to rr', and the shadow of the vase is cast on the horizontal plane of projection.

The chief difficulty is with the line of separation on the upper or concave part of the vase. $a'b'c'd'$ and $e'f'g'h'k'$ are the elevations of the

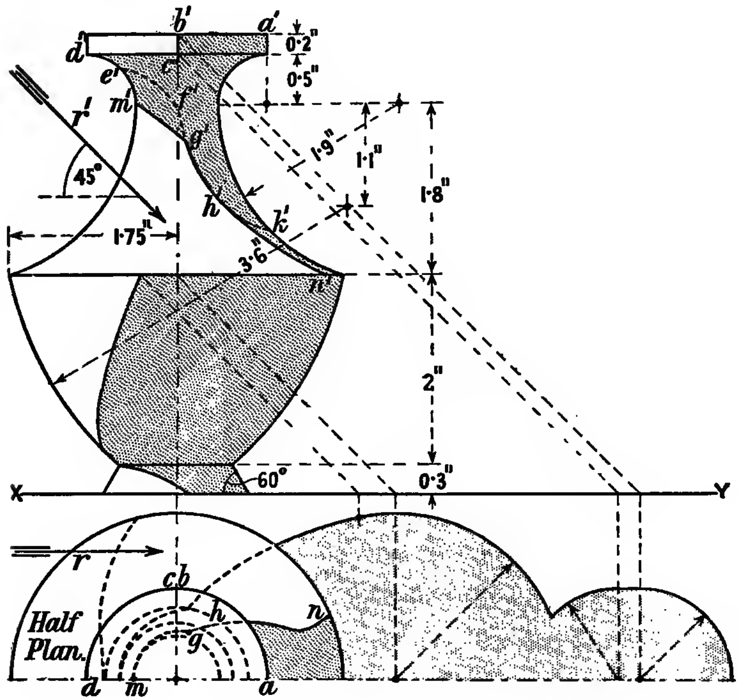


Fig. 780.

lines of contact of the cylinder or cylinders enveloping the upper part of the solid. A part of the line CD casts a shadow MG on the neck of the vase. A part of CD and a part of the solid in the neighbourhood of H cast a shadow on a part of the solid below H and the outline of this shadow is the curve HN. The curves MG and HN are obtained by the construction of Art. 334, p. 411.

If the rays of light proceed from a point the problem may be solved by the application of the construction of Art. 293, p. 328.

Exercises XXVII

1. The projections of a rectangle ABCD and a square MNOP are given in Fig. 781. Determine the shadow cast by the square on the rectangle and the shadows cast by both figures on the planes of projection. The rays of light are parallel to *rr'*.

2. Determine the shadows cast by the triangle *abc a'b'c'* (Fig. 782) on the planes of projection. The rays of light proceed from the point *ss'*.

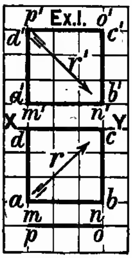


FIG. 781.

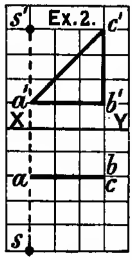


FIG. 782.

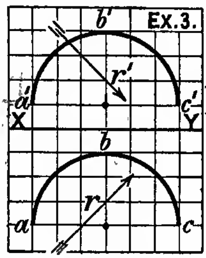


FIG. 783.

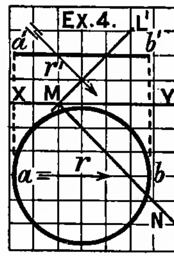


FIG. 784.

In reproducing the above diagrams take the small squares as of half inch side.

3. The semicircles *abc* and *a'b'c'* (Fig. 783) are the plan and elevation respectively of a certain curve. Show the shadows cast by this curve on the planes of projection when the rays of light are parallel to *rr'*.

4. The plan *ab* and elevation *a'b'* of a horizontal circle are given in Fig. 784. An oblique plane L'MN is also given. Determine the shadows cast by the circle on the oblique plane and on the horizontal plane when the rays of light are parallel to *rr'*.

5. *ab*, the plan of a straight line makes 45° with XY. *a* and *b* are $\frac{1}{2}$ inch $1\frac{1}{2}$ inches respectively below XY. A and B are 2 inches and 1 inch respectively above the H.P. The shadow of AB on the V.P. is a horizontal line 3 inches long. (1) Determine the directions, in plan and elevation, of parallel rays of light which would cast this shadow. (2) The shadow being cast by light from a luminous point which is in a plane perpendicular to XY and which contains the point A, determine the plan and elevation of the luminous point. Show the shadow in each case.

6. A cube of $1\frac{1}{2}$ inches edge has its base horizontal and 1 inch above the H.P. Determine the shadow cast by the cube on the H.P. when the rays of light are parallel to a diagonal of the solid.

7. A solid of the form of the letter T stands on the floor in the angle of two vertical walls as shown in Fig. 785. Determine the shadows cast by the solid on the floor and on the walls, the rays of light being parallel to *rr'*.

8. Fig. 786 represents a square pyramid penetrating a square prism. Determine the shadows cast by these solids on one another and on the planes of projection, the rays of light being parallel to *rr'*.

9. Determine the shadow cast by the object shown in Fig. 787 on itself and on the horizontal plane. The rays of light are parallel to *rr'*.

10. Determine the shadow cast by the object shown in Fig. 788 on itself and on the horizontal plane. The rays of light are parallel to *rr'*.

11. *o* is the centre of a circle 2 inches in diameter. *oab* is a straight line cutting the circle at *c*. *oa* = 0·5 inch, *ob* = 2 inches. The circle is the plan of a right circular cylinder standing on the H.P. Height of cylinder, 2 inches. *ab* is the plan of a straight line which touches the upper end of the cylinder at C and has the end B on the H.P. Show in plan and elevation the shadow cast by the

line on the cylinder and on the H.P. The ground line for the elevation makes 75° with ab. The plans of the rays of light make 45° with the ground line and 30° with ab, and the elevations of the rays are perpendicular to their plans.

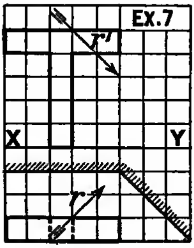


FIG. 785.

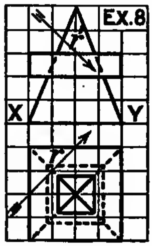


FIG. 786.

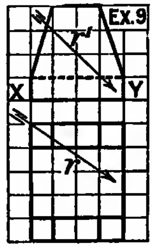


FIG. 787.

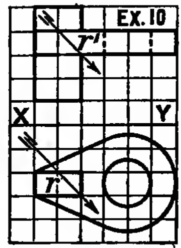


FIG. 788.

In reproducing the above diagrams take the small squares as of half inch side.

12. The solid shown in Fig. 789 is illuminated by rays of light which are parallel to rr'. Determine the parts of the surface of the solid which are in shadow.

13. A solid made up of two cylinders is shown in Fig. 790. Determine the shadow cast by this solid on the V.P. when the rays of light proceed from the point ss'. Also, indicate the parts of the surface of the solid which are in shadow.

14. A cone and cylinder are shown in Fig. 791. Determine the shadows cast by these solids on the H.P., also the shadow cast by the cone on the cylinder. Indicate the parts of the surfaces of the solids which are not illuminated. The rays of light are parallel to rr'.

15. A right cone, base 3 inches diameter, axis 3·5 inches long, lies on the ground, one of the generating lines of the cone being the line of contact. The rays of light being parallel, inclined at 45° to the ground, and their plans inclined

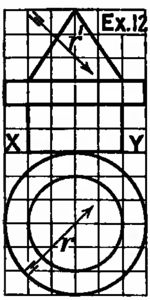


FIG. 789.

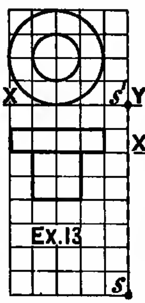


FIG. 790.

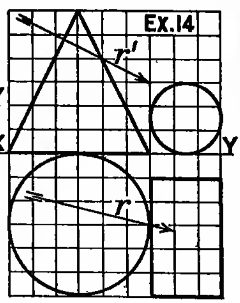


FIG. 791.

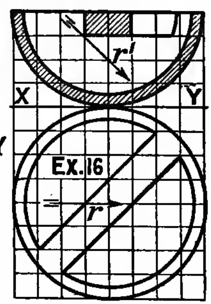


FIG. 792.

In reproducing the above diagrams take the small squares as of half inch side.

at 35° to the plan of the axis of the cone, and the base of the cone being in shadow, determine the shadow cast by the cone on the ground.

16. Fig. 792 shows a hemispherical cup with a cross bar at the top. Show on the plan the shadow cast on the inside surface of the cup. The rays of light are parallel to rr'.

17. The curved surface of a bowl is a zone of a sphere. Larger diameter of zone, 2·75 inches. Smaller diameter, 1·75 inches. Height, 1 inch. The bowl stands with its base on the ground and is illuminated from a point which is 2·25 inches above the ground and whose plan is 2 inches from the centre of the plan

of the bowl. Determine the shadow of the bowl on the ground and the portions of the bowl in shade, both in plan and elevation. The ground line for the elevation makes 30° with the line joining the plan of the source of light and the centre of the plan of the bowl. The thickness of the bowl is to be neglected.

18. a is the centre of a circle 3·25 inches in diameter. b is the centre of a circle 2 inches in diameter. $ab = 1·75$ inches. ca is a line making the angle $cab = 10°$. The larger circle is the plan of a hemispherical hole in the ground. The smaller circle is the plan of a sphere resting on the ground. Determine the plan of the shadows cast in the hole and on the sphere by parallel rays of light. The plans of the rays of light are parallel to ca and the rays are inclined at 35° to the ground.

19. Front and side elevations of a corbel projecting from a vertical wall are given in Fig. 793. Show on the front elevation the shadow cast by the corbel on the wall, also the part of the corbel in shade. The direction of the parallel rays of light is given.

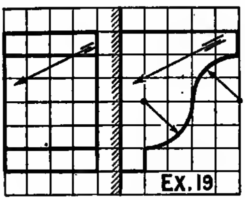


FIG. 793.

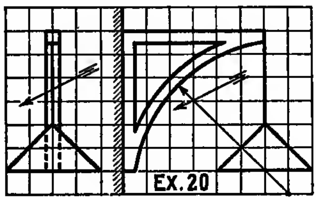


FIG. 794.

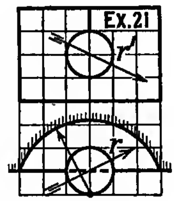


FIG. 795.

In reproducing the above diagrams take the small squares as of half inch side.

20. Two elevations of a wall bracket and a hanging conical lamp shade are given in Fig. 794. Show on the left hand elevation the shadow cast by the bracket and shade on the wall. One of the parallel rays of light is shown.

21. Fig. 795 shows a suspended sphere and a niche in a vertical wall. The

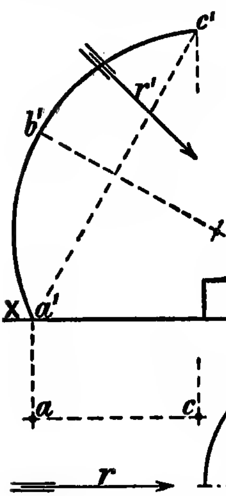


FIG. 796.

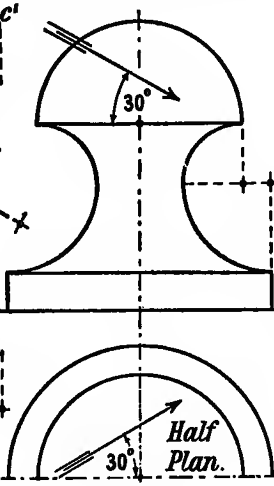


FIG. 797.

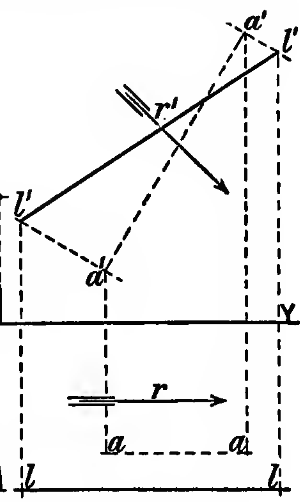


FIG. 798.

These diagrams are to be reproduced twice this size.

surface of the niche is cylindrical. Determine the shadows cast in the niche and the part of the sphere in shade. The rays of light are parallel to rr'.

22. The circular arc ABC (Fig. 796) rotates about its chord AC. Draw the elevation and plan of the figure generated. Determine the shadow cast on the horizontal plane by parallel rays of light, one of which R is given. Show the margin of light and shade on the surface of the solid. [B.E.]

23. The elevation and half plan of a solid of revolution are given in Fig. 797, the axis being vertical. Find the projections of the limits of light and shade on the solid, and the complete outline of the shadow thrown by the solid on the horizontal plane. The arrows indicate the direction of the parallel rays of light. [B.E.]

24. The line LL (Fig. 798) rotates about the axis AA. Draw the plan and elevation of the figure generated. Determine the shadow cast on the horizontal plane by parallel rays of light, one of which R is given. [B.E.]

25. An annulus or anchor ring is cut in two by an axial plane. One of the halves is shown in plan (Fig. 799), with its section ends resting on the horizontal plane. Draw the elevation of this semi-annulus on XY. Determine also the shadow cast on the horizontal plane by rays of light, parallel to the vertical plane, and inclined at 30° to the horizontal plane, one of which is shown in plan at r. And show the projections of the limits of light and shade on the surface of the solid. [B.E.]

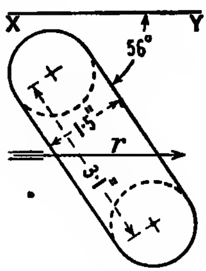


Fig. 799.

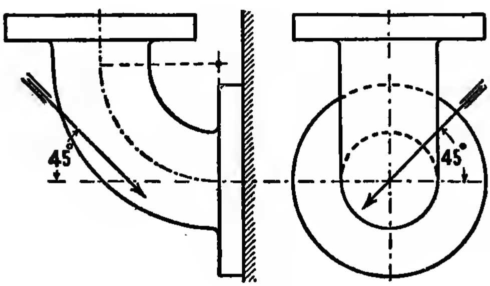


Fig. 800.

26. Two elevations of a pipe bend, with two circular flanges, are shown in Fig. 800. Reproduce these elevations three times the size given, and determine the projections of the limits of light and shade on the solid when it is illuminated by parallel rays of light inclined as shown. Show also on the right hand elevation the shadow cast by the whole solid on the vertical plane containing the face of the vertical flange. To avoid the interference of the left hand elevation with the shadow cast on the vertical plane the two elevations may be placed further apart.

27. The plan of an anchor ring resting on the horizontal plane is two concentric circles, the larger being 3·1 inches in diameter, and the smaller 1·1 inches in diameter. The smaller circle is also the plan of a vertical right cylinder (height 1·6 inches) standing on the horizontal plane. Determine the shadow cast on the horizontal plane, and the unilluminated portion of the anchor ring, taking a point as a source of light which is 2·7 inches above the horizontal plane and whose plan is 2·2 inches from the centre of the plan of the ring.

CHAPTER XXVIII

MISCELLANEOUS PROBLEMS IN SOLID GEOMETRY

341. The Regular Dodecahedron.—The *regular dodecahedron* is one of the five regular solids and has twelve faces all equal and regular pentagons. This solid is shown in its simplest position, in relation to the planes of projection, in Fig. 801. One face ABCDE is on the horizontal plane and AB, one edge of that face, is at right angles to XY. Let *o* be the centre of the circumscribing circle of the regular pentagon *abcde*. The sides of the pentagon *abcde* are the horizontal traces of five of the faces of the dodecahedron, and these five faces meet in straight lines whose plans pass through the angular points *a*, *b*, *c*, *d*, and *e*, and if produced these plans pass through *o*.

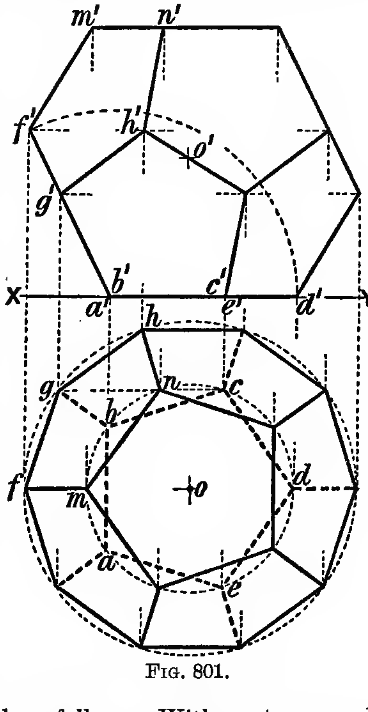


FIG. 801.

Consider the two faces whose horizontal traces are *ab* and *bc*; these faces meet in a line whose plan *gb*, when produced, passes through *o*. To find the point *g*, imagine the face ABG to revolve about AB in the clockwise direction until it is in the horizontal plane. It will then evidently coincide with the pentagon *abcde* and the plan of the point G will travel in the line *gc* perpendicular to *ab* Hence a line through *c* perpendicular to *ab* to meet a radial line *obg* at *g* determines the point *g*.

The plan may now be completed as follows. With centre *o* and radius *og* describe a circle. Divide the circumference of this circle into ten equal parts at the points *f*, *g*, *h*, etc. Draw the circumscribing circle of the pentagon *abcde*. Draw the radial lines *fm*, *hn*, etc. and join the various points thus found, as shown.

To draw the elevation: with centre a' and radius $a'd'$ describe the arc $d'f'$ to cut the projector from f at f'. Join $a'f'$. The line $a'f'$ is the elevation of the face whose horizontal trace is ab. A projector from g to meet $a'f'$ at g' determines the elevation of the point G. Points whose plans are the alternate angular points of the outer polygon, beginning with g, have their elevations at the same level as g', and the points whose plans are the remaining angular points of this outer polygon have their elevations at the same level as f'. Points such as M and N on the top face are at a height above F equal to the height of G above the horizontal plane.

From the plan and elevation thus determined other projections may be drawn in the usual way. For example, if a ground line be taken perpendicular to $m'd'$, the elevation of one of the axes of the solid, a plan of the solid, when that axis is vertical, may be projected from the elevation already drawn. But the student is recommended to try and draw directly the plan and an elevation of the dodecahedron when its axis is vertical, without first drawing it with one face on the horizontal plane.

When an axis of the solid is vertical it should be noticed that the planes of the three faces meeting at an extremity of that axis will be equally inclined to the ground, and being equally inclined to one another, their lines of intersection will in plan make 120° with one another.

As an exercise in drawing the projections of the dodecahedron the edges of the solid may be conveniently taken, say, 1·25 inches long.

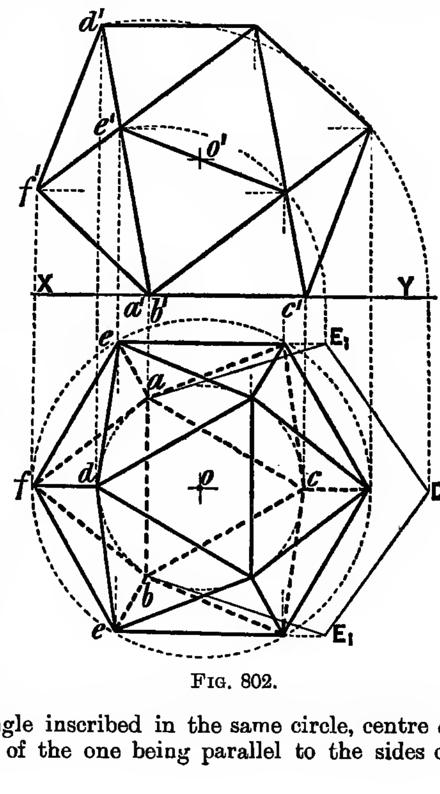


FIG. 802.

342. The Regular Icosahedron.—The *regular icosahedron* is another one of the five regular solids and has twenty faces all equal equilateral triangles. Fig. 802 shows a regular icosahedron, in plan and elevation, when one face ABC is on the horizontal plane and AB one edge of that face is perpendicular to XY. The plan of the top face, which is also horizontal, is an equilateral triangle inscribed in the same circle, centre o, as the triangle abc, the sides of the one being parallel to the sides of the other, as shown.

A property of the icosahedron, which leads to simple constructions for drawing it, is, that any angular point of the solid is the vertex of a right pentagonal pyramid, the five faces meeting at the vertex being faces of the icosahedron. Thus the faces meeting at the point F are the faces of a right pyramid whose base is the regular pentagon ABEDE, and the sides of the pentagon are edges of the icosahedron.

To complete the plan proceed as follows. Draw on *ab* as base the regular pentagon $abE_1D_1E_1$. Through E_1 draw E_1e perpendicular to *ab* to meet the radial line *oae* at *e*. With centre *o* and radius *oe* describe a circle. A regular hexagon inscribed in this circle, one angular point being at *e*, is the boundary line of the plan, which is completed by joining the angular points of the hexagon to the angular points of the two central equilateral triangles, as shown. The pentagon $abE_1D_1E_1$ is the rabatment, on the horizontal plane, of the pentagon of which *abede* is the plan. Keeping this in mind the construction of the elevation easily follows.

The student should also draw directly a plan and an elevation of the icosahedron when an axis of the solid is vertical. The extremities of the axis in question will be the vertices of two of the right pentagonal pyramids previously mentioned. The pentagonal bases of these pyramids will be horizontal and their plans will be inscribed in the same circle.

As an exercise in drawing the projections of the icosahedron the edges may be conveniently taken, say, 1·75 inches long.

343. Solids Inscribed in the Sphere.—A solid is said to be inscribed in a sphere when all its angular points are on the surface of the sphere.

Fig. 803 shows the constructions for finding the length of an edge of each of the five regular solids when inscribed in a sphere of given diameter.

AB is the diameter of the sphere. On this a semicircle is described.

Take BD equal to one-third of AB. Draw DE at right angles to AB to meet the semicircle at E. Join AE and BE. AE is the edge of the inscribed tetrahedron, and BE is the edge of the inscribed cube.

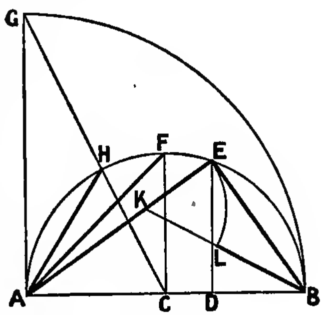


Fig. 803.

C being the middle point of AB, draw CF at right angles to AB to meet the semicircle at F. Join AF. AF is the edge of the inscribed octahedron.

Draw AG at right angles to AB and make AG equal to AB. Draw CG cutting the semicircle at H. Join AH. AH is the edge of the inscribed icosahedron.

On EA make EK equal to half of BE. Join BK. With centre K and radius KE describe an arc to cut BK at L. BL is the edge of the inscribed dodecahedron.

The following facts should be kept in view when working problems on the regular solids inscribed in the sphere.

The plane containing one edge of a tetrahedron and the centre of the circumscribing sphere bisects the opposite edge at right angles.

The rectangle which has for its diagonals two of the diagonals of a cube is inscribed in a great circle of the circumscribing sphere.

The octahedron can be divided into two square pyramids in three different ways, and the square bases of all these pyramids will be inscribed in great circles of the circumscribing sphere.

344. Solids Circumscribing the Sphere.—A solid is said to circumscribe a sphere, or a sphere to be inscribed in a solid, when all the faces of the solid are tangential to the surface of the sphere.

A plane bisecting the angle between any two faces of the solid passes through the centre of the inscribed sphere.

In the case of any one of the five regular solids the centres of the inscribed and circumscribing spheres coincide.

345. The Sphere, Cylinder and Cone in Contact.—If two spheres touch one another, the point of contact and the centres of the spheres are in the same straight line.

If a sphere touches a cylinder or cone it will do so at a point, and if the cylinder or cone be a right circular cylinder or a right circular cone, the centre of the sphere the point of contact, and the axis of the cylinder or cone are in the same plane.

If two cylinders or two cones or a cylinder and a cone touch one another, they will do so either along a straight line or at a point. If the cylinders or cones are right circular cylinders or right circular cones and they touch one another along a straight line, the line of contact and the axes of the surfaces are in the same plane; the axes will therefore either be parallel or they will intersect.

When two right circular cylinders touch one another at a point the common perpendicular to their axes passes through the point of contact.

Two surfaces which touch one another have a common tangent plane.

346. Projections of Four Spheres in Mutual Contact.—Denote the spheres and their centres by A, B, C, and D, and let their radii be r_1, r_2, r_3, and r_4 respectively. Assume that the spheres A, B, and C are resting on the horizontal plane and that the line joining the centres A and B is parallel to the vertical plane of projection. The elevations of the spheres A and B (Fig. 804) are circles touching one another and XY and after these are drawn the plans may be projected from them as shown. Next determine the horizontal distances between the centres A and C, and B and C as shown on the elevation. This determines the centre cc' and the plan and elevation of the sphere C may then be drawn.

The centre D of the fourth sphere will be at a distance $r_1 + r_4$ from the centre A of the first sphere, and at a distance $r_2 + r_4$ from the centre B of the second sphere. If the triangle ADB be conceived to rotate about the side AB, the point D will describe a circle which will be

the locus of the centre of a sphere of radius r_4 which touches the spheres A and B.

Remembering that ab, $a'b'$ is parallel to the vertical plane of projection, with centres a' and b' and radii equal to $r_1 + r_4$ and $r_2 + r_4$ respectively, describe arcs intersecting at d_1'. Draw $d_1'e'$ to intersect $a'b'$ at right angles at e'. Then $e'd_1'$ is the radius of the circle described by the point D when the triangle ADB is rotated about AB. The plan of this circle is an ellipse whose semi-minor axis is ed_1, the projection

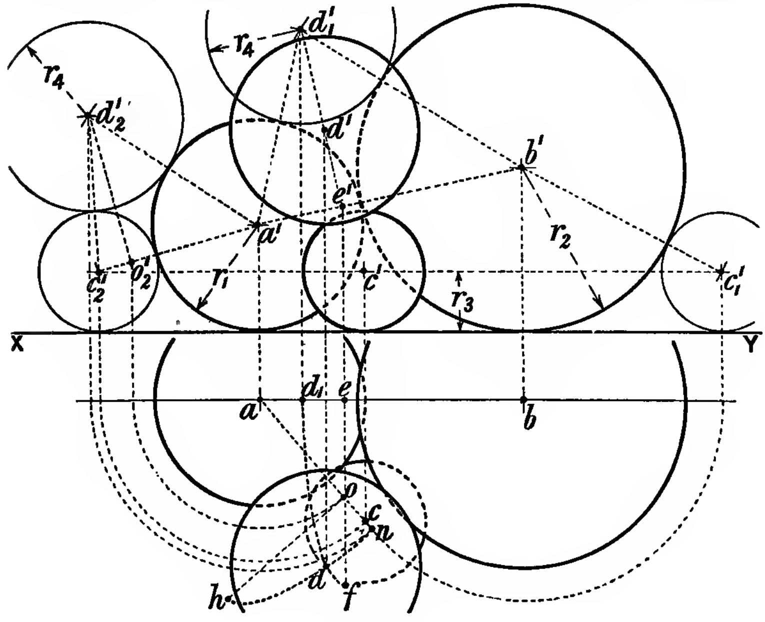


FIG. 804.

of $e'd_1'$ on ab. The semi-major axis ef is at right angles to ab and equal to $e'd_1'$. A quarter of this ellipse is shown.

Again, the centre D of the fourth sphere will be at a distance $r_1 + r_4$ from the centre A of the first sphere, and at a distance $r_3 + r_4$ from the centre C of the third sphere. If the triangle ADC be conceived to rotate about the side AC, the point D will describe a circle which will be the locus of the centre of a sphere of radius r_4 which touches the spheres A and C. The circle whose centre is c_2' is the elevation of the sphere C when that sphere is brought round into the position in which it touches the sphere A and the horizontal plane and has its centre in a plane parallel to the vertical plane of projection and containing the centre A of the first sphere. With centres a' and c_2' and radii $r_1 + r_4$ and $r_3 + r_4$ respectively, describe arcs intersecting at d_2'. Draw $d_2'o_2'$ to intersect $a'c_2'$ at right angles at o_2'. Then $o_2'd_2'$ is the radius of the circle

described by the point D when the triangle ADC is rotated about AC. The plan of this circle is an ellipse whose semi-minor axis *on* is on *ac* and is obtained as shown. The semi-major axis *oh* is at right angles to *ac* and equal to $o_2'd_2'$. A quarter of this ellipse is shown.

The two circles which have been referred to as described by the point D intersect at two points one of which has the point *d* for its plan, *d* being on the two ellipses which are the plans of the circles. The elevation *d'* is in $e'd_1'$ and in the projector from *d*. This determines the centre of the fourth sphere and the plan and elevation of that sphere may now be drawn.

The two circles referred to as described by the point D are on the surface of a sphere whose centre is A and radius $r_1 + r_4$. The ellipses which are the plans of these circles may intersect at four points, but only two of these are plans of the points of intersection of the circles. The student should have no difficulty in deciding on which of the points of intersection of the ellipses are to be taken.

347. Spherical Roulettes.—If two cones be placed in line contact with their vertices coinciding and if one cone be made to roll on the other, which is fixed, any point carried by the rolling cone will describe a *spherical roulette.* The describing point carried by the rolling cone may be outside, or inside, or on the surface of that cone and it will be at a constant distance from the common vertex of the two cones; hence the describing point will move on the surface of a fixed sphere.

When the two cones are right circular cones, and the describing point is on the surface of the rolling cone, the spherical roulette becomes a *spherical epicycloid* or a *spherical hypocycloid* according as the rolling cone rolls outside or inside the fixed cone.

Spherical roulettes are tortuous curves, and no single projection of a tortuous curve can show its true form. A spherical roulette has therefore to be represented by two projections.

Projections of a spherical epicycloid and a spherical hypocycloid are shown in Fig. 805. The description which follows applies to either curve. *v'a'a'* is the elevation of the fixed cone the axis of which is vertical, and the semicircle *aba* is the half plan of this cone. *v'a'e'* is the elevation of the rolling cone when its position is such that its axis is parallel to the vertical plane of projection.

Let the rolling cone start from the position in which *vb*, *v'b'* is the line of contact and let the point of contact *bb'* of the bases of the cones in this position be the initial position of the describing point. Next suppose that the rolling cone rolls into the position in which the line of contact has *vm* for its plan. Draw a circle mnr_1 having a radius equal to the radius of the base of the rolling cone and touching the plan of the fixed cone at *m*. Consider this circle to be the rabatment of the base of the rolling cone on the plane of the base of the fixed cone when the rolling cone is in the position now being considered. The rabatment of the describing point will be r_1, the position of r_1 being such that the arc mnr_1 is equal to the arc *mb*. Restoring the base of the rolling cone to its inclined position the plan *r* of the describing point

will be in the line through r_1 parallel to mv. To fix the position of r in the line r_1r let the circle mnr_1 be carried round about the centre v into the position as_1e_1, the diameter ae_1 being in line with va and parallel to the ground line. r_1 will move to s_1. Draw the projector s_1s_1' to meet the ground line at s_1'. With centre a' and radius $a's_1'$ describe the arc $s_1's'$ to meet $a'e'$ at s'. Draw the projector $s's$ to meet s_1s parallel to ae_1 at s. With centre v and radius vs describe the arc sr to cut r_1r at r.

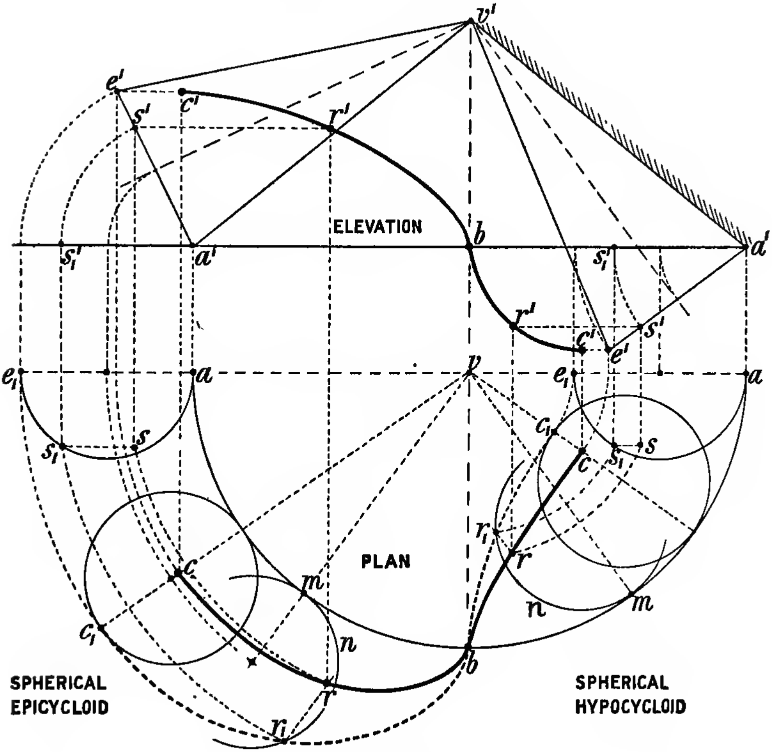


FIG. 805.

Draw the projector rr' to meet the horizontal line $s'r'$ at r'. Then rr' is a point on the spherical epicycloid or on the spherical hypocycloid. In like manner any number of points may be found. The length of curve shown is that due to half a revolution of the rolling cone.

The points such as r_1 lie on a curve, shown dotted, which is either a plane epicycloid or a plane hypocycloid.

The tangent to the spherical epicycloid or spherical hypocycloid at a point R lies on the tangent plane to the sphere whose centre is V and radius VA or VR at the point R. It also lies on the tangent plane to the sphere whose centre is M and radius MR at the point

R. The tangent is therefore the line of intersection of these two planes.

The spherical epicycloid and the spherical hypocycloid occur in connection with the formation of the teeth of bevel wheels.

348. Reflections.—When a ray of light impinges on a polished surface at a point Q, it is reflected so that the incident and reflected rays and the normal to the reflecting surface at Q are in the same plane; also the incident and reflected rays are equally inclined to the normal.

Referring to Figs. 806 and 807, let the plane of the paper be the plane containing the incident ray PQ and the normal QT to the surface whose section by the plane of the paper is AB. Then the reflected ray QR will be in the plane of the paper and will make with QT an angle θ equal to the angle between PQ and QT.

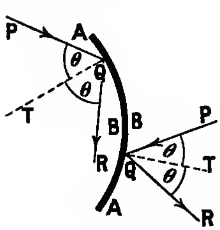


FIG. 806.

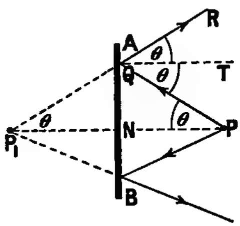


FIG. 807.

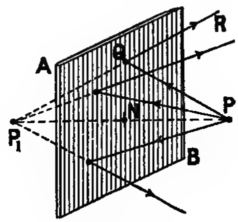


FIG. 808.

If the reflecting surface is a plane (Figs. 807 and 808) it is obvious that all incident rays from a point P will be reflected from the plane as if they came direct from a point P_1 on the other side of the plane and in the normal PN to the plane, P_1N being equal to PN. A knowledge of this fundamental theorem makes the solution of problems on reflections from plane mirrors quite simple. The point P_1 is called the *image* of the point P in the mirror AB. It should be noticed that although the image of P is in PN produced it is not necessary that the mirror should extend to the point N in order that the image may be seen from the point R.

Fig. 809 shows how to trace the path of a ray of light which passes from a fixed point P and is reflected in turn from a number of plane mirrors AB, BC, CD, and DE and then passes through a fixed point S, the mirrors being at right angles to the plane of the paper. P_1 is the image of P in AB, P_2 is the image of P_1 in BC, P_3 is the image of P_2 in CD, and P_4 is the image of P_3 in DE. The ray which is reflected from DE and passes through S must evidently be in the line P_4S. The incident ray on DE is the reflected ray from CD and must therefore be in the straight line from P_3, and so on the ray is traced back to the point P as shown.

Fig. 810 illustrates the general case of the problem: given the incident ray and a plane from which it is reflected, to determine the reflected ray. PQ is the given incident ray and H.T. and V.T. are the

traces of the given plane. Find Q the point of intersection of PQ and the plane. From a point P in PQ draw the normal PNP_1 to the plane and find N the point of intersection of this normal and the plane. Make NP_1 equal to NP. Join P_1Q and produce it. QR the produced part of this line is the reflected ray required.

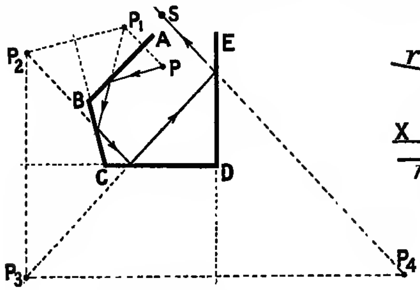


FIG. 809.

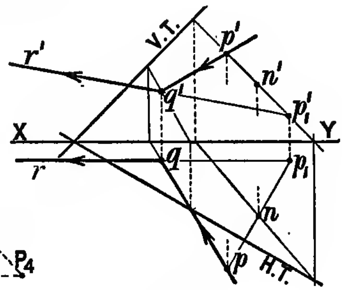


FIG. 810.

If the given surface from which a given incident ray is reflected is a curved surface, the point of intersection of the incident ray and the surface must be determined by the method for finding the intersection of a straight line and the particular curved surface given. The normal to the surface at the point where the incident ray strikes it must then be found and the plane containing it and the incident ray determined. It will then generally be necessary to obtain a projection of the incident ray and the normal on a plane parallel to their plane, or to obtain a rabatment of the incident ray and normal into one of the planes of projection, in order to draw the reflected ray which must make with the normal an angle equal to the angle between the incident ray and the normal.

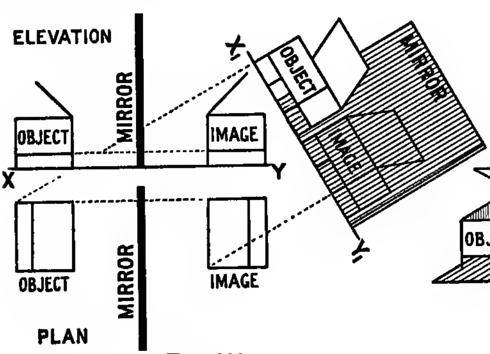


FIG. 811.

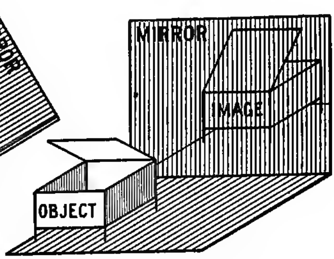


FIG. 812.

An example on the projection of an object and its image in a plane mirror is illustrated by Figs. 811 and 812. The object and its image and the mirror are shown in plan and elevation to the left in Fig. 811,

the mirror being at right angles to the planes of projection. The same Fig. shows to the right an elevation on a second ground line X_1Y_1 inclined to the first ground line XY. The pictorial projection in Fig. 812 illustrates further the relative positions of the object, mirror, and image.

Exercises XXVIII

1. Draw a circle, centre o and radius 1·75 inches. Take a point a 1 inch from o. Draw a straight line ax making an angle of 60° with oa. The circle is the plan of a sphere and a is the plan of a point A on its upper surface. A is one corner of a cube inscribed in the sphere and ax is the direction of the plan of one edge containing the point A. Complete the plan of the cube.

2. Referring to the preceding exercise, make $oa = 1{\cdot}5$ inches and the angle $oax = 60°$. a is the plan of one angular point of a tetrahedron inscribed in the sphere and ax is the direction of the plan of one edge containing the point A. Complete the plan of the tetrahedron.

3. Same as exercise 2, except that the solid is an octahedron instead of a tetrahedron.

4. The base of a pyramid is an equilateral triangle ABC of 2·5 inches side. The vertex V of the pyramid is in a line through A at right angles to the plane of ABC, and VA is 2·5 inches long. Placing the pyramid with its base on the ground and AB parallel to XY, draw the projections of the inscribed and circumscribing spheres.

5. A right circular cone, base 2·5 inches in diameter and altitude 2·5 inches, stands with its base on the ground. A cylinder 2 inches in diameter lies on the ground in contract with the cone, the axis of the cylinder being horizontal. A sphere 1·25 inches in diameter rests on the ground in contact with the cone and cylinder. Draw a plan of the group and an elevation on a vertical plane parallel to the axis of the cylinder showing the projections of the points of contact of the solids.

6. Three spheres, two of them 1·5 inches in diameter, and one 2 inches in diameter, rest on the horizontal plane, each in contact with the other two. A cylinder 2 inches in diameter rests, with its axis horizontal, on top of the spheres, touching each of them. Draw the plan of the group.

7. Two straight lines ab and cd bisect one another at right angles. $ab = 4$ inches and $cd = 3{\cdot}5$ inches. These lines are the plans of the axes of two cylinders whose diameters are equal and which touch one another. The heights of the points A, B, C, and D above the horizontal plane are 0·7 inch, 2·8 inches, 2·4 inches, and 4·5 inches respectively. Draw the plan of the cylinders and an elevation on a ground line parallel to ab, showing the point of contact.

8. abc is a triangle, $ab = 2$ inches. Angle bac = angle $abc = 30°$. a is the plan of the centre of a sphere of 1·3 inches radius which rests on the horizontal plane. b is the plan of the centre of a second sphere which rests on the horizontal plane and touches the first sphere. c is the plan of a point on the upper surface of the first sphere. Draw the plan of a third sphere which touches the other two, c being its point of contact with the first sphere.

9. A cone of revolution, vertical angle 64°, of indefinite length, lies with its curved surface on the ground; draw its plan. Determine a sphere of 0·6 inch radius, which rests on the ground and touches the cone at a point 2·5 inches from its vertex. Show the indexed plan of the point of contact, and determine the common tangent plane at this point. [B.E.]

10. A right circular cone, base 2·5 inches in diameter and axis 3 inches long, lies with its slant side on the ground. A sphere, 2 inches in diameter, moves in contact with the ground and the surface of the cone. Draw the plan of the locus of the point of contact.

11. vab is a triangle. $va = 3{\cdot}4$ inches, $vb = 2{\cdot}4$ inches, and $ab = 2$ inches. a is the plan of the centre of a sphere of 2·8 inches diameter, A being 2·5 inches

above the ground. b is the plan of the centre of a sphere of 1·75 inches diameter, B being 1·5 inches above the ground. v is the plan of the vertex of a cone whose semi-vertical angle is 20° and which touches the two spheres, V being 2 inches above the ground. Draw the plan of the group and an elevation on a ground line parallel to av, showing the points of contact of the cone and spheres.

12. Same as preceding exercise except that a cylinder 1·5 inches in diameter is to take the place of the cone, V being a point on the axis of the cylinder.

13. A cone, in elevation $v'h'k'$ (Fig. 813) lies on the horizontal plane, its axis VC parallel to the vertical plane. A cylinder of 1 inch radius, its axis parallel to the line de, $d'e'$, *touches* the cone at a point whose height is represented by the horizontal $g'b'$. Draw the plan of the surfaces, and the projections of their point of contact, showing the trace of the cylinder on the horizontal plane, and the projections of its generator in contact with the cone. [B.E.]

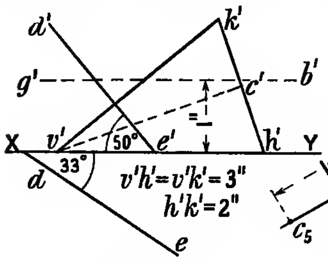


FIG. 813.

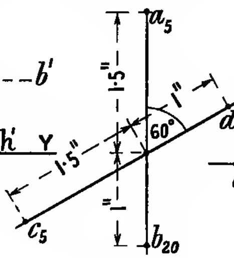


FIG. 814.

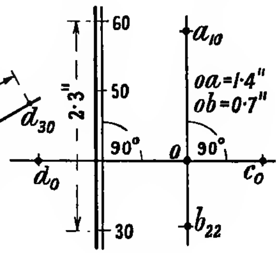


FIG. 815.

14. The straight line AB (Fig. 814) is tangential to a sphere of 1 inch radius whose centre is on the straight line CD. Draw the figured plan of the lines and sphere and show the point of contact of the sphere and the line AB. Unit for indices 0·1 inch.

15. Two lines AB and CD are given in Fig. 815 by their figured plans, and a plane is given by its scale of slope. Determine a sphere of 1·25 inches radius touching the lines and plane. Unit 0·1 inch. The sphere is intended to be *below* the plane. [B.E.]

16. $v'a'o'$ (Fig. 816) is the half elevation of a fixed right circular cone (1) whose axis VO is vertical and whose vertex is V. $v'a'e'$ is the elevation of a second right circular cone (2) in line contact with cone (1) and having its vertex at V. The cone (2) rolls on the cone (1) and a point on the circumference of the base of the rolling cone describes a spherical epicycloid of which $a'c'b'$ (above $a'o'$) is the elevation, or a spherical hypocycloid of which $a'c'b'$ (below $a'o'$) is the elevation. Draw the plan and elevation of these spherical roulettes.

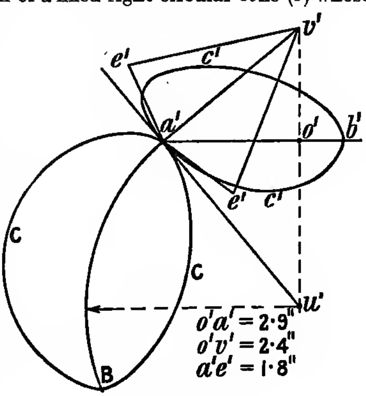


FIG. 816.

Next take a right circular cone (3) which has the base of the cone (1) for a circular section and u' for the elevation of its vertex, the angle $v'a'u'$ being a right angle. Take the spherical roulette, of which $a'c'b'$ is the elevation, to be the directing curve of a cone (4) with its vertex at V. Determine the curve of intersection (5) of the cones (3) and (4). Develop the surface of the cone (3) with the curve (5) on it. a'CB is a sketch of this development.

Lastly, draw on the circular arc a'B as base a plane epicycloid and a plane hypocycloid with a rolling circle 1·8 inches in diameter (the diameter of the base of the rolling cone) for comparison with the curves a'CB on opposite sides of the arc a'B.

17. Taking the fixed and rolling cones (1) and (2) as given in the preceding exercise and Fig. 816, draw the plan and elevation of the spherical epitrochoid described by a point on a diameter of the base of the rolling cone produced, the describing point to be 1·4 inches from the centre of the base of the rolling cone.

18. The ray of light rr' (Fig. 817) impinges on the horizontal plane and is reflected on to the vertical plane of projection from which it is again reflected. Show the path of the ray in plan and elevation.

19. A ray of light rr' (Fig. 818) impinges on the horizontal plane and is reflected on to the vertical plane HTV from which it is again reflected. Draw the plan and elevation of the path of the ray. Show also in plan and elevation the path of a ray which after reflection from the horizontal plane and the plane HVT passes through the point nn', the incident ray being parallel to rr'.

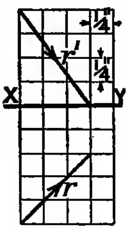


Fig. 817.

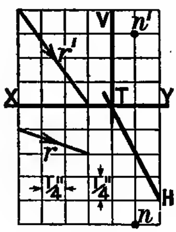


Fig. 818.

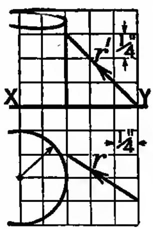


Fig. 819.

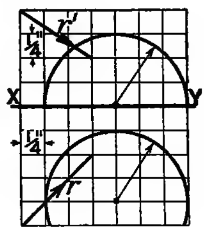


Fig. 820.

20. A polished right circular cylinder (Fig. 819) has its axis vertical. The ray of light rr' strikes this cylinder and is reflected on to the vertical plane of projection. Draw the plan and elevation of the path of the ray.

21. The surface of a shield (Fig. 820) is spherical. A projectile strikes this shield and is deflected. rr' is the line of flight before impact. Show the line of flight after impact, assuming that the projectile is deflected like a ray of light.

22. A right circular cone, diameter of base 2 inches, altitude 2 inches, stands with its base on the ground. A ray of light is parallel to the ground line and 0·75 inch above the ground, and its plan is at a perpendicular distance of 0·25 inch from the centre of the plan of the cone. This ray of light impinges on the surface of the cone and is reflected. Draw the plan and elevation of the reflected ray.

23. An object and a mirror are given in Fig. 821. Draw an elevation of the

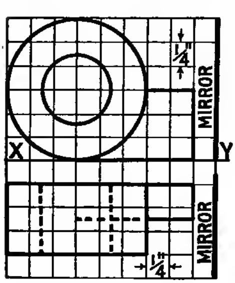


Fig. 821.

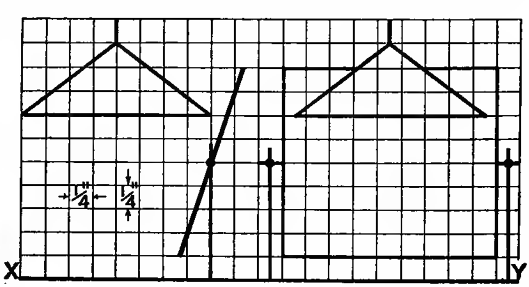


Fig. 822.

object and its image in the mirror on a ground line inclined at 60° to XY. Assume that the mirror is large enough to show the whole of the image in the elevation asked for.

24. Fig. 822 represents a hanging lamp shade and a tilted mirror. Draw the two elevations and the plan of the image of the conical shade in the mirror. From the plan project on a ground line inclined at 70° to XY the elevation of the mirror, the shade, and as much of the image as would be seen within the boundary of the mirror. [B.E.]

25. The point *cc'* (Fig. 823) is the centre of a polished sphere of 1 inch radius. A ray of light parallel to *rr'* impinges on the sphere and is reflected, the reflected ray passing through the point *pp'*. Draw the projections of the path of the ray.

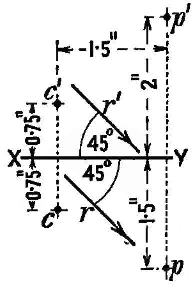


FIG. 823.

APPENDIX

MATHEMATICAL TABLES

The Mathematical Tables on the following five pages are those supplied to Candidates at the Examinations of the Board of Education, and are reprinted by permission of the Controller of H.M. Stationery Office.

Angle.		Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
Degrees.	Radians.								
0°	0	0	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·062	·0523	·0624	19·0811	·9886	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1046	·1051	9·5144	·9946	1·338	1·4661	81
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
8	·1671	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	6·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	6·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	76
16	·2793	·278	·2766	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3067	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9611	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3684	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9206	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
26	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4648	·5543	1·8040	·8746	1·015	1·0647	61
30	·6236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·6411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·6586	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	67
34	·5934	·585	·5592	·6745	1·4826	·8290	·839	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	65
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·669	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·5981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cosine	Co-tangent	Tangent	Sine	Chord	Radians	Degrees
								Angle	

Logarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	6	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3386	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	16	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3827	3945	3962	2	4	6	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	6	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4602	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4883	4688	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	6024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	6092	5105	5119	5132	6145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	6263	5276	6289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	6366	5378	5391	5403	6416	6428	1	3	4	6	6	8	9	10	11
35	5441	5463	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	6587	6599	6611	5623	5635	5647	5658	6670	1	2	4	5	6	7	8	10	11
37	5682	5694	6705	6717	5729	6740	5752	5763	5775	5786	1	2	3	6	6	7	8	9	10
38	5798	6809	6821	5832	6843	5855	5886	6877	5888	6899	1	2	3	5	6	7	8	9	10
39	6911	6922	5933	6944	5855	5966	6977	5988	5999	6010	1	2	3	4	6	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6365	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6622	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6681	8571	6680	6590	6599	8609	6618	1	2	3	4	6	6	7	8	9
46	6628	6637	6646	6656	8665	6675	6684	6693	8702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	8767	6776	6786	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	8857	6866	6875	6884	6893	1	2	3	4	4	6	6	7	8
49	6902	6911	6920	6928	6937	6948	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7060	7059	7067	1	2	3	3	4	5	6	7	8

Logarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7207	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7469	7466	7474	1	2	2	3	4	5	6	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7876	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8565	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	6	5
73	8633	8639	8845	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8916	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9366	9360	9385	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9466	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9652	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9758	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9796	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9864	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1161	1163	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1260	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1488	1489	1493	1496	1500	1603	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1628	1531	1536	1538	1642	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1583	1687	1670	1674	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1689	1692	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1837	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1887	1671	1675	1679	1683	1687	1690	1894	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1768	1762	1768	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1864	1868	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1906	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1960	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2076	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2188	2173	2178	2163	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344.	2350	2355	2360	2386	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	6
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2468	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2659	2564	1	1	2	2	3	4	4	6	5
·41	2670	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	6	5
·42	2630	2636	2642	2649	2655	2661	2887	2673	2679	2685	1	1	2	2	3	4	4	6	6
·43	2892	2898	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2811	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2961	2958	2966	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3065	3062	3089	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3166	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3189	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3367	3965	3373	3381	1	2	2	3	4	5	6	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3469	1	2	2	3	4	5	6	6	7
·54	3487	3476	3483	3491	3499	3608	3516	3624	3532	3640	1	2	2	3	4	5	6	6	7
·55	3648	3556	3666	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	9846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3938	3946	3954	3963	3972	1	2	9	4	5	5	6	7	6
·60	3881	3990	3999	4009	4018	4027	4036	4046	4055	4084	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4169	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4268	4276	4286	4296	4305	4316	4325	4335	4345	4955	1	2	3	4	5	6	7	8	9
·64	4366	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4629	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4888	4909	4920	4932	4943	4965	4966	4977	4989	6000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	6058	6070	6082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	6129	5140	5152	6164	5176	5188	5200	5212	6224	6238	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	6284	6297	5309	5321	5333	5346	5368	1	2	4	5	6	7	9	10	11
·73	5370	6383	6396	6408	5420	6433	5445	5458	5470	6483	1	3	4	5	6	8	9	10	11
·74	6495	5508	6621	5534	5546	5569	5672	5685	6598	6610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	6808	6821	5834	5848	5861	5676	1	3	4	5	7	8	9	11	12
·77	6888	5902	6818	6929	5943	5967	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	6	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6261	6296	1	3	4	6	7	9	10	11	13
·80	6310	6324	8339	6363	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6466	6501	6516	6631	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6746	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6962	6998	7015	7031	7047	7063	2	3	6	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7296	7311	7328	7345	7362	7379	7996	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7484	7482	7499	7516	7534	7551	7568	2	3	6	7	9	10	12	14	16
·88	7586	7603	7821	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7825	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8366	8375	8395	8414	8433	8463	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	6690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8760	8770	8790	8810	8831	8861	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9067	9078	9099	2	4	6	8	10	12	16	17	19
·96	9120	9141	9162	9183	9204	8226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9378	9397	9419	9441	9462	9484	9508	8528	2	4	7	9	11	13	16	17	20
·98	9650	9672	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9854	9877	2	5	7	9	11	14	16	18	20

INDEX

Numbers refer to pages

THE END

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